Math 2270-1 Midterm 1, January 30, 2009

Solutions

Problem 1 (20 points)**.** Solve the linear system

$$\begin{array}{rrrl} x & +2y & -3z & = -5 \\ 3x & -y & -z & = 1 \\ x & +y & -z & = -1 \end{array}$$

using Gauss elimination.

Solution: The augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & -5 \\ 3 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

Multiplying the first row by 3 and subtracting from the second, and subtracting it from third we get

$$\begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & -7 & 8 & 16 \\ 0 & -1 & 2 & 4 \end{bmatrix}.$$

Switching the last two rows and multiplying by -1 we get

$$\begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & -2 & -4 \\ 0 & -7 & 8 & 16 \end{bmatrix}.$$

Multiplying the second by 7 and adding to the last we get

$$\begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -6 & -12 \end{bmatrix}.$$

Dividing the last by -6 we get

$$\begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Multiplying the last row by 2 and adding to the second row, and by 3 and adding it to the first row, we get

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Finally, multiplying the second row by 2 and subtracting from the first row we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Hence, the solution is $x = 1$, $y = 0$ and $z = 2$.

Problem 2 (20 points)**.** Find all solutions of the linear system

$$\begin{array}{rrrrr} x & +2y & +z & +5t & = -2 \\ -x & -y & -z & +5t & = 1 \\ x & +y & -z & -5t & = -1 \end{array}$$

using Gauss elimination.

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -2 \\ -1 & -1 & -1 & 5 & 1 \\ 1 & 1 & -1 & -5 & -1 \end{bmatrix}.$$

Adding the first row to the second row, and subtracting it from the third row we get

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -2 \\ 0 & 1 & 0 & 10 & -1 \\ 0 & -1 & -2 & -10 & 1 \end{bmatrix}.$$

Adding the second row to the third row we get

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -2 \\ 0 & 1 & 0 & 10 & -1 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix}.$$

After dividing the third row by -2 we get

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -2 \\ 0 & 1 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Subtracting the third row from the first row we have

$$\begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 1 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Finally, multiplying the second row by 2 and subtracting it from the first row we get

$$\begin{bmatrix} 1 & 0 & 0 & -15 & 0 \\ 0 & 1 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

This is the reduced row echelon form of the augmented matrix. We see that $z = 0$, $y = -1 - 10t$ and $x = 15t$. Hence, the system has infinitely many solutions which are parametrized by the free variable t.

Problem 3 (20 points)**.** Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 7 & 9 & 11 \\ 7 & 8 & 9 & 10 \end{bmatrix}$$

Solution:

Multiplying the first row by 5 and subtracting from the second row, and multiplying it by 7 and subtracting from the third row we get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 6 & 12 & 18 \end{bmatrix}.$$

Dividing the second row by -3 and the third by 6 we get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

Hence by subtracting the second row from the third, we get:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Multiplying the second row by 2 and subtracting it from the first one we get

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the reduced row echelon form of A. Therefore rank of A is 2.

Problem 4 (20 points)**.** Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

Find $AB - AB^{-1}$.

Solution:

We know that

$$B^{-1} = -\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}.$$

Hence we have

$AB - BA^{-1} =$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 6 \\ 3 & 9 \end{bmatrix}.$$

Problem 5 (20 points)**.** Find the inverse matrix A^{-1} of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Solution:

We consider the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{bmatrix}.$$

If we subtract the first row from the second and third row, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{bmatrix}.$$

If we multiply the second row by 2 and subtract from the third row, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}.$$

Now we subtract the last row from the first row and multiply the last row by 2 and subtract from the second row to get

$$\left[\begin{array}{rrrrrr} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right].$$

Finally, we subtract the second row from the first to get

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right].$$

Hence, the inverse of A is

$$A^{-1} = \left[\begin{array}{rrr} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{array}\right].$$