Problem 1. Let $M_2(\mathbb{R})$ be the set of all real $2 \times 2$ matrices with usual addition and multiplication.

(a) Show that $M_2(\mathbb{R})$ is a ring with zero equal to
\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]
and identity
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

(b) Show that the matrix
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
is invertible if and only if $ad - bc \neq 0$.

(c) Show that $M_2(\mathbb{R})$ is not a field.

Problem 2. Consider the space $M_{nm}(\mathbb{R})$ of all real matrices with $n$ rows and $m$ columns. It is a real vector space with standard addition and multiplication.

(a) Let $E_{ij}$ be a matrix in $M_{nm}(\mathbb{R})$ with all matrix coefficients equal to 0 except the $(i, j)$ coefficient which is equal to 1. Show that
\[
\{ E_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m \}
\]
is a basis of the vector space $M_{nm}(\mathbb{R})$.

(b) What is the dimension of the vector space $M_{nm}(\mathbb{R})$?

Problem 3. Let $P$ be a linear map from vector space $V$ into itself. Assume that $P^2 = P$. Show that $V$ is the direct sum of $\ker P$ and $\text{im } P$. 
Problem 4. Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 1 \\
3 & 2 & 3 \\
\end{bmatrix}
\]

Find:

(a) A basis of the kernel of this matrix;
(b) the dimension of the kernel;
(c) a basis of the image;
(d) the dimension of the image.

Problem 5. Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \) with positive definite scalar product \((\cdot | \cdot)\). Let \( \| \cdot \| \) be the corresponding norm. Prove that:

(a) the norm satisfies the parallelogram law:
\[
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)
\]
for any two vectors \( u, v \in V \);
(b) The scalar product satisfies
\[
(u|v) = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2)
\]
i.e., it is completely determined by the norm.