Math 2270-1 Midterm 3, March 27, 2009

Solutions

Problem 1 (20 points)**.** Consider the vectors

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^4$. Find

a) their lengths $\|u\|$ and $\|v\|$;
b) the angle θ between u and v.

Solution: We have

$$\|u\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$$

and

$$\|v\| = \sqrt{(-1)^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2.$$

Moreover,

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{1}{4}(1(-1) + 1^2 + 1^2 + 1^2) = \frac{1}{2}.$$

Hence, the angle θ is 60 degrees.

Problem 2 (20 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & x \end{bmatrix}.$$

Find

a) x such that this matrix is orthogonal;
b) in this case, find A^{-1}.

Solution: The third column has to be orthogonal to the first one. Hence

$$2 + 2 + 2x = 0$$

and $x = -2$. Clearly, now third column is also orthogonal to the second one, since $(-2) + 2 \cdot 2 + (-2) = 0$. Moreover, the lengths of all three columns are equal to 1 and the matrix is orthogonal.

Since the inverse of an orthogonal matrix is equal to its transpose, we have

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

Problem 3 (20 points)**.** Find the matrix A of the orthogonal projection onto the line in $\mathbb{R}^4$ spanned by the vector

$$u = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: We have

$$A = uu^T = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Problem 4 (20 points)**.** Find the determinant

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$$

using Gauss elimination.

Solution: By subtracting the first row from second, third and fourth we get

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix}.$$

Now, by subtracting second row from the third one, we get

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & -2 \end{vmatrix}.$$

finally, by subtracting the third row from the fourth one, we get

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 2 \cdot 2 \cdot (-4) = -16.$$

Problem 5 (20 points)**.** Solve the linear system

$$\begin{array}{rrrcr} x & +2y & -3z & = & -5 \\ 3x & -y & -z & = & 1 \\ x & +y & -z & = & -1 \end{array}$$

using Cramer's rule.

Solution: The determinant of the system is equal to

$$\begin{vmatrix} 1 & 2 & -3 \\ 3 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1\cdot(-1)^2+2\cdot(-1)+(-3)\cdot 3-(-3)\cdot(-1)-(-1)-2\cdot 3\cdot(-1) = -6$$

by Sarrus rule.

On the other hand,

$$x = -\frac{1}{6}\begin{vmatrix} -5 & 2 & -3 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= (-5)\cdot(-1)^2+2\cdot(-1)^2+(-3)-(-3)\cdot(-1)^2-(-5)\cdot(-1)-2\cdot(-1) = -\frac{6}{6} = 1.$$

Also,

$$y = -\frac{1}{6}\begin{vmatrix} 1 & -5 & -3 \\ 3 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= (-1)+(-5)\cdot(-1)+(-3)\cdot 3\cdot(-1)-(-3)-(-1)^2-(-5)\cdot 3\cdot(-1) = 0,$$

and

$$z = -\frac{1}{6}\begin{vmatrix} 1 & 2 & -5 \\ 3 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (-1)^2 + 2 + (-5)\cdot 3 - (-5)\cdot(-1) - 1 - 2\cdot 3\cdot(-1) = \frac{12}{6} = 2.$$