Math 2270-2 Midterm 2, February 27, 2009

Solutions

Problem 1 (20 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}.$$

Find:

a) A basis of the kernel of this linear transformation;
b) The nullity of this linear transformation.

Solution: Multiplying the first row by 4 and subtracting from the second row, and multiplying it by 7 and subtracting from the third row we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -6 & -3 \\ 0 & -12 & -6 \end{bmatrix}.$$

Dividing the second row by -6 and the third by -12 we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

Hence by subtracting the second row from the third, we get:

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiplying the second row by 3 and subtracting it from the first one we get

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

What is the reduced row echelon form of A. Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be a vector in the kernel. Then x, y and z satisfy the equations

$$\begin{array}{ccc} x & & +\frac{1}{2}z & = 0 \\ & y & +\frac{1}{2}z & = 0 \end{array}.$$

Hence, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}z \\ -\frac{1}{2}z \\ z \end{bmatrix} = -\frac{1}{2}z \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

and the kernel is spanned by the vector

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

In particular, nullity of A is equal to 1.

Problem 2 (20 points)**.** Consider the linear transformation from the first problem.

Find:

a) A basis of the image of this linear transformation;
b) The rank of this linear transformation.

Solution: Since the reduced row echelon form of A is equal to

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

we know that the rank of A is 2 and a basis of the image of A is given by the first two columns of A, i.e., by the vectors

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

by dividing the second vector by 3 we get a slightly simpler basis

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Problem 3 (20 points)**.** Consider the vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

in $\mathbb{R}^3$. Are they linearly independent or not? If not, find which one is redundant.

Solution:

Consider the equation

$$c_1 u + c_2 v + c_3 w = 0.$$

This means that

$$\begin{array}{llll} c_1 & +2c_2 & & = 0 \\ 2c_1 & +c_2 & +3c_3 & = 0 \\ 3c_1 & +3c_2 & +3c_3 & = 0 \end{array}.$$

Dividing the last equation by 3, we get the augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

By multiplying the first row by 2 and subtracting it from the second, and subtracting the first row from the third, we get

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

After division of the second row by -3. we get

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

If we add the second row to the third, we have

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, if we multiply the second row by 2 and subtract from the first one we get

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the reduced row echelon form of the augmented matrix. This implies that the solutions are $c_1 = -2t$, $c_2 = t$ and $c_3 = t$ for arbitrary t. It follows that u, v and w are linearly dependent. Moreover, u and v are linearly independent (since $c_3 = 0$ implies $t = 0$). Hence w is redundant (since $w = 2u - v$).

Problem 4 (20 points)**.** Consider the linear space P_2 of all polynomials in t of degree ≤ 2. Show that $T(f) = f'' - 2f'$ is a linear transformation from P_2 into P_2. Find the matrix of this linear transformation in the basis $\{1, t, t^2\}$.

a) Find the image and the kernel of this linear transformation.
b) Is this linear transformation an isomorphism?

(Explain your answer!).

Solution:

We have $T(1) = 0$, $T(t) = -2$ and $T(t^2) = 2 - 4t$. Hence the matrix is equal to

$$A = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$T(a + bt + ct^2) = 2c - 2(b + 2ct) = 2(c - b) - 4ct.$$

Hence, the kernel consists of all constant polynomials. The image consists of all polynomials of degree ≤ 1.

Since the kernel is nonzero, this map is not an isomorphism.

Problem 5 (20 points)**.** Consider the linear transformation T from $\mathbb{R}^2$ into $\mathbb{R}^2$ given by the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Find the matrix B of this linear transformation in the basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

a) by direct calculation;
b) using the change of basis matrix S.

Solution: a) We have

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence, we have

$$B = \begin{bmatrix} -2 & -5 \\ 2 & 4 \end{bmatrix}.$$

b) On the other hand,

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Finally, we have

$$B = S^{-1}AS$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 2 & 4 \end{bmatrix}.$$