Math 2270-1 Final, May 4, 2009

Solutions

Problem 1 (25 points)**.** Find all solutions of the linear system

$$\begin{array}{rrrrl} x & +y & +z & +t & = -3 \\ -x & +y & -z & +t & = 1 \\ 2x & +y & +2z & +t & = -7 \end{array}$$

using Gauss elimination.

Solution: The augmented matrix of the system is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -3 \\ -1 & 1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 1 & -7 \end{bmatrix}.$$

By adding the first row to the second and subtracting its double from the last we get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -3 \\ 0 & 2 & 0 & 2 & -2 \\ 0 & -1 & 0 & -1 & -1 \end{bmatrix}.$$

By dividing the second row by 2 and adding it to the last we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Dividing the last row by -2 and adding it to the second and adding its triple to the first we get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Subtracting the second row from the first we get the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, the system has no solutions.

Problem 2 (25 points)**.** Calculate the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: Consider the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

By subtracting the first row from the third and its double from the second we get

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

By multiplying the third row by -1 and switching it with the second we get:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -3 & -2 & -2 & 1 & 0 \end{bmatrix}.$$

Multiplying the second row by 3 and adding it to the last we get:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{bmatrix}.$$

Subtracting the last row from the second and its double from the first we get:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & -2 & 6 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{bmatrix}.$$

Subtracting the double of the second row from the first we finally get:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{bmatrix}.$$

Hence the inverse matrix is:

$$A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & -3 \end{bmatrix}.$$

Problem 3 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}.$$

Find:

a) A basis of the kernel of this linear transformation.
b) The nullity of this linear transformation.

Solution: To find the kernel we have to solve the linear system

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

By multiplying the first row of A by 2, resp. 3, and subtracting it from the second and the third row, we get:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{bmatrix}.$$

By multiplying the second row by -1 and adding it to the last, we get:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

By subtracting the second row from the first, we finally get the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $x = z$ and $y = -3z$. It follows that the kernel is spanned by the vector

$$\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

Hence, the nullity of A is 1.

Problem 4 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix}.$$

Find:

a) A basis of the image of this linear transformation.
b) The rank of this linear transformation.

Solution: We have to find the reduced row echelon form of A. If we switch the first two rows we get:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

If we subtract the first row from the second and its double from the third, we get:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Hence, but subtracting the second row from the third, we get:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

By dividing the second row by 2, we get the reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the image is spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the rank of A is equal to 2.

Problem 5 (25 points)**.** Find the matrix A of the orthogonal projection onto the plane in $\mathbb{R}^3$ spanned by the vectors

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Find the kernel of A.

Solution:
The subspace spanned by u and v is equal to the subspace spanned by the standard basis vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, the projection is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Its kernel is the subspace spanned by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6 (25 points)**.** Find the determinant

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix}.$$

Solution: By Gauss elimination, we have

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \\ 0 & -2 & 2 & 0 \end{vmatrix}$$

by subtracting the first row from the second and the fourth and adding it to the third. By taking the common factor 2 from the last three rows and switching the second and third row we get

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix} = -8 \begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{vmatrix}.$$

By adding the second row to the last row we get

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix} = -8 \begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

Finally, by subtracting the third row from the last, we get

$$\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{vmatrix} = -8 \begin{vmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -16.$$

Problem 7 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find

a) the eigenvalues of A;
b) the eigenvectors of A;
c) the algebraic and geometric multiplicity of these eigenvalues.

Does the matrix A have an eigenbasis? Is it similar to a diagonal matrix?

Solution: The characteristic polynomial is:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3.$$

Hence, the only eigenvalue of A is 1.

To find the eigenvectors we have to solve the linear system:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Hence, $y = 0$ and x and z are arbitrary. It follows that the eigenspace for the eigenvalue 1 is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the algebraic multiplicity of 1 is 3, and the geometric multiplicity is 2.

It follows that the eigenbasis of A doesn't exist and A is not similar to a diagonal matrix.

Problem 8 (25 points)**.** Consider the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ given by the symmetric matrix

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

Find

a) the eigenvalues of A;
b) an orthonormal basis consisting of eigenvectors of A;
c) the "change of basis" matrix S from the standard basis to the basis above;
d) the diagonal matrix $D = S^{-1}AS$.

Solution: The characteristic polynomial is:

$$\begin{vmatrix} 4-\lambda & -2 & 0 \\ -2 & 3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)(2-\lambda) - 4(4-\lambda) - 4(2-\lambda)$$

$$= (4-\lambda)(3-\lambda)(2-\lambda) - 4(6-2\lambda) = (4-\lambda)(3-\lambda)(2-\lambda) - 8(3-\lambda)$$

$$= (3-\lambda)\big((4-\lambda)(2-\lambda) - 8\big) = (3-\lambda)(\lambda^2 - 6\lambda) = -\lambda(\lambda-3)(\lambda-6).$$

Hence, the eigenvalues are 0, 3 and 6.

The eigenvectors for 0 satisfy

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Hence, $2x = y$ and $y = z$, i.e., the solutions are proportional to

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The corresponding normalized vector is

$$\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The eigenvectors for 3 satisfy

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Hence, $x = 2y$ and $x = -z$, i.e., the solutions are proportional to

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

The corresponding normalized vector is

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

The eigenvectors for 6 satisfy

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Hence, $x = -y$ and $2z = -y$, i.e., the solutions are proportional to

$$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

The corresponding normalized vector is

$$\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, the "change of basis" matrix S is

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Since it is orthogonal and symmetric, $S^{-1} = S$.

Hence, we have

$$D = S^{-1}AS = SAS$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & -6 \\ 12 & -12 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 54 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$