Lectures on Representation Theory

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CHAPTER I

Representations of finite groups

1. Category of representations of finite groups

1.1. Category of group representations. Let $G$ be a group. Let $V$ be a vector space over $\mathbb{C}$. Denote by $\text{GL}(V)$ the general linear group of $V$, i.e., the group of all linear automorphisms of $V$.

A representation $(\pi, V)$ of $G$ on the vector space $V$ is a group homomorphism $\pi : G \rightarrow \text{GL}(V)$. A morphism $\phi : (\pi, V) \rightarrow (\nu, U)$ of representation $(\pi, V)$ into $(\mu, U)$ is a linear map $\phi : V \rightarrow U$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\pi(g)} & V \\
\phi \downarrow & & \downarrow \phi \\
U & \xrightarrow{\nu(g)} & U
\end{array}
$$

commutes for all $g \in G$. Morphisms of representations are also called intertwining maps. The set of all morphisms of $(\pi, V)$ into $(\nu, U)$ is denoted by $\text{Hom}_G(V, U)$.

It is easy to check that all representations of $G$ form a category $\text{Rep}(G)$ of representations of $G$.

An isomorphism $\phi : (\pi, V) \rightarrow (\nu, U)$ in this category is a morphism of representations which is a linear isomorphism of the vector space $V$ with $U$. If two representations are isomorphic, we say that they are equivalent.

Let $(\pi, V)$ and $(\nu, U)$ be two representations of $G$. Let $U$ be a subspace of $V$ which is invariant for $G$, i.e., $\pi(g)(U) \subseteq U$ for all $g \in G$. Then the linear maps $\nu(g)$ restricted to $U$ define linear maps $\nu(g)$, $g \in G$. Clearly, $(\nu, U)$ is a representation of $G$. We call it the subrepresentation of $\pi$ on $U$.

Let $\phi : (\pi, V) \rightarrow (\nu, U)$ be a morphism of representations. Then, $\ker \phi \subseteq V$ is a $G$-invariant subspace of $V$. Hence, $\ker \phi$ is a subrepresentation of $(\pi, V)$. Also, $\text{im} \phi$ is a $G$-invariant subspace of $U$, so $\text{im} \phi$ is a subrepresentation of $(\nu, U)$.

Let $(\pi, V)$ be a representation of $G$. Let $U$ be an invariant subspace of $V$. For each $g \in G$ we define a linear operator $\rho(g)$ on the quotient space $V/U$ by...
\[ \rho(g)(v + U) = \pi(g)v + U \] for any \( g \in G \). Then \((\rho, V/U)\) is a quotient representation of \((\pi, V)\).

Clearly, the category \( \text{Rep}(G) \) is an abelian category.

If the vector space \( V \) is equipped with an inner product \( \langle \cdot, \cdot \rangle \) and all linear operators \( \pi(g), g \in G \), are unitary with respect to this inner product structure, we say that the representation \((\pi, V)\) is unitary.

### 1.2. Representations of finite groups.

Let \( G \) be a group. We say that \( G \) is a finite group, if \( G \) is a finite set.

In this section we assume that the group \( G \) is finite. We put \([G] = \text{Card}(G)\).

A representation \((\pi, V)\) of \( G \) is finite-dimensional if \( V \) is a finite-dimensional vector space. We put \( \dim \pi = \dim_{\mathbb{C}} V \).

#### 1.2.1. Lemma.

Let \((\pi, V)\) be a representation of \( G \). Let \( v \in V, v \neq 0 \). Then there exists a finite-dimensional subrepresentation \((\nu, U)\) of \((\pi, V)\) such that \( v \in U \).

**Proof.** Let \( U \) be the vector subspace of \( V \) generated by vectors \( \pi(g)v, g \in G \). Then \( U \) is \( G \)-invariant and finite-dimensional. Moreover, \( v = \pi(1)v \) is in \( U \). \( \square \)

A representation \((\pi, V)\) of \( G \) is called irreducible if the only \( G \)-invariant subspaces in \( V \) are \( \{0\} \) and \( V \).

#### 1.2.2. Theorem.

Let \((\pi, V)\) be an irreducible representation of \( G \). Then \( \pi \) is finite-dimensional.

**Proof.** Let \( v \in V, v \neq 0 \). By 1.2.1, \( V \) contains a finite-dimensional \( G \)-invariant subspace \( U \) such that \( v \in U \). If \( \pi \) is irreducible, we must have \( U = V \) and \( V \) is finite-dimensional. \( \square \)

#### 1.2.3. Corollary.

Every representation \((\pi, V)\) of \( G \) contains an irreducible subrepresentation.

The main result on representations of finite groups is the following observation.

#### 1.2.4. Theorem (Mascke).

Let \((\pi, V)\) be a representation of \( G \). Let \((\nu, U)\) be a subrepresentation of \((\pi, V)\). Then there exists a subrepresentation \((\rho, W)\) of \((\pi, V)\) such that \( \pi = \nu \oplus \rho \).

**Proof.** Let \( P \) be a projector of \( V \) onto \( U \). Consider the linear map

\[
Q = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) P \pi(g)
\]

on \( V \). Clearly, since \( U \) is \( G \)-invariant, \( Q(V) \subset U \). Moreover, for any \( u \in U \), we have

\[
Q u = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) P \pi(g) u = \frac{1}{[G]} \sum_{g \in G} u = u.
\]

Therefore, \( U = \text{im} Q \) and \( Q^2 = Q \). It follows that \( Q \) is a projection onto \( U \) along \( \ker Q \).

In addition, we have

\[
Q \pi(h) = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) P \pi(gh) = \frac{1}{[G]} \sum_{g \in G} \pi(hg^{-1}) P \pi(g) = \pi(h)Q
\]

for all \( h \in G \), i.e., \( Q \) is a morphism of \((\pi, V)\) into \((\nu, U)\). Hence \( W = \ker Q \) is a \( G \)-invariant subspace and \( V = U \oplus W \). \( \square \)
Therefore, the category $\text{Rep}(G)$ is semisimple.

1.3. **Schur Lemma.** Let $(\nu, U)$ and $(\pi, V)$ be two irreducible representations of $G$. Let $A$ be a morphism of $\nu$ into $\pi$. Then, for $u \in \ker A$, we have $A(\nu(g)u) = \pi(g)Au = 0$ for any $g \in G$. Hence, $\nu(g)u \in \ker A$ for any $g \in G$, and $\ker A$ is a $G$-invariant subspace of $U$.

Since $\nu$ is irreducible, $\ker A$ is equal to either $\{0\}$ or $U$. In the latter case, we see that $A = 0$. In the first case, $A$ is injective. Moreover, if $v \in \im A$, we have $v = Au$ for some $u \in U$. Therefore, $\pi(g)v = \pi(g)Au = A(\nu(g)u)$ is in $\im A$ for any $g \in G$. Hence, $\im A$ is a $G$-invariant subspace of $V$. Since $A$ is injective, $\im A$ is not $\{0\}$. It follows that $\im A = V$, and $A$ is an isomorphism.

This implies the following result.

1.3.1. **Proposition.** Let $(\nu, U)$ and $(\pi, V)$ be two irreducible representations of $G$. Assume that $\pi$ and $\nu$ are not equivalent. Then $\text{Hom}_G(U, V) = \{0\}$.

In addition we have the following result.

1.3.2. **Theorem (Schur Lemma).** Let $(\pi, V)$ be an irreducible representation of $G$. Then $\text{Hom}_G(V, V) = \mathbb{C}I$.

**Proof.** Let $A$ be an endomorphism of $\pi$. Since $V$ is finite-dimensional, $A$ has an eigenvalue $\lambda \in \mathbb{C}$. Therefore, $B = A - \lambda I$ is an endomorphism of $\pi$ which is not injective. By the above discussion, it must be equal to 0. Hence, we have $A = \lambda I$. $\square$

1.4. **Regular representation.** Let $G$ be a finite group. Denote by $\mathbb{C}[G]$ the space of all complex valued functions on $G$. Clearly, $\dim \mathbb{C}[G] = |G|$. The vector space $\mathbb{C}[G]$ has a structure of inner product space with the inner product

$$(f | f') = \frac{1}{|G|} \sum_{g \in G} f(g)f'(g)$$

for $f, f' \in \mathbb{C}[G]$.

For $g \in G$ and $f \in \mathbb{C}[G]$ define the function $R(g)f$ by $(R(g)f)(h) = f(gh)$ for any $h \in G$. Clearly, $R(g) : f \mapsto R(g)f$ is a linear map on $\mathbb{C}[G]$.

Moreover, for $g, h \in G$, we have

$$(R(gh)f)(k) = f(kgh) = (R(h)f)(k) = (R(g)R(h)f)(k)$$

for any $k \in K$. Therefore $R(gh) = R(g)R(h)$. Clearly, $R(1) = I$. It follows that $(R, \mathbb{C}[G])$ is a representation of $G$. We call it the (right) regular representation of $G$.

1.4.1. **Lemma.** The right regular representation is unitary.

**Proof.** Clearly, for $g \in G$, we have

$$(R(g)f | R(g)f') = \frac{1}{|G|} \sum_{h \in G} f(hg)f'(hg) = \frac{1}{|G|} \sum_{h \in G} f(h)f'(h) = (f | f')$$

for any $f, f' \in \mathbb{C}[G]$. Therefore, $R(g)$, $g \in G$, are unitary operators. $\square$

The following property of regular representation is critical.

1.4.2. **Lemma.** Let $g \in G$, $g \neq 1$. Then $R(g) \neq I$. 

Proof. Denote by $\delta_h$ the function on $G$ which is 1 at point $h \in G$ and zero everywhere else. Then we have
\[
(R(g)\delta_1)(h) = \delta_1(hg) = \delta_{g^{-1}}(h)
\]
for any $h \in G$, i.e., $R(g)\delta_1 = \delta_{g^{-1}} \neq \delta_1$.

Since $R$ is a direct sum of irreducible representations of $G$, this result has a following consequence.

1.4.3. Theorem. Let $g \in G$, $g \neq 1$. Then there exists an irreducible representation $\pi$ of $G$ such that $\pi(g) \neq I$.

In other words, irreducible representations of $G$ separate points in $G$.

1.5. Abelian finite groups. Let $G$ be a finite group. Let $\pi$ be an one-dimensional representation of $G$. Then $\pi(g) = \lambda(g)I$, where $\lambda : G \rightarrow \mathbb{C}^*$ is group homomorphism of $G$ into the multiplicative group of complex numbers different than zero. This implies that $g \mapsto |\lambda(g)|$ is a homomorphism of $G$ into the multiplicative group of positive real numbers $\mathbb{R}^*$. Since 1 is the only element of that group of finite order, we conclude that $|\lambda(g)| = 1$, i.e., $\lambda$ is a homomorphism of $G$ into the group of complex numbers of absolute value equal to 1. We call such homomorphisms the characters of $G$.

Assume that $G$ is abelian finite group. Let $(\pi, V)$ be an irreducible representation of $G$. Let $g \in G$. Then $\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$ for all $h \in G$. Therefore, by Schur Lemma, we see that $\pi(g) = \lambda(g)I$ for some complex number $\lambda(g) \neq 0$. By the above discussion, $\lambda$ is a character of $G$. This in turn implies that $\dim \pi = 1$.

1.5.1. Proposition. Let $G$ be a finite group. Then the following conditions are equivalent.

(i) $G$ is abelian;
(ii) all irreducible representations of $G$ are one-dimensional.

Proof. We already proved that (i) implies (ii).

Assume that all irreducible representations are one-dimensional. Let $g, h \in G$. Consider the element $a = ghg^{-1}h^{-1}$. Let $\pi$ be an irreducible representation of $G$. Then $\pi$ is one-dimensional and
\[
\pi(a) = \pi(ghg^{-1}h^{-1}) = \pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = I
\]
since $\pi(g)$ and $\pi(h)$ commute. By 1.4.3, this implies that $a = 1$, i.e., $ghg^{-1}h^{-1} = 1$. It follows that $gh = hg$ for all $g, h \in G$, i.e., $G$ is abelian.

Hence, all irreducible representations of an abelian finite group are characters. Let $\phi$ and $\psi$ be two characters of $G$. Then we have
\[
\phi(g)(\phi \mid \psi) = \frac{1}{|G|} \sum_{h \in G} \phi(h)\overline{\psi(h)} = \frac{1}{|G|} \sum_{h \in G} \phi(h)\overline{\psi(g^{-1}h)} = \psi(g)(\phi \mid \psi)
\]
for any $g \in G$. Hence, if $\phi$ and $\psi$ are different, they are orthogonal to each other. Moreover, for a character $\phi$ we have
\[
\|\phi\|^2 = (\phi \mid \phi) = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\phi(g)} = \frac{1}{|G|} \sum_{g \in G} \phi(g)\phi(g^{-1}) = 1.
\]
Hence, the characters form an orthonormal family of functions in $\mathbb{C}[G]$. Moreover, we have the following result.

1.5.2. Proposition. Characters form an orthonormal basis of $\mathbb{C}[G]$.

Proof. Since irreducible representations of $G$ are characters, $R$ is a direct sum of characters. This implies that there is a basis $e_i, 1 \leq i \leq |G|$, and characters $\phi_i, 1 \leq i \leq |G|$, such that $R(g)e_i = \phi_i(g)e_i$ for any $g \in G$. This in turn implies that $e_i(g) = (R(g)e_i)(1) = \phi_i(g)e_i(1)$ for all $g \in G$. Since $e_i$ is a nonzero vector, we must have $e_i(1) \neq 0$. Hence $e_i$ is proportional to $\phi_i$. Therefore, $\mathbb{C}[G]$ is spanned by characters. □

Let $\hat{G}$ be the set of all characters of $G$. Let $\phi, \psi$ be two characters of $G$. Define their product as $(\phi \cdot \psi)(g) = \phi(g)\psi(g)$ for all $g \in G$. This defines a binary operation on $\hat{G}$. It is easy to check that $\hat{G}$ is an abelian group with this operation. By the above result, $\hat{G}$ is finite and $|\hat{G}| = \dim \mathbb{C}[G] = |G|$. We call $\hat{G}$ the dual group of $G$.

Applying the above discussion twice, we get $|\hat{\hat{G}}| = |\hat{G}| = |G|$.

Let $g \in G$. Then the map $\phi \mapsto \phi(g)$ is a character of $\hat{G}$. This defines a map $\alpha$ from $G$ into $\hat{\hat{G}}$. Moreover,

$$\alpha(gh)(\phi) = \phi(gh) = \alpha(g)(\phi)\alpha(h)(\phi) = (\alpha(g) \cdot \alpha(h))(\phi)$$

for all $\phi \in \hat{G}$, i.e., $\alpha : G \rightarrow \hat{\hat{G}}$ is a group morphism.

Assume that $\alpha(g) = 1$. Then $\alpha(g)(\phi) = \phi(g) = 1$ for all $\phi \in \hat{G}$. By 1.4.3, it follows that $g = 1$. Therefore, $\alpha$ is an injective morphism. Hence, $\alpha : G \rightarrow \hat{\hat{G}}$ is a group isomorphism.

1.5.3. Theorem. Let $G$ be an abelian finite group and $\hat{G}$ its dual group. Then

(i) $|\hat{\hat{G}}| = |G|$;

(ii) $\alpha : G \rightarrow \hat{\hat{G}}$ is an isomorphism.

This is a special case of Pontryagin duality.

Since characters form an orthonormal basis of $\mathbb{C}[G]$, any function $f$ in $\mathbb{C}[G]$ can be written as

$$f = \sum_{\phi \in \hat{G}} (f | \phi)\phi.$$

By Bessel equality, we have

$$\|f\|^2 = \sum_{\phi \in \hat{G}} |(f | \phi)|^2.$$

We define the Fourier transform $\mathcal{F}f$ of $f$ as the function on $\hat{G}$ given by

$$(\mathcal{F}f)(\phi) = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\phi(g)}, \quad \phi \in \hat{G}.$$

Therefore, the inverse Fourier transform is given by

$$f(g) = \sum_{\phi \in \hat{G}} (\mathcal{F}f)(\phi)\phi(g), \quad g \in G.$$
The above equality then implies that  
\[ \|f\|^2 = \sum_{\phi \in \hat{G}} |(Ff)(\phi)|^2. \]

This is a special case of Plancherel theorem.

### 1.6. Unitarity.

Let \((\pi, V)\) be a finite-dimensional representation of \(G\). Let \(\langle \cdot | \cdot \rangle\) be an inner product on \(V\).

Put  
\[ (u | v) = \frac{1}{|G|} \sum_{g \in G} \langle \pi(g)u | \pi(g)v \rangle. \]

Clearly, \((u, v) \mapsto (u | v)\) is a linear in first and antilinear in the second variable. Moreover, we have \((u | v) = (v | u)\). In addition,

\[ (v | v) = \frac{1}{|G|} \sum_{g \in G} \langle \pi(g) | \pi(g)v \rangle \geq 0 \]

for any \(v \in V\). If \((v | v) = 0\), we have \(\langle \pi(g)v | \pi(g)v \rangle = 0\) for all \(g \in G\). In particular \((v | v) = 0\), and \(v = 0\). Hence, \((\cdot | \cdot)\) is an inner product on \(V\).

### 1.6.1. Lemma. **Inner product \((\cdot | \cdot)\) is \(G\)-invariant.**

**Proof.** Let \(g \in G\). Then we have

\[ \langle \pi(g)u | \pi(g)v \rangle = \frac{1}{|G|} \sum_{h \in G} \langle \pi(hg)u | \pi(hg)v \rangle = \frac{1}{|G|} \sum_{h \in G} \langle \pi(h)u | \pi(h)v \rangle = (u | v). \]

Therefore, there exists an inner product on \(V\) such that \((\pi, V)\) is a unitary representation.

### 1.7. Orthogonality relations.

Let \((\nu, U)\) and \((\pi, V)\) be two irreducible representations of \(G\). Let \(A: U \rightarrow V\) be a linear map. Define

\[ B = \frac{1}{|G|} \sum_{g \in G} \pi(g)A\nu(g^{-1}). \]

Then, \(B\) is also a linear map from \(U\) into \(V\).

Let \(g \in G\). Then

\[ \pi(g)B = \frac{1}{|G|} \sum_{h \in G} \pi(gh)A\nu(h^{-1}) = \frac{1}{|G|} \sum_{h \in G} \pi(h)A\nu(h^{-1}g) = B\nu(g). \]

Hence, it follows that \(B \in \text{Hom}_G(U, V)\).

If \(\nu\) and \(\pi\) are not equivalent, by Schur Lemma, we have \(B = 0\).

### 1.7.1. Lemma. **Let \((\nu, U)\) and \((\pi, V)\) be two inequivalent irreducible representations of \(G\). Then**

\[ \frac{1}{|G|} \sum_{g \in G} \langle \pi(g)A\nu(g^{-1}) | \nu \rangle = 0 \]

**for any linear operator** \(A: U \rightarrow V\).
Consider now an irreducible representation \((\pi, V)\) and a linear map \(A : V \to V\). Let
\[
B = \frac{1}{|G|} \sum_{g \in G} \pi(g) A \pi(g^{-1}).
\]
Then \(B \in \text{Hom}_G(V, V)\). By Schur Lemma, we conclude that \(B = \lambda I\) for some \(\lambda \in \mathbb{C}\).

Moreover, we have
\[
\text{tr} B = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\pi(g) A \pi(g^{-1})) = \frac{1}{|G|} \sum_{g \in G} \text{tr} A = \text{tr} A.
\]
This implies the following result.

1.7.2. **Lemma.** Let \((\pi, V)\) be an irreducible representation of \(G\). Then
\[
\frac{1}{|G|} \sum_{g \in G} \pi(g) A \pi(g^{-1}) = \frac{\text{tr} A \dim \pi}{\dim \pi} I
\]
for any linear operator \(A : V \to V\).

By 1.6.1, we can assume that \(U\) and \(V\) are equipped with \(G\)-invariant inner products. Let \((e_i; 1 \leq i \leq \dim \nu)\) and \((f_j; 1 \leq j \leq \dim \pi)\), be two orthonormal bases of \(U\) and \(V\) respectively. Denote by \(\nu(g)_{pq}\) and \(\pi(g)_{rs}\), the matrix coefficients of \(\nu(g)\) and \(\pi(g)\) respectively. Then we first observe that
\[
\sum_{s=1}^{\dim \pi} \sum_{p=1}^{\dim \nu} \frac{1}{|G|} \sum_{g \in G} \nu(g)_{rs} A_{sp} \nu(g^{-1})_{pq} = 0,
\]
where \(A_{sp}\) are matrix coefficients of \(A\). Since \(A\) is arbitrary, we conclude that
\[
\frac{1}{|G|} \sum_{g \in G} \nu(g)_{rs} \nu(g^{-1})_{pq} = 0
\]
for all \(p, q, r, s\). Clearly, since \((\nu(g^{-1})_{pq})\) is a unitary matrix, we have \(\nu(g^{-1})_{pq} = \nu(g)_{qp}\) for all \(p, q\). Hence, we conclude that
\[
\frac{1}{|G|} \sum_{g \in G} \pi(g)_{rs} \nu(g)_{pq} = 0
\]
for all \(p, q, r, s\).

Let \((\pi, V)\) be an irreducible representation of \(G\). Denote by \(M(\pi)\) the vector subspace of \(\mathbb{C}[G]\) spanned by matrix coefficients of \(\pi\). This subspace is independent of choice of the basis of \(V\). Moreover, it depends only on the equivalence class of \(\pi\).

1.7.3. **Proposition.** Let \((\pi, V)\) be an irreducible representation of \(G\). Then the subspace \(M(\pi)\) is an invariant subspace of the regular representation \((R, \mathbb{C}[G])\).

**Proof.** Let \((e_1, e_2, \ldots, e_n)\) be a basis of \(V\). Denote by \(g \mapsto \pi(g)_{ij}, 1 \leq i, j \leq n\), the matrix coefficients of \(\pi\) in this basis. Then \(M(\pi)\) is spanned by these functions.

Let \(1 \leq p, q \leq n\). Put \(f(g) = \pi(g)_{pq}\) for \(g \in G\). Then we have
\[
(R(g)f)(h) = f(hg) = \pi(hg)_{pq} = \sum_{s=1}^{n} \pi(h)_{ps} \pi(g)_{sq}
\]
for all \( h \in G \). Therefore, \( R(g)f \) is a linear combination of matrix coefficients of \( \pi \), i.e., \( R(g)f \in M(\pi) \). It follows that \( M(\pi) \) is invariant for \( R(g) \). □

The above calculation proves the following result.

1.7.4. Proposition. Let \( \nu \) and \( \pi \) be two inequivalent irreducible representations of \( G \). Then \( M(\nu) \perp M(\pi) \).

Consider now an irreducible representation \((\pi, V)\). As above, we have

\[
\sum_{s=1}^{\dim \pi} \sum_{p=1}^{\dim \pi} \frac{1}{|G|} \sum_{g \in G} \pi(g)_{rs} A_{sp} \pi(g^{-1})_{pq} = \frac{\tr A}{\dim \pi} \delta_{rq}.
\]

By selecting \( A \) such that \( A_{kl} = 1 \) for some \( k \neq l \), and all other entries are 0, we get

\[
\frac{1}{|G|} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lq} = 0.
\]

If we select \( A \) such that \( A_{kk} = 1 \) for some \( k \), and all other entries are 0, we get

\[
\frac{1}{|G|} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{kq} = \frac{1}{\dim \pi} \delta_{kq}.
\]

Therefore, we have

\[
\frac{1}{|G|} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lj} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}
\]

and

\[
\frac{1}{|G|} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{ql} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}
\]

for all \( 1 \leq k, l, q, r \leq \dim \pi \). These are Schur orthogonality relations. This implies that \( \{\pi(g)_{ij}; 1 \leq i, j \leq \dim \pi\} \) is an orthogonal basis of \( M(\pi) \).

1.7.5. Theorem. Let \((\pi, V)\) be an irreducible representation of \( G \). Then \( \dim M(\pi) = (\dim \pi)^2 \).

The next result describes the structure of regular representation.

1.7.6. Theorem. We have

\[
\mathbb{C}[G] = \bigoplus_{\pi \in \hat{G}} M(\pi).
\]

Proof. By 1.7.3, the subspaces \( M(\pi) \), \( \pi \in \hat{G} \), are invariant subspaces of \((R, \mathbb{C}[G])\). Therefore, their orthogonal sum \( M = \bigoplus_{\pi \in \hat{G}} M(\pi) \) is an invariant subspace in \((R, \mathbb{C}[G])\).

Let \( M^\perp \) be the orthogonal complement of \( M \). Then \( M^\perp \) is also an invariant subspace since \( R \) is unitary. Assume that \( M^\perp \) is different from \( \{0\} \). Then it contains an irreducible representation \((\nu, U)\) of \( G \) by 1.2.3. Let \((f_1, f_2, \ldots, f_m)\) be a basis of \( U \). Then we have

\[
\nu(g)f_i = \sum_{j=1}^{m} \pi(g)_{ji} f_j.
\]
Therefore, we have
\[ f_i(g) = (R(g)f_i)(1) = (\nu(g)f_i)(1) = \sum_{j=1}^{m} \nu(g)j_if_j(1) \]
for all \( g \in G \). Hence, we have \( f_i \in M(\nu) \subset M \). Therefore, \( f_i \) is orthogonal on itself, and \( f_i = 0 \). This contradicts our choice. It follows that \( M_\perp = \{0\} \), i.e., \( M = \mathbb{C}[G] \). \( \square \)

This has the following consequence.

1.7.7. Corollary. We have
\[ [G] = \sum_{\pi \in \hat{G}} (\dim(\pi))^2. \]

1.8. Characters and central functions. Let \((\pi, V)\) be a finite-dimensional representation of \( G \). Define the function \( \text{ch}(\pi) : G \rightarrow \mathbb{C} \) by
\[ \text{ch}(\pi)(g) = \text{tr} \pi(g) \]
for \( g \in G \). The function \( \text{ch}(\pi) \) on \( G \) is called the character of \( \pi \). The character of \( \pi \) depends only on the equivalence class of \( \pi \).

1.8.1. Example. Let \(((R, \mathbb{C}[G]))\) be the regular representation of \( G \). For any \( g \in G \), define the function \( \delta_g \) which is equal 1 at \( g \) and 0 everywhere else. Clearly, \((\delta_g, g \in G)\) is a basis of \( \mathbb{C}[G] \).

Let \( g \in G \). Then we have
\[ (R(g)\delta_h)(k) = \delta_h(kg) = \begin{cases} 1, & \text{if } k = hg^{-1}; \\ 0, & \text{if } k \neq hg^{-1} \end{cases} = \delta_{hg^{-1}}(k) \]
for all \( k \in G \). Hence \( R(g)\delta_h = \delta_{hg^{-1}} \) for all \( h \in G \). It follows that the matrix of \( R(g) \) has nonzero coefficients on the diagonal if and only if \( g = 1 \). Hence we see that \( \text{tr} R(g) = 0 \) if \( g \neq 1 \) and \( \text{tr} R(1) = \dim(R) = [G] \). Therefore, we have \( \text{ch}(R) = [G] \cdot \delta_1 \).

Moreover, if \( \pi = \nu \oplus \rho \) we have
\[ \text{ch}(\pi) = \text{ch}(\nu) + \text{ch}(\rho). \]

Hence, the character map defines a homomorphism of the Grothendieck group of \( \mathcal{R}_{\text{fd}}(G) \) into functions on \( G \).

1.8.2. Theorem. (i) Let \((\pi, V)\) and \((\nu, U)\) be two irreducible representations of \( G \). If \( \pi \) is not equivalent to \( \nu \) we have \( (\text{ch}(\pi) \mid \text{ch}(\nu)) = 0 \).

(ii) Let \((\pi, V)\) be irreducible representation of \( G \). Then we have \( (\text{ch}(\pi) \mid \text{ch}(\pi)) = 1 \).

Proof. This follows immediately from Schur orthogonality relations. \( \square \)

Therefore, \((\text{ch}(\pi) : \pi \in \hat{G})\) is an orthonormal family of functions in \( \mathbb{C}[G] \).

Moreover we see that
\[ \dim \text{Hom}_G(U, V) = (\text{ch}(\nu) \mid \text{ch}(\pi)) \]
for any two finite-dimensional representations of \( G \).

Clearly, if \( g, h \in G \) we have
\[ \text{ch}(\pi)(hgh^{-1}) = \text{tr}(\pi(hgh^{-1})) = \text{tr}(\pi(h)\pi(g)\pi(h)^{-1}) = \text{tr}(\pi(g)) = \text{ch}(\pi)(g). \]
Hence, characters are constant on conjugacy classes in $G$.

This has the following consequence.

1.8.3. **Proposition.** Let $(\pi, V)$ be an irreducible representation of $G$. Let $f$ be a matrix coefficient of $\pi$. Then

$$
\frac{1}{|G|} \sum_{h \in G} f(gh^{-1}) = \frac{f(1)}{\dim \pi} \text{ch}(\pi)(g)
$$

for any $g \in G$.

**Proof.** Clearly, both sides of the equality are linear forms in $f$ on the space $M(\pi)$. Therefore, it is enough to check the equality on a basis of $M(\pi)$.

By 1.6.1 we can assume that $\pi$ is unitary. Let $(e_i; 1 \leq i \leq \dim \pi)$, be an orthonormal basis of $V$. Let $g \mapsto \pi(g)_{ij}$ the matrix coefficients of $\pi$ in that basis.

Then they are a basis of $M(\pi)$.

For these functions we have

$$
\frac{1}{|G|} \sum_{h \in G} \pi(hg^{-1})_{ij} = \frac{1}{|G|} \sum_{h \in G} \left( \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(h)_{ik} \pi(g)_{kl} \pi(h^{-1})_{lj} \right) = \frac{1}{\dim \pi} \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(g)_{kl} \delta_{ij} \delta_{kl}
$$

using Schur orthogonality relations.

We say that a function $f$ on $G$ is central if it is constant on conjugacy classes in $G$. Denote by $C(G)$ the vector subspace of $\mathbb{C}[G]$ consisting of all central functions. Clearly, the dimension of $C(G)$ is equal to the number of conjugacy classes in $G$.

By 1.8.2, $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family of functions in $C(G)$.

1.8.4. **Theorem.** $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal basis of $C(G)$.

**Proof.** We already know that $(\text{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family in $C(G)$.

Let $f$ be a central function on $G$ orthogonal on all characters $\text{ch}(\pi)$, $\pi \in \hat{G}$. Let $\phi \in M(\pi)$, then we have

$$
(\phi | f) = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{f(g)} = \frac{1}{|G|} \sum_{g \in G} \left( \frac{1}{|G|} \sum_{h \in G} \phi(g) \overline{f(h^{-1}gh)} \right) = \frac{1}{|G|} \sum_{h \in G} \left( \frac{1}{|G|} \sum_{g \in G} \phi(hgh^{-1}) \right) \overline{f(g)},
$$

since $f$ is a central function. By 1.8.3, it follows that

$$
(\phi | f) = \frac{f(1)}{\dim \pi} \frac{1}{|G|} \sum_{g \in G} \text{ch}(\pi)(g) \overline{f(g)} = \frac{f(1)}{\dim \pi} (\text{ch}(\pi)|f) = 0.
$$
Hence \( f \) is orthogonal to \( M(\pi) \) for all \( \pi \in \hat{G} \). By 1.7.6, it follows that \( f \) is orthogonal to \( C[G] \). Hence \( f = 0 \). Therefore, \((ch(\pi); \pi \in \hat{G})\) is a maximal orthonormal family in \( C(G) \), i.e., it is an orthonormal basis. \( \square \)

Therefore, \( \dim C(G) \) is equal to \( \text{Card}(\hat{G}) \). This implies the following result.

1.8.5. Corollary. \( \text{Card}(\hat{G}) \) is equal to the number of conjugacy classes in \( G \).

## 2. Frobenius reciprocity

### 2.1. Restriction functor

Let \( G \) be a a finite group. Let \( H \) be the a subgroup of \( G \). Denote by \( \text{Rep}(G) \), resp. \( \text{Rep}(H) \), the categories of representations of \( G \), resp. \( H \).

Let \((\pi, V)\) be a representation in \( \text{Rep}(G) \). Denote by \( \nu \) the restriction of function \( \pi : G \to \text{GL}(V) \) to \( H \). Then \((\nu, V)\) is a representation in \( \text{Rep}(H) \). This representation is called the restriction of \( \pi \) to \( H \) and denoted by \( \text{Res}^\pi_H(\pi) \) (when there is no ambiguity we shall just write \( \text{Res}(\pi) \)).

Clearly, \( \text{Res}^\pi_H \) is an exact functor form the abelian category \( \text{Rep}(G) \) into the abelian category \( \text{Rep}(H) \).

### 2.2. Induction functor

Let \((\nu, U)\) be a representation of \( H \). Denote by \( V = \text{Ind}(U) \) the space of all functions \( F : G \to U \) such that \( F(hg) = \nu(h)F(g) \) for all \( h \in H \) and \( g \in G \). Let \( F \) be the function in \( V \) and \( g \in G \). Then the function \( \rho(g)F : G \to U \) defined by \((\rho(g)F)(g') = F(g'g)\) for all \( g' \in G \), satisfies

\[
(\rho(g)F)(hg') = F(hg'g) = \nu(h)F(g'g) = \nu(h)(\rho(g)F)(g')
\]

for all \( h \in H \) and \( g' \in G \). Therefore \( \rho(g)F \) is a function in \( V \).

Clearly \( \rho(g) \) is a linear operator on \( V \) for any \( g \in G \). Moreover, \( \rho(1) \) is the identity on \( V \). For any \( F \in V \) we have

\[
(\rho(gg')F)(g'') = F(g''gg') = (\rho(g'F)(g')g) = (\rho(g)(\rho(g')F))(g'')
\]

for all \( g'' \in G \), i.e., we have

\[
\rho(gg')F = \rho(g)(\rho(g')F)
\]

for \( g, g' \in G \). Therefore, \( \rho(gg') = \rho(g)(\rho(g') \) for any \( g, g' \in G \) and \( \rho \) is a representation of \( G \) on \( V \).

The representation \((\rho, V)\) of \( G \) is called the induced representation and denoted by \( \text{Ind}^G_H(\nu) \).

If \( H \) is the identity subgroup and \( \nu \) is the trivial representation, the corresponding induced representation is the regular representation of \( G \).

Let \((\nu, U) \) and \((\nu', U') \) be two representations of \( H \) and \( \phi \) a morphism of \( \nu \) into \( \nu' \). Let \( F \) be a function in \( \text{Ind}(U) \). Then \( \Phi(F)(g) = \phi(F(g)) \) for all \( g \in G \) is a function from \( G \) into \( U' \). Moreover, we have

\[
\Phi(F)(hg) = \Phi(F(hg)) = \phi(\nu(h)F(g)) = \nu'(h)\phi(F(g)) = \nu'(h)\Phi(F)(g)
\]

for all \( h \in H \) and \( g \in G \). Hence, \( \Phi(F) \) is in \( \text{Ind}(U') \). Clearly, \( \Phi \) is a linear map from \( \text{Ind}(U) \) into \( \text{Ind}(U') \).

Moreover, we have

\[
(\rho'(g)\Phi(F))(g') = \Phi(F)(g'g) = \phi(F(g'g)) = \phi((\rho(g)F)(g')) = \Phi(\rho(g)F)(g')
\]
for all $g' \in G$. Therefore, $\rho'(g) \circ \Phi = \Phi \circ \rho(g)$ for all $g \in G$, and $\Phi$ is a morphism of $\text{Ind}^G_H(\nu)$ into $\text{Ind}^G_H(\nu')$. We put $\text{Ind}^G_H(\phi) = \Phi$. It is straightforward to check that in this way $\text{Ind}^G_H$ becomes an additive functor from $\text{Rep}(H)$ into $\text{Rep}(G)$.

We call $\text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G)$ the \textit{induction functor}.

The next result is a functorial form of Frobenius reciprocity.

2.2.1. \textbf{Theorem.} The induction functor $\text{Ind}^G_H : \text{Rep}(H) \to \text{Rep}(G)$ is a right adjoint functor of the restriction functor $\text{Res}^G_H : \text{Rep}(G) \to \text{Rep}(H)$.

\textbf{Proof.} Let $(\nu, U)$ a representation of $H$. Consider the induced representation $\text{Ind}^G_H(\nu)$ of $G$. The evaluation map $\epsilon : \text{Ind}(U) \to U$ given by $\epsilon(F) = F(1)$ for $F \in \text{Ind}(U)$, satisfies

$$\epsilon(\rho(h)F)(1) = (\rho(h)F)(1) = F(h) = \nu(h)F(1) = \nu(h)e\epsilon(F)$$

for all $F \in \text{Ind}(U)$, i.e., $\epsilon$ is a morphism of representations of $H$.

Let $(\pi, V)$ be a representation of $G$. Let $\Psi : V \to \text{Ind}(U)$ be a morphism of representations of $G$. Then the composition $\epsilon \circ \Psi$ is a morphism of $\text{Res}^G_H(\pi)$ into $\nu$. Denote the linear map $\Psi \to e \circ \Psi$ from $\text{Hom}_G(\pi, \text{Ind}^G_H(\nu))$ into $\text{Hom}_H(\text{Res}^G_H(\pi), \nu)$ by $A$.

Let $\phi : V \to U$ be a morphism of representations of $H$. Let $v \in V$. Then we consider the function $F_v : G \to U$ given by $F_v(g) = \phi(\pi(g)v)$ for any $g \in G$. First, for $h \in H$, we have

$$F_v(hg) = \phi(\pi(hg)v) = \phi(\pi(h)\pi(g)v) = \nu(h)\phi(\pi(g)v) = \nu(h)F_v(g)$$

for all $g \in G$. Hence $F_v$ is a function in $\text{Ind}(U)$. Consider the map $\Phi : V \to \text{Ind}(U)$ defined by $\Phi(v) = F_v$. Clearly,

$$\Phi(v + v')(g) = F_{v+v'}(g) = \phi(\pi(g)(v + v')) = \phi(\pi(g)v) + \phi(\pi(g)v')$$

$$= F_v(g) + F_{v'}(g) = \Phi(v)(g) + \Phi(v')(g)$$

for any $g \in G$, hence we have $\Phi(v + v') = \Phi(v) + \Phi(v')$ for all $v, v' \in V$. In addition,

$$\Phi(\alpha v)(g) = \alpha\phi(\pi(g)v) = \alpha\Phi(v)(g)$$

for all $g \in G$, hence we have $\Phi(\alpha v) = \alpha\Phi(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. It follows that $\Phi$ is a linear map from $V$ into $\text{Ind}(U)$. Moreover, we have

$$\Phi(\pi(g)v)(g') = \phi(\pi(g')\pi(g)v) = \phi(\pi(g')\pi(g)v) = \Phi(g')(\pi(g)\Phi(v)) = (\rho(g)\Phi(v))(g')$$

for all $g' \in V$. Hence, we have $\Phi(\pi(g)v) = \rho(g)\Phi(v)$ for all $g \in G$ and $v \in V$. Therefore, $\Phi$ is a morphism of representations $(\pi, V)$ and $\text{Ind}^G_H(\nu)$ of $G$. Denote the map $\phi \to \Phi$ from $\text{Hom}_H(\text{Res}^G_H(\pi), \nu)$ into $\text{Hom}_G(\pi, \text{Ind}^G_H(\nu))$ by $B$.

Clearly, for $\phi \in \text{Hom}_H(\text{Res}^G_H(\pi), \nu)$, we have

$$((A \circ B)(\phi))(v) = (A(\Phi))(v) = \Phi(v)(1) = F_v(1) = \phi(v)$$

for all $v \in V$. Therefore, $A \circ B$ is the identity map.

In addition, for $\Psi \in \text{Hom}_G(\pi, \text{Ind}^G_H(\nu))$, we have

$$(((B \circ A)(\Psi))(v))(g) = (B(\Psi))(v))(g) = A(\Psi)(\pi(g)v)$$

$$= ((\Psi(\pi(g)v))(1) = (\rho(g)(\Psi(v)))(1) = \Psi(\pi(g)v))$$

for all $g \in G$. Hence, we have $((B \circ A)(\Psi))(v) = \Psi(v)$ for all $v \in V$, i.e., $(B \circ A)(\Psi) = \Psi$ for all $\Psi$ and $B \circ A$ is also the identity map. \qed
By Maschke’s theorem, \( \text{Rep}(H) \) is semisimple, and every short exact sequence splits. Therefore we have the following result.

2.2.2. **Theorem.** The induction functor \( \text{Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G) \) is exact.

2.3. **Induction in stages.** Let \( K \) be a subgroup of \( H \). Then we have \( \text{Res}_K^G = \text{Res}_K^H \circ \text{Res}_H^G \) as functors from \( \text{Rep}(G) \) into \( \text{Rep}(K) \). Since induction functors are right adjoints, this immediately implies the following result which is called the **induction in stages**.

2.3.1. **Theorem.** Let \( H \) be a subgroup of \( G \) and \( K \) a subgroup of \( H \). Then the functors \( \text{Ind}_K^G \) and \( \text{Ind}_H^G \circ \text{Ind}_K^H \) are isomorphic.

2.4. **Frobenius Reciprocity.** Obviously, the restriction functor \( \text{Res}_H^G \) maps finite-dimensional representations into finite-dimensional representations. From the following result we see that the induction functor \( \text{Ind}_H^G \) does the same.

2.4.1. **Proposition.** Let \( (\nu, U) \) be a finite-dimensional representation of \( H \). Then
\[
\dim \text{Ind}_H^G(\nu) = \text{Card}(H \setminus G) \cdot \dim(\nu).
\]

**Proof.** Let \( C \) be a right \( H \)-coset in \( G \). Let \( g_C \) be an element in \( C \). Then the functions
\[
F_{C,v}(g) = \begin{cases} 
\nu(gg_C^{-1})v & \text{for } g \in Hg_C; \\
0 & \text{for } g \notin Hg_C;
\end{cases}
\]
span \( \text{Ind}(U) \). If \( e_1, e_2, \ldots, e_m \) is a basis of \( U \), the family \( F_{C,e_i}, C \in H \setminus G, 1 \leq i \leq m \), is a basis of \( \text{Ind}(U) \). \( \square \)

Let \( (\pi, V) \) be an irreducible representation of \( G \) and \( \nu \) an irreducible representation of \( H \). Then \( \text{Ind}_H^G(\nu) \) is finite-dimensional by 2.4.1 and a direct sum of irreducible representations of \( G \). The multiplicity of \( \pi \) in this direct sum is \( \dim_C \text{Hom}_G(\pi, \text{Ind}_H^G(\nu)) \) by Schur Lemma. By 2.2.1, we conclude that
\[
\dim_C \text{Hom}_G(\pi, \text{Ind}_H^G(\nu)) = \dim_C \text{Hom}_H(\text{Res}_H^G(\pi), \nu).
\]
The latter expression is the multiplicity of \( \nu \) in \( \text{Res}_H^G(\pi) \).

This leads to the following version of Frobenius reciprocity for representations of finite groups.

2.4.2. **Theorem.** Let \( \pi \) be an irreducible representation of \( G \) and \( \nu \) an irreducible representation of \( H \). Then the multiplicity of \( \pi \) in \( \text{Ind}_H^G(\nu) \) is equal to the multiplicity of \( \nu \) in \( \text{Res}_H^G(\pi) \).

2.5. **An example.** Let \( S_3 \) be the symmetric group in three letters. We shall show how above results allow us to construct irreducible representations of \( S_3 \).

The order of \( S_3 \) is \( 3! = 6 \). It contains the normal subgroup \( A_3 \) consisting of all even permutations which is of order 3. The quotient group \( S_3/A_3 \) consists of two elements.

The identity element is \((1 2 3)\). The other two even permutations are \((2 3 1)\) and \((3 1 2)\). We have \((2 1 3)^2 = 1\) and
\[
(2 1 3)(2 3 1)(2 1 3) = (3 1 2).
\]
Hence nontrivial even permutations form a conjugacy class.
The odd permutations are \((2\ 1\ 3), (1\ 3\ 2)\) and \((3\ 2\ 1)\). Since \((2\ 1\ 3)(1\ 3\ 2)\(2\ 1\ 3) = (3\ 1\ 2), (1\ 3\ 2)\) and \((3\ 2\ 1)\) are conjugate. On the other hand, \((1\ 3\ 2)^2 = 1\) and \((1\ 3\ 2)(3\ 1\ 2)(1\ 3\ 2) = (3\ 2\ 1), (2\ 3\ 1)\) and \((3\ 2\ 1)\) are conjugate. Therefore all odd permutations form a conjugacy class. It follows that \(S_3\) has three conjugacy classes. Therefore \(S_3\) has three irreducible representations.

Clearly, two irreducible representations of \(S_3\) are the trivial representation and the sign representation. Since \(1^2 + 1^2 + 2^2 = 6\), by Burnside theorem, the third irreducible representation \(\pi\) is two-dimensional. By 1.8.1, the character of regular representation is 6 at the identity element and 0 on all other elements. By Burnside theorem the character of \(\pi\) is one half of the difference of the characters of regular representation and the direct sum of trivial and sign representation. The latter character is 2 on even elements and 0 on odd elements. Therefore, the character of \(\pi\) is 2 at the identity, \(-1\) on nontrivial even elements and 0 at odd elements. It follows that the character of \(\pi\) is supported on \(A_3\).

The group \(A_3\) is cyclic with three elements. It has two nontrivial one-dimensional representations. If we pick a generator \(a = (2\ 3\ 1)\) of \(A_3\) one character maps \(a\) into \(e^{i\frac{2\pi}{3}}\) and the other maps \(a\) to \(e^{-i\frac{2\pi}{3}}\). We call the first one \(\nu\). By a direct calculation we see that \((2\ 1\ 3)a(2\ 1\ 3) = a^{-1}\). The restriction of \(\pi\) to \(A_3\) is a direct sum of two characters of \(A_3\). Since we know that \(\operatorname{ch}(\pi)(a) = -1\) we see that it must be

\[
\nu(a) + \nu(a)^{-1} = e^{\frac{i\pi}{6}} + e^{-\frac{i\pi}{6}} = \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) = -1.
\]

Therefore, \(\operatorname{Res}^{S_3}_{A_3}(\pi) = \nu \oplus \nu^{-1}\).

By Frobenius reciprocity, we have

\[
\dim \operatorname{Hom}_{S_3}(\pi, \operatorname{Ind}^{S_3}_{A_3}(\nu)) = \dim \operatorname{Hom}_{A_3}(\operatorname{Res}^{S_3}_{A_3}(\pi), \nu) = 1.
\]

Hence, \(\pi\) is an equivalent to a subrepresentation of \(\operatorname{Ind}^{S_3}_{A_3}(\nu)\). Since their dimensions are equal, we have \(\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu)\). Analogously, we prove that \(\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})\).

Therefore we proved that the dual of \(S_3\) consists of the classes of the trivial representation, sign representation and the induced representation \(\operatorname{Ind}^{S_3}_{A_3}(\nu) \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})\).

2.6. Characters of induced representations. Let \((\nu, U)\) be a finite-dimensional representation of \(H\). Let \((e_i; 1 \leq i \leq n)\) be a basis of \(U\). In the proof of 2.4.1, we constructed a basis \((F_{C,i}; C \in H\backslash G, 1 \leq i \leq n)\) of \(\operatorname{Ind}(U)\). Let \(C \in H\backslash G\) and \(1 \leq i \leq n\). Let \(g \in G\). Then

\[
(\rho(g)F_{C,i})(g') = F_{C,i}(g'g)
\]

for all \(g' \in G\), i.e., \(\rho(g)F_{C,i}\) is supported on the coset \(D = C \cdot g^{-1}\). Therefore, it is a linear combination of \(F_{D,j}\), \(1 \leq j \leq n\), i.e.,

\[
\rho(g)F_{C,i} = \sum_{j=1}^{n} c_j F_{D,j}.
\]

Hence, \(\rho(g)F_{C,i}\) is a linear combination of \(F_{C,j}\), \(1 \leq j \leq n\), if and only if \(D = C\), i.e., \(gC\) and \(ggC\) are in the same \(H\)-coset. This implies that \(gCg = hgC\) for some \(h \in H\), i.e., \(ggggC^{-1} = h \in H\). Conversely, if \(ggggC^{-1} \in H\) for some \(C\), we have

\[
C = HgC = HggC = C \cdot g
\]
and \( g_C \) and \( g_C g \) are in the same \( H \)-coset. Moreover, we have

\[
(\rho(g)F_{C,i})(g) = F_{C,i}(g_C) = F_{C,i}(hgC) = \nu(h)F_{C,i}(g_C)
\]

\[
= \nu(h)e_i = \sum_{j=1}^{n} \nu(h)_{j,i}e_j = \sum_{j=1}^{n} \nu(h)_{j,j}F_{C,j}(g_C).
\]

This in turn implies that

\[
\rho(g)F_{C,i} = \sum_{j=1}^{n} \nu(h)_{j,j}F_{C,j}
\]

if \( C \cdot g^{-1} = C \). Therefore, the matrix of \( \rho(g) \) has a nonzero diagonal entry in the basis \( \{F_{C,i}, C \in H \setminus G, 1 \leq i \leq n\} \), only if \( C = C \cdot g \) and then these entries are \( \nu(h)_{jj}, 1 \leq j \leq n \). This implies that

\[
\text{ch(Ind}_G^H(\nu))(g) = \frac{1}{|H|} \sum_{g \in H} \text{ch(Ind}_G^H(\nu))(g_Cg^{-1}_C - g^{-1})
\]

\[
= \frac{1}{|H|} \sum_{g \in H} \text{ch(Ind}_G^H(\nu))(hgCg^{-1}_C - h^{-1})
\]

\[
= \frac{1}{|H|} \sum_{g'g^{-1} \in H} \text{ch(Ind}_G^H(\nu))(g'g^{-1}).
\]

We extend the character of \( \nu \) to a function \( \chi_\nu \) on \( G \) which vanishes outside \( H \). Then we get the following result.

2.6.1. Theorem. The character of induced representation \( \text{Ind}_H^G(\nu) \) is equal to

\[
\text{ch(Ind}_H^G(\nu))(g) = \frac{1}{|H|} \sum_{g' \in G} \chi_\nu(g'g^{-1}).
\]

Therefore the character of the induced representation is proportional to the average of the function \( \chi_\nu \) on the equivalence classes in \( G \).

In particular we have the following result.

2.6.2. Corollary. The character of \( \text{Ind}_H^G(\nu) \) is supported in the union of conjugacy classes in \( G \) which intersect \( H \).

The result is particularly simple if \( H \) is a normal subgroup of \( G \).

2.6.3. Corollary. Let \( H \) be a normal subgroup of \( G \). Then:

(i) the support of the character of \( \text{Ind}_H^G(\nu) \) is in \( H \);

(ii) we have

\[
\text{ch(Ind}_H^G(\nu))(h) = \frac{1}{|H|} \sum_{g \in G} \text{ch(Ind}_H^G(\nu))(ghg^{-1})
\]

for any \( h \in H \).
2.7. An example. Consider again the representation \( \pi \cong \text{Ind}_{A_3}^{S_3}(\nu) \). By the above formula, its character vanishes outside of \( A_3 \) and is equal to
\[
\text{ch}(\pi)(h) = \frac{1}{3} \sum_{g \in S_3} \nu(ghg^{-1})
\]
for \( h \in A_3 \). If \( h = 1 \), we see that \( \text{ch}(\pi)(1) = \frac{6}{3} = 2 \).

If \( h = a \), we have \( gag^{-1} = a \) for \( g \in A_3 \). If \( g \) is not in \( A_3 \), it is in the other \( A_3 \)-coset. Therefore, it is in the coset represented by \( (2 \ 1 \ 3) \). By the calculation done before, \( gag^{-1} = a^{-1} \) for \( g \notin A_3 \). Therefore, we have
\[
\text{ch}(\pi)(a) = \frac{1}{3} \sum_{g \in S_3} \nu(gag^{-1}) = \nu(a) + \nu(a^{-1}) = -1.
\]
This agrees with the calculation of the character of \( \pi \) done before.

2.8. Characters and Frobenius reciprocity. Now we are going to give a proof of 2.4.2 based on character formula for the induced representation and the orthogonality relations.

We denote by \((\cdot | \cdot)_G\) the inner product on \( \mathbb{C}[G] \) and by \((\cdot | \cdot)_H\) the inner product on \( \mathbb{C}[H] \). Let \( \pi \) be a finite-dimensional representation of \( G \) and \( \nu \) a finite-dimensional representation of \( H \). Then we have
\[
(ch(\text{Ind}_H^G(\nu)) | ch(\pi))_G = \frac{1}{|G|} \sum_{g \in G} ch(\text{Ind}_H^G(\nu))(g) ch(\pi)(g)
\]
\[
= \frac{1}{|G||H|} \sum_{g \in G} \left( \sum_{g' \in G} \chi_\nu(g'gg'^{-1}) ch(\pi)(g) \right) = \frac{1}{|H|} \sum_{g' \in G} \frac{1}{|G|} \left( \sum_{g \in G} \chi_\nu(g'gg'^{-1}) ch(\pi)(g) \right)
\]
\[
= \frac{1}{|H|} \sum_{g' \in G} \frac{1}{|G|} \left( \sum_{g \in G} \chi_\nu(g) ch(\pi)(g) \right) = \frac{1}{|H|} \sum_{h \in H} ch(\nu)(h) ch(\pi)(h)
\]
\[
= (ch(\nu) | ch(\text{Res}_H^G(\pi)))_H.
\]
CHAPTER II

Representations of compact groups

1. Haar measure on compact groups

1.1. Compact groups. Let $G$ be a group. We say that $G$ is a topological group if $G$ is equipped with hausdorff topology such that the multiplication $(g, h) \mapsto gh$ from the product space $G \times G$ into $G$ and the inversion $g \mapsto g^{-1}$ from $G$ into $G$ are continuous functions.

Let $G$ and $H$ be two topological groups. A morphism of topological groups $\varphi : G \to H$ is a group homomorphism which is also continuous.

Topological groups and morphisms of topological groups for the category of topological groups.

Let $G$ be a topological group. Let $G^{opp}$ be the topological space $G$ with the multiplication $(g, h) \mapsto g \ast h = h \cdot g$. Then $G^{opp}$ is also a topological group which we call the opposite group of $G$. Clearly, the inverse of an element $g \in G$ is the same as the inverse in $G^{opp}$. Moreover, the map $g \mapsto g^{-1}$ is an isomorphism of $G$ with $G^{opp}$. Clearly, we have $(G^{opp})^{opp} = G$.

A topological group $G$ is compact, if $G$ is a compact space. The opposite group of a compact group is compact.

We shall need the following fact. Let $G$ be a topological group. We say that a function $\phi : G \to \mathbb{C}$ is right (resp. left) uniformly continuous on $G$ if for any $\epsilon > 0$ there exists an open neighborhood $U$ of 1 such that $|\phi(g) - \phi(h)| < \epsilon$ for any $g, h \in G$ such that $gh^{-1} \in U$ (resp. $g^{-1}h \in U$). Clearly, a right uniformly continuous function on $G$ is left uniformly continuous function on $G^{opp}$.

1.1.1. Lemma. Let $G$ be a compact group. Let $\phi$ be a continuous function on $G$. Then $\phi$ is right and left uniformly continuous on $G$.

Proof. By the above discussion, it is enough to prove that $\phi$ is right uniformly continuous.

Let $\epsilon > 0$. Let consider the set $A = \{(g, g') \in G \times G \mid |\phi(g) - \phi(g')| < \epsilon\}$. Then $A$ is an open set in $G \times G$. Let $U$ be an open neighborhood of 1 in $G$ and $B_U = \{(g, g') \in G \times G \mid g'g^{-1} \in U\}$. Since the function $(g, g') \mapsto g'g^{-1}$ is continuous on $G \times G$ the set $B_U$ is open. It is enough to show that there exists an open neighborhood $V$ of 1 in $G$ such that $B_V \subset A$.

Clearly, $B_U$ are open sets containing the diagonal $\Delta$ in $G \times G$. Moreover, under the homomorphism $\kappa$ of $G \times G$ given by $\kappa(g, g') = (g, gg')$, the sets $B_U$ correspond to the sets $G \times U$. In addition, the diagonal $\Delta$ corresponds to $G \times \{1\}$. Assume that the open set $O$ corresponds to $A$.

By the definition of product topology, for any $g \in G$ there exist neighborhoods $U_g$ of 1 and $V_g$ of $g$ such that $V_g \times U_g$ is a neighborhood of $(g, 1)$ contained in $O$. Clearly, $(V_g; g \in G)$ is an open cover of $G$. Since $G$ is compact, there exists a finite subcovering $(V_{g_i}; 1 \leq i \leq n)$ of $G$. Put $U = \bigcap_{i=1}^n U_{g_i}$. Then $U$ is an open
neighborhood of 1 in $G$. Moreover, $G \times U$ is an open set in $G \times G$ contained in $O$. Therefore $B_U \subset A$.

Therefore, we can say that a continuous function on $G$ is uniformly continuous.

1.2. A compactness criterion. Let $X$ be a compact space. Denote by $C(X)$ the space of all complex valued continuous functions on $X$. Let $\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. Then $f \mapsto \|f\|$ is a norm on $C(X)$, $C(X)$ is a Banach space.

Let $\mathcal{S}$ be a subset of $C(X)$.

We say that $\mathcal{S}$ is equicontinuous if for any $\epsilon > 0$ and $x \in X$ there exists a neighborhood $U_x$ of $x$ such that $|f(y) - f(x)| < \epsilon$ for all $y \in U_x$ and $f \in \mathcal{S}$.

We say that $\mathcal{S}$ is pointwise bounded if for any $x \in X$ there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{S}$.

The aim of this section is to establish the following theorem.

1.2.1. Theorem (Arzelà-Ascoli). Let $\mathcal{S}$ be a pointwise bounded, equicontinuous subset of $C(X)$. Then the closure of $\mathcal{S}$ is a compact subset of $C(X)$.

Proof. We first prove that $\mathcal{S}$ is bounded in $C(X)$. Let $\epsilon > 0$. Since $\mathcal{S}$ is equicontinuous, for any $x \in X$, there exists an open neighborhood $U_x$ of $x$ such that $y \in U_x$ implies that $|f(y) - f(x)| < \epsilon$ for all $f \in \mathcal{S}$. Since $X$ is compact, there exists a finite set of points $x_1, x_2, \ldots, x_n \in X$ such that $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover $X$.

Since $\mathcal{S}$ is pointwise bounded, there exists $M \geq 2\epsilon$ such that $|f(x_i)| \leq \frac{M}{2}$ for all $1 \leq i \leq n$ and all $f \in \mathcal{S}$. Let $x \in X$. Then $x \in U_{x_i}$ for some $1 \leq i \leq n$. Therefore, we have

$$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < \frac{M}{2} + \epsilon \leq M$$

for all $f \in \mathcal{S}$. It follows that $\|f\| \leq M$ for all $f \in \mathcal{S}$. Hence $\mathcal{S}$ is contained in a closed ball of radius $M$ centered at 0 in $C(X)$.

Now we prove that $\mathcal{S}$ is contained in a finite family of balls of fixed small radius centered in elements of $\mathcal{S}$. We keep the choices from the first part of the proof. Let $D = \{z \in \mathbb{C} \mid |z| \leq M\}$. Then $D$ is compact. Consider the compact set $D^n$. It has natural metric given by $d(z, y) = \max_{1 \leq i \leq n} |z_i - y_i|$. There exist points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $D^n$ such that the balls $B_i = \{z \in D^n \mid d(\alpha_i, z) < \epsilon\}$ cover $D^n$.

Denote by $\Phi$ the map from $\mathcal{S}$ into $D^n$ given by $f \mapsto (f(x_1), f(x_2), \ldots, f(x_n))$. Then we can find a subfamily of the above cover of $D^n$ consisting of balls intersecting $\Phi(\mathcal{S})$. After a relabeling, we can assume that these balls are $B_i$ for $1 \leq i \leq k$. Let $f_1, f_2, \ldots, f_k$ be functions in $\mathcal{S}$ such that $\Phi(f_i)$ is in the ball $B_i$ for any $1 \leq i \leq k$. Denote by $C_i$ the open ball of radius $2\epsilon$ centered in $\Phi(f_i)$. Let $\beta \in B_i$. Then we have $d(\beta, \alpha_i) < \epsilon$ and $d(\Phi(f_i), \alpha_i) < \epsilon$. Hence, we have $d(\beta, \Phi(f_i)) < 2\epsilon$, i.e., $B_i \subset C_i$. It follows that $\Phi(\mathcal{S})$ is contained in the union of $C_1, C_2, \ldots, C_k$.

Differently put, for any function $f \in \mathcal{S}$, there exists $1 \leq i \leq k$ such that $|f(x_j) - f_i(x_j)| < 2\epsilon$ for all $1 \leq j \leq n$.

Let $x \in X$. Then $x \in U_{x_j}$ for some $1 \leq j \leq n$. Hence, we have

$$|f(x) - f_i(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| < 4\epsilon,$$

i.e., $\|f - f_i\| < 4\epsilon$.

Now we can prove the compactness of the closure $\overline{\mathcal{S}}$ of $\mathcal{S}$. Assume that $\mathcal{S}$ is not compact. Then there exists an open cover $\mathcal{U}$ of $\mathcal{S}$ which doesn’t contain a finite subcover. By the above remark, $\mathcal{S}$ can be covered by finitely many closed balls $\{f \in C(X) \mid \|f - f_i\| \leq 1\}$ with $f_i \in \mathcal{S}$. Therefore, there exists a set $K_1$
1. HAAR MEASURE ON COMPACT GROUPS

which is the intersection of \( \bar{S} \) with one of the closed balls and which is not covered by a finite subcover of \( \mathcal{U} \). By induction, we can construct a decreasing family \( K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots \) of closed subsets of \( \bar{S} \) which are contained in closed balls of radius \( \frac{1}{n} \) centered in some point of \( S \), such that none of \( K_n \) is covered by a finite subcover of \( \mathcal{U} \).

Let \( (F_n; n \in \mathbb{N}) \) be a sequence of functions such that \( F_n \in K_n \) for all \( n \in \mathbb{N} \). Then \( F_p, F_q \in K_n \) for all \( p, q \) greater than \( n \). Since \( K_n \) are contained in closed balls of radius \( \frac{1}{n} \), \( \|F_p - F_q\| \leq \frac{2}{n} \) for all \( p, q \) greater than \( n \). Hence, \( (F_n) \) is a Cauchy sequence in \( C(X) \). Therefore, it converges to a function \( F \in C(X) \). This function is in \( \mathcal{S} \) and therefore in one element \( V \) of the open cover \( \mathcal{U} \). Therefore, for sufficiently large \( n \), there exists a closed ball of radius \( \frac{2}{n} \) centered in \( F \) which is contained in \( V \). Since \( F \) is also in \( K_n \), we see that \( K_n \) is in \( V \). This clearly contradicts our construction of \( K_n \). It follows that \( \bar{S} \) must be compact. \( \qed \)

1.3. Haar measure on compact groups. Let \( C_R(G) \) be the space of real valued functions on \( G \). For any function \( f \in C_R(G) \) we define the maximum \( M(f) = \max_{g \in G} f(g) \) and minimum \( m(f) = \min_{g \in G} f(g) \). Moreover, we denote by \( V(f) = M(f) - m(f) \) the variation of \( f \).

Clearly, the function \( f \) is constant on \( G \) if and only if \( V(f) = 0 \).

Let \( f, f' \in C_R(G) \) be two functions such that \( \|f - f'\| < \epsilon \). Then

\[
(f(g) - \epsilon < f'(g) < f(g) + \epsilon
\]

for all \( g \in G \). This implies that

\[
m(f) - \epsilon < f'(g) < M(f) + \epsilon
\]

for all \( g \in G \), and

\[
m(f) - \epsilon < m(f') < M(f') < M(f) + \epsilon.
\]

Hence

\[
V(f') = M(f') - m(f') < M(f) - m(f) + 2\epsilon = V(f) + 2\epsilon,
\]

i.e., \( V(f') - V(f) < 2\epsilon \). By symmetry, we also have \( V(f) - V(f') < 2\epsilon \). It follows that \( |V(f) - V(f')| < 2\epsilon \).

Therefore, we have the following result.

1.3.1. Lemma. The variation \( V \) is a continuous function on \( C_R(G) \).

Let \( f \in C_R(G) \) and \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \) a finite sequence of points in \( G \). We define the (right) mean value \( \mu(f, \mathbf{a}) \) of \( f \) with respect to \( \mathbf{a} \) as

\[
\mu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(ga_i)
\]

for all \( g \in G \). Clearly, \( \mu(f, \mathbf{a}) \) is a continuous real function on \( G \).

If \( f \) is a constant function, \( \mu(f, \mathbf{a}) = f \).

Clearly, mean value \( f \mapsto \mu(f, \mathbf{a}) \) is a linear map. Moreover, we have the following result.

1.3.2. Lemma. (i) The linear map \( f \mapsto \mu(f, \mathbf{a}) \) is continuous. More precisely, we have

\[
\|\mu(f, \mathbf{a})\| \leq \|f\|
\]

for any \( f \in C_R(G) \);
(ii) \[ M(\mu(f, a)) \leq M(f) \]
for any \( f \in C_\mathbb{R}(G) \);

(iii) \[ m(\mu(f, a)) \geq m(f) \]
for any \( f \in C_\mathbb{R}(G) \);

(iv) \[ V(\mu(f, a)) \leq V(f) \]
for any \( f \in C_\mathbb{R}(G) \).

Proof. (i) Clearly, we have
\[ \|\mu(f, a)\| = \max_{g \in G} |\mu(f, a)| \leq \frac{1}{n} \sum_{g \in G} \max_{g \in G} |f(ga_i)| = \|f\|. \]

(ii) We have
\[ M(\mu(f, a)) = \frac{1}{n} \max_{g \in G} \left( \sum_{i=1}^{n} f(ga_i) \right) \leq \frac{1}{n} \sum_{i=1}^{n} \max_{g \in G} f(ga_i) = M(f). \]

(iii) We have
\[ m(\mu(f, a)) = \frac{1}{n} \min_{g \in G} \left( \sum_{i=1}^{n} f(ga_i) \right) \geq \frac{1}{n} \sum_{i=1}^{n} \min_{g \in G} f(ga_i) = m(f). \]

(iv) By (ii) and (iii), we have
\[ V(\mu(f, a)) = M(\mu(f, a)) - m(\mu(f, a)) \leq M(f) - m(f) = V(f). \]

\[ \square \]

Denote by \( M_f \) the set of mean values of \( f \) for all finite sequences in \( G \).

1.3.3. LEMMA. The set of functions \( M_f \) is uniformly bounded and equicontinuous.

Proof. By 1.3.2 (ii) and (iii), it follows that
\[ m(f) \leq m(\mu(f, a)) \leq \mu(f, a)(g) \leq M(\mu(f, a)) \leq M(f). \]

This implies that \( M_f \) is uniformly bounded.

Now we want to prove that \( M_f \) is equicontinuous. First, by 1.1.1, the function \( f \) is uniformly continuous. Hence, for any \( \epsilon > 0 \), there exists an open neighborhood \( U \) of 1 in \( G \) such that \( |f(g) - f(h)| < \epsilon \) if \( gh^{-1} \in U \). Since, this implies that \((ga_i)(ha_i)^{-1} = gh^{-1} \in U\) for any \( 1 \leq i \leq n \), we see that
\[ |\mu(f, a)(g) - \mu(f, a)(h)| = \frac{1}{n} \left| \sum_{i=1}^{n} (f(ga_i) - f(ha_i)) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |f(ga_i) - f(ha_i)| < \epsilon \]
for \( g \in hU \). Hence, the family \( M_f \) is equicontinuous. \[ \square \]

By 1.2.1, we have the following consequence.

1.3.4. LEMMA. The set \( M_f \) of all right mean values of \( f \) has compact closure in \( C_\mathbb{R}(G) \).

We need another result on variation of mean value functions. Clearly, if \( f \) is a constant function \( \mu(f, a) = f \) for any \( a \).
1.3.5. Lemma. Let $f$ be a function in $C_2(G)$. Assume that $f$ is not a constant. Then there exists $a$ such that $V(\mu(f,a)) < V(f)$.

PROOF. Since $f$ is not constant, we have $m(f) < M(f)$. Let $C$ be such that $m(f) < C < M(f)$. Then there exists an open set $V$ in $G$ such that $f(g) \leq C$ for all $g \in V$. Since the right translates of $V$ cover $G$, by compactness of $G$ we can find $a = (a_1, a_2, \ldots, a_n)$ such that $(V a_i^{-1}, 1 \leq i \leq n)$ is an open cover of $G$. For any $g \in V a_i^{-1}$ we have $ga_i \in V$ and $f(ga_i) \leq C$. Hence, we have

$$\mu(f,a)(g) = \frac{1}{n} \sum_{j=1}^{n} f(ga_j) = \frac{1}{n} \left( f(ga_i) + \sum_{j \neq i} f(ga_j) \right) \leq \frac{1}{n} (C + (n-1)M(f)) < M(f).$$

On the other hand, by 1.3.2.(iii) we know that $m(\mu(f,a)) \geq m(f)$ for any $a$. Hence we have

$$V(\mu(f,a)) = M(\mu(f,a)) - m(\mu(f,a)) < M(f) - m(f) = V(f).$$

\square

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_m)$ be two finite sequences in $G$. We define $a \cdot b = (a_i b_j; 1 \leq i \leq n, 1 \leq j \leq m)$.

1.3.6. Lemma. We have

$$\mu(\mu(f,b),a) = \mu(f,b \cdot a).$$

PROOF. We have

$$\mu(\mu(f,b),a) = \frac{1}{m} \sum_{i=1}^{m} \mu(f,b)(ga_i) = \frac{1}{mn} \sum_{j=1}^{m} \sum_{i=1}^{n} f(ga_i b_j) = \mu(f,a \cdot b).$$

\square

1.3.7. Lemma. For any $f \in C_2(G)$, the closure $\overline{M}_f$ contains a constant function on $G$.

PROOF. By 1.3.4, we know that $\overline{M}_f$ is compact. Since, by 1.3.1, the variation $V$ is continuous on $C_2(G)$, it attains its minimum $\alpha$ at some $\varphi \in \overline{M}_f$.

Assume that $\varphi$ is not a constant. By 1.3.5, there exists $a$ such that $V(\mu(\varphi,a)) < V(\varphi)$. Let $\alpha - V(\mu(\varphi,a)) = \epsilon > 0$.

Since $V$ and $\mu(\cdot,a)$ are continuous maps by 1.3.1 and 1.3.2.(i), this implies that there is $b$ such that $|V(\mu(\varphi,a)) - V(\mu(f,b),a)| < \frac{\epsilon}{2}$. Therefore, we have

$$V(\mu(\mu(f,b),a)) \leq V(\mu(\varphi,a)) + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2}.$$

By 1.3.6, we have

$$V(\mu(f,a \cdot b)) < \alpha - \frac{\epsilon}{2}$$

counter to our choice of $\alpha$.

It follows that $\varphi$ is a constant function. In addition $\alpha = 0$.  \square
Consider now left mean values of a function \( f \in C_\mathbb{R}(G) \). We define the left mean value of \( f \) with respect to \( a = (a_1, a_2, \ldots, a_n) \) as the function

\[
\nu(f, a)(g) = \frac{1}{n} \sum_{i=1}^{n} f(a_i g)
\]

for \( g \in G \). We denote by \( \mathcal{N}_f \) the set of all left mean values of \( f \).

Let \( G^{opp} \) be the compact group opposite to \( G \). Then the left mean values of \( f \) on \( G \) are the right mean values of \( f \) on \( G^{opp} \).

Hence, from 1.3.7, we deduce the following result.

1.3.8. Lemma. For any \( f \in C_\mathbb{R}(G) \), the closure \( \mathcal{N}_f \) contains a constant function on \( G \).

By direct calculation we get the following result.

1.3.9. Lemma. For any \( f \in C_\mathbb{R}(G) \) we have

\[
\nu(\mu(f, a), b) = \mu(\nu(f, b), a)
\]

for any two finite sequences \( a \) and \( b \) in \( G \).

Proof. We have

\[
\nu(\mu(f, a), b)(g) = \frac{1}{m} \sum_{j=1}^{m} \mu(f, a)(b_j g) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(b_j g a_i) = \frac{1}{n} \sum_{i=1}^{n} \nu(f, b)(g a_i) = \mu(\nu(f, b), a)(g)
\]

for any \( g \in G \).

Putting together these results, we finally get the following.

1.3.10. Proposition. For any \( f \in C_\mathbb{R}(G) \), the closure \( \mathcal{M}_f \) contains a unique function constant on \( G \).

This function is also the unique constant function in \( \mathcal{N}_f \).

Proof. Let \( \varphi \) and \( \psi \) be two constant functions such that \( \varphi \) is in the closure of \( \mathcal{M}_f \) and \( \psi \) is in the closure of \( \mathcal{N}_f \). For any \( \epsilon > 0 \) we have \( a \) and \( b \) such that \( \|\mu(f, a) - \varphi\| < \frac{\epsilon}{2} \) and \( \|\nu(f, b) - \psi\| < \frac{\epsilon}{2} \).

On the other hand, we have

\[
\|\nu(\mu(f, a), b) - \varphi\| = \|\nu(\mu(f, a), b) - \nu(\varphi, b)\| = \|\nu(\mu(f, a) - \varphi, b)\| \leq \|\mu(f, a) - \varphi\| < \frac{\epsilon}{2}.
\]

In the same way, we also have

\[
\|\mu(\nu(f, b), a) - \psi\| = \|\mu(\nu(f, b), a) - \mu(\psi, a)\| = \|\mu(\nu(f, b) - \psi, a)\| \leq \|\nu(f, b) - \psi\| < \frac{\epsilon}{2}.
\]

By 1.3.9, this immediately yields

\[
\|\varphi - \psi\| \leq \|\nu(\mu(f, a), b) - \varphi\| + \|\mu(\nu(f, b), a) - \psi\| < \epsilon.
\]

This implies that \( \varphi = \psi \). Therefore, any constant function in the closure of \( \mathcal{M}_f \) has to be equal to \( \psi \).
The value of the unique constant function in the closure of $\mathcal{M}_f$ is denoted by $\mu(f)$ and called the mean value of $f$ on $G$. In this way, we get a function $f \mapsto \mu(f)$ on $C_G$. 

Let $\gamma : C_G \rightarrow \mathbb{R}$ be a linear form. We say that $\gamma$ is positive if for any $f \in C_G$ such that $f(g) \geq 0$ for any $g \in G$ we have $\gamma(f) \geq 0$.

1.3.11. Lemma. The function $\mu$ is a positive linear form on $C_G$.

To prove this result we need some preparation.

1.3.12. Lemma. Let $f \in C_G$. Then, for any $a$ we have 
\[
\mu(\mu(f,a)) = \mu(f).
\]

Proof. Let $\mu(f) = \alpha$. Let $\varphi$ be the function equal to $\alpha$ everywhere on $G$. Fix $\epsilon > 0$. Then there exists a finite sequence $b$ such that 
\[
\|\nu(f,b) - \varphi\| < \epsilon.
\]
This implies that 
\[
\|\nu(f - \varphi, b)\| = \|\nu(f,b) - \nu(\varphi,b)\| = \|\nu(f,b) - \varphi\| < \epsilon.
\]
This, by 1.3.2.(i), implies that 
\[
\|\mu(\nu(f - \varphi, b), a)\| \leq \|\nu(f - \varphi, b)\| < \epsilon
\]
for any finite sequence $a$.

By 1.3.9, we have
\[
\|\nu(\mu(f, a), b)\| = \|\mu(f, a)\| < \epsilon,
\]
and
\[
\|\nu(\mu(f, a), b) - \varphi\| = \|\nu(\mu(f, a) - \varphi, b)\| = \|\nu(\mu(f - \varphi, a), b)\| < \epsilon.
\]
Therefore, if we fix $a$, we see that $\varphi$ is in the closure of $N_{\mu(f,a)}$. By 1.3.10, this proves our assertion. 

Let $f$ and $f'$ be two functions in $C_G$. Let $\alpha = \mu(f)$ and $\beta = \mu(f')$. Denote by $\varphi$ and $\psi$ the corresponding constant functions. Let $\epsilon > 0$.

Clearly, there exists $a$ such that 
\[
\|\mu(f, a) - \varphi\| < \frac{\epsilon}{2}.
\]
This, by 1.3.2.(ii) implies, that we have 
\[
\|\mu(\mu(f, a), b) - \varphi\| = \|\mu(f, a) - \varphi, b)\| < \frac{\epsilon}{2}
\]
for arbitrary $b$. By 1.3.6, this in turn implies that 
\[
\|\mu(f, a \cdot b) - \varphi\| < \frac{\epsilon}{2}.
\]

On the other hand, by 1.3.12, we have $\mu(\mu(f', a)) = \mu(f') = \beta$. Therefore, there exists a finite sequence $b$ such that 
\[
\|\mu(\mu(f', a), b) - \psi\| < \frac{\epsilon}{2}.
\]
This, by 1.3.6, implies that 
\[
\|\mu(f', a \cdot b) - \psi\| < \frac{\epsilon}{2}.
\]
Hence, we have
\[ \| \mu(f + f', a \cdot b) - (\varphi + \psi) \| \leq \| \mu(f, a \cdot b) - \varphi \| + \| \mu(f', a \cdot b) - \psi \| < \epsilon. \]
Therefore, \( \varphi + \psi \) is in the closure of \( \mathcal{M}_{f + f'} \). It follows that
\[ \mu(f + f') = \alpha + \beta = \mu(f) + \mu(f'), \]
i.e., \( \mu \) is additive.

Let \( c \in \mathbb{R} \) and \( f \in C_{\mathbb{R}}(G) \). Then \( \mu(cf, a) = c \mu(f, a) \) for any \( a \). Therefore, \( \mathcal{M}_{cf} = c \mathcal{M}_f \). This immediately implies that \( \mu(cf) = c \mu(f) \). Therefore \( \mu \) is a linear form.

Assume that \( f \) is a function in \( C_{\mathbb{R}}(G) \) such that \( f(g) \geq 0 \) for all \( g \in G \). Then \( \mu(f, a)(g) \geq 0 \) for any \( a \) and \( g \in G \). Hence, any function \( \phi \in \mathcal{M}_f \) satisfies \( \phi(g) \geq 0 \) for all \( g \in G \). This immediately implies that \( \phi(g) \geq 0 \), for any \( \phi \) in the closure of \( \mathcal{M}_f \). It follows that \( \mu(f) \geq 0 \). Hence, we \( \mu \) is a positive linear form. This completes the proof of 1.3.11.

Clearly, \( \mu(1) = 1 \). Let \( f \in C_{\mathbb{R}}(G) \). Then we have
\[ -\|f\| \leq f(g) \leq \|f\| \]
for any \( g \in G \). Since \( \mu \) is a positive linear form, we see that
\[ -\|f\| = \mu(-\|f\|) \leq \mu(f) \leq \mu(\|f\|) = \|f\|. \]
Therefore, we have
\[ |\mu(f)| \leq \|f\| \]
for any \( f \in C_{\mathbb{R}}(G) \). In particular, \( \mu \) is a continuous linear form on \( C_{\mathbb{R}}(G) \).

By Riesz representation theorem, the linear form \( \mu : C_{\mathbb{R}}(G) \rightarrow \mathbb{R} \) defines a regular positive measure \( \mu \) on \( G \) such that
\[ \mu(f) = \int_G f \, d\mu. \]
Clearly, we have
\[ \mu(G) = \int_G d\mu = \mu(1) = 1. \]
so we say that \( \mu \) is normalized.

Denote by \( R \) (resp. \( L \)) the right regular representation (resp. left regular representation) of \( G \) on \( C(G) \) given by \( (R(g)f)(h) = f(hg) \) (resp. \( (L(g)f)(h) = f(g^{-1}h) \)) for any \( f \in C(G) \) and \( g, h \in G \).

1.3.13. Lemma. Let \( f \in C_{\mathbb{R}}(G) \) and \( g \in G \). Then
\[ \mu(R(g)f) = \mu(L(g)f) = \mu(f). \]

Proof. Let \( g = (g) \). Clearly, we have
\[ \mu(f, g)(h) = f(hg) = (R(g)f)(h) \]
for all \( h \in G \), i.e., \( R(g)f = \mu(f, g) \). By 1.3.12, we have
\[ \mu(R(g)f) = \mu(\mu(f, g)) = \mu(f). \]

This statement for \( G^{opp} \) implies the other equality. \( \square \)

We say that the linear form \( \mu \) is biinvariant, i.e., right invariant and left invariant.

The above result implies that the measure \( \mu \) is biinvariant, i.e., we have the following result.
1.3.14. **Lemma.** Let $A$ be a measurable set in $G$. Then $gA$ and $Ag$ are also measurable and

$$
\mu(gA) = \mu(Ag) = \mu(A)
$$

for any $g \in G$.

**Proof.** Since $C_R(G)$ is dense in $L^1(\mu)$, the invariance from 1.3.13 holds for any function $f \in L^1(\mu)$. Applying it to the characteristic function of the set $A$ implies the result. □

A normalized biinvariant positive measure $\mu$ on $G$ is called a **Haar measure** on $G$.

We proved the existence part of the following result.

1.3.15. **Theorem.** Let $G$ be a compact group. Then there exists a unique Haar measure $\mu$ on $G$.

**Proof.** We constructed a Haar measure on $G$. It remains to prove the uniqueness. Let $\nu$ be another Haar measure on $G$. Then, by left invariance, we have

$$
Z_G \int_G f \, d\nu = \frac{1}{n} \sum_{i=1}^n \int_G f(ga_i) \, d\nu = \int_G f \, d\nu
$$

for any $a$. Hence the integral with respect to $\nu$ is constant on $M_f$. By continuity, it is also constant on its closure. Therefore, we have

$$
\int_G f \, d\nu = \mu(f) \int_G f \, d\nu = \mu(f) = \int_G f \, d\mu
$$

for any $C_R(G)$. This in turn implies that $\nu = \mu$. □

1.3.16. **Lemma.** Let $\mu$ be the Haar measure on $G$. Let $U$ be a nonempty open set in $G$. Then $\mu(U) > 0$.

**Proof.** Since $U$ is nonempty, $(Ug; g \in G)$ is an open cover of $G$. It contains a finite subcover $(Ug_i; 1 \leq i \leq n)$. Therefore we have

$$
1 = \mu(G) = \mu \left( \bigcup_{i=1}^n U g_i \right) \leq \sum_{i=1}^n \mu(Ug_i) = \sum_{i=1}^n \mu(U) = n \mu(U)
$$

by 1.3.14. This implies that $\mu(U) \geq \frac{1}{n}$. □

1.3.17. **Lemma.** Let $f$ be a continuous function on $G$. Then

$$
\int_G f(g^{-1}) \, d\mu(g) = \int_G f(g) \, d\mu(g).
$$

**Proof.** Clearly, it is enough to prove the statement for real-valued functions. Therefore, we can consider the linear form $\nu : f \mapsto \int_G f(g^{-1}) \, d\mu(g)$. Clearly, this a positive continuous linear form on $C_R(G)$. Moreover,

$$
\nu(f) = \int_G f(h^{-1}) \, d\mu(h) = \int_G f((hg)^{-1}) \, d\mu(h)
$$

$$
= \int_G f(g^{-1}h^{-1}) \, d\mu(h) = \int_G (L(g)f)(h^{-1}) \, d\mu(h) = \nu(L(g)f)
$$
and
\[ \nu(f) = \int_G f(h^{-1}) \, d\mu(h) = \int_G f((g^{-1}h)^{-1}) \, d\mu(h) \]
\[ = \int_G f(h^{-1}g) \, d\mu(h) = \int_G (R(g)f)(h^{-1}) \, d\mu(h) = \nu(R(g)f) \]
for any \( g \in G \). Hence, this linear form is left and right invariant. By the uniqueness of the Haar measure we get the statement. \( \square \)

2. Algebra of matrix coefficients

2.1. Topological vector spaces. Let \( E \) be a vector space over \( \mathbb{C} \). We say that \( E \) is a topological vector space over \( \mathbb{C} \), if it is also equipped with a topology such that the functions \((u,v) \mapsto u + v \) from \( E \times E \) into \( E \), and \((\alpha,u) \mapsto \alpha u \) from \( \mathbb{C} \times E \) into \( E \) are continuous.

A morphism \( \phi : E \to F \) of topological vector space \( E \) into \( F \) is a continuous linear map from \( E \) to \( F \).

We say that \( E \) is a hausdorff topological vector space if the topology of \( E \) is hausdorff.

The vector space \( \mathbb{C}^n \) with its natural topology is a hausdorff topological vector space.

Let \( E \) be a topological vector space and \( F \) a vector subspace of \( E \). Then \( F \) is a topological vector space with the induced topology. Moreover, if \( E \) is hausdorff, \( F \) is also hausdorff.

2.1.1. Lemma. Let \( E \) be a topological vector space over \( \mathbb{C} \). Then the following conditions are equivalent:

(i) \( E \) is hausdorff;

(ii) \( \{0\} \) is a closed set in \( E \).

Proof. Assume that \( E \) is hausdorff. Let \( v \in E \), \( v \neq 0 \). Then there exist open neighborhoods \( U \) of 0 and \( V \) of \( v \) such that \( U \cap V = \emptyset \). In particular, \( V \subset E - \{0\} \). Hence, \( E - \{0\} \) is an open set. This implies that \( \{0\} \) is closed.

Assume now that \( \{0\} \) is closed in \( E \). Then \( E - \{0\} \) is an open set. Let \( u \) and \( v \) be different vectors in \( E \). Then \( u - v \neq 0 \). Since the function \((x,y) \mapsto x + y \) is continuous, there exist open neighborhoods \( U \) of \( u \) and \( V \) of \( v \) such that \( U - V \subset E - \{0\} \). This in turn implies that \( U \cap V = \emptyset \). \( \square \)

The main result of this section is the following theorem. It states that hausdorff finite-dimensional topological vector spaces have unique topology.

2.1.2. Theorem. Let \( E \) be a finite-dimensional hausdorff topological vector space over \( \mathbb{C} \). Let \( v_1, v_2, \ldots, v_n \) be a basis of \( E \). Then the linear map \( \mathbb{C}^n \to E \) given by

\[ (c_1, c_2, \ldots, c_n) \mapsto \sum_{i=1}^n c_i v_i \]

is an isomorphism of topological vector spaces.

Clearly, the map

\[ \phi(z) = \sum_{i=1}^n z_i v_i, \]
for all \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \), is a continuous linear isomorphism of \( \mathbb{C}^n \) onto \( E \). Therefore, it is enough to show that that map is also open.

We start with an elementary lemma.

2.1.3. Lemma. Let \( E \) and \( F \) be topological vector spaces and \( \phi : E \to F \) a linear map. Let \( (U_i, i \in I) \) be a fundamental system of neighborhoods of 0 in \( E \). If \( \phi(U_i), i \in I, \) are neighborhoods of 0 in \( F, \phi \) is an open map.

Proof. Let \( U \) be an open set in \( E \). For any \( u \in U \) there exists \( i \in I \) such that \( u + U_i \) is a neighborhood of \( u \) contained in \( U \). Therefore, \( \phi(u + U_i) = \phi(u) + \phi(U_i) \) is a neighborhood of \( \phi(u) \) contained in \( \phi(U) \). Hence, \( \phi(u) \) is an interior point of \( \phi(U) \). This implies that \( \phi(U) \) is open. \( \square \)

We consider on \( \mathbb{C}^n \) the standard euclidean norm \( \| \cdot \| \). Let \( B_1 = \{ z \in \mathbb{C}^n \mid \|z\| < 1 \} \) be the open unit ball in \( \mathbb{C}^n \). Then the balls \( \{ cB_1 \mid 0 < c < \infty \} \) form a fundamental system of neighborhoods of 0 in \( \mathbb{C}^n \). By 2.1.3, to show that the above map \( \phi : \mathbb{C}^n \to E \) is open it is enough to show that \( \phi(B_1) \) is a neighborhood of 0 in \( E \).

Let \( S = \{ z \in \mathbb{C}^n \mid \|z\| = 1 \} \) be the unit sphere in \( \mathbb{C}^n \). Then, \( S \) is a bounded and closed set in \( \mathbb{C}^n \). Hence it is compact. This implies that \( \phi(S) \) is a compact set in \( E \). Since 0 is not in \( S \), 0 is not in \( \phi(S) \). Since \( E \) is hausdorff, \( \phi(S) \) is closed and \( E - \phi(S) \) is an open neighborhood of 0 in \( E \). By continuity of multiplication by scalars at \( (0,0) \), there exists \( \epsilon > 0 \) and an open neighborhood \( U \) of 0 in \( E \) such that \( zU \subset \epsilonE - \phi(S) \), i.e., \( zU \cap \phi(S) = \emptyset \) for all \( |z| \leq \epsilon \).

Let \( v \in U - \{ 0 \} \). Then we have

\[
v = \sum_{i=1}^{n} c_i v_i.
\]

Let \( c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \). Then, \( \frac{1}{\|c\|} c \in S \), and \( \frac{1}{\|c\|} v \in \phi(S) \). By our construction, we must have \( \frac{1}{\|c\|} > \epsilon \). Hence, we have \( \|c\| < \frac{1}{\epsilon} \), i.e., \( c \in B_{\frac{1}{\epsilon}} \). This in turn yields \( v \in \phi(\frac{1}{\epsilon}B_1) \). Therefore, we have

\[
\phi\left(\frac{1}{\epsilon}B_1\right) \supset U,
\]

i.e., \( \phi(B_1) \supset \epsilon U \). Hence, \( \phi(B_1) \) is a neighborhood of 0 in \( E \). This completes the proof of 2.1.2.

2.1.4. Corollary. Let \( E \) be a hausdorff topological vector space over \( \mathbb{C} \). Let \( F \) be a finite-dimensional vector subspace of \( E \). Then \( F \) is closed in \( E \).

Proof. Clearly, the topology of \( E \) induces a structure of hausdorff topological vector space on \( F \). Let \( v_1, v_2, \ldots, v_n \) be a basis of \( F \). Assume that \( F \) is not closed. Let \( w \) be a vector in the closure of \( F \) which is not in \( F \). Then \( w \) is linearly independent of \( v_1, v_2, \ldots, v_n \). Let \( F' \) be the direct sum of \( F \) and \( \mathbb{C}w \). Then \( F' \) is a \( (n+1) \)-dimensional hausdorff topological vector space. By 2.1.2, we know that

\[
(c_1, c_2, \ldots, c_n, c_{n+1}) \mapsto \sum_{i=1}^{n} c_i v_i + c_{n+1} w
\]

is an isomorphism of the topological vector space \( \mathbb{C}^{n+1} \) onto \( F' \). This isomorphism maps \( \mathbb{C}^n \times \{ 0 \} \) onto \( F \). Therefore, \( F \) is closed in \( F' \), and \( w \) is not in the closure of \( F \). Hence, we have a contradiction. \( \square \)
2.15. Lemma. Let $E$ and $F$ be two finite-dimensional Hausdorff topological vector spaces. Then any linear map $\phi : E \to F$ is continuous.

2.2. Representations on topological vector spaces. Let $G$ be a compact group. Let $E$ be a Hausdorff topological vector space over $\mathbb{C}$. We denote by $\text{GL}(E)$ the group of all automorphisms of $E$.

A (continuous) representation of $G$ on $E$ is a group homomorphism $\pi : G \to \text{GL}(E)$ such that $(g, v) \mapsto \pi(g)v$ is continuous from $G \times E$ into $E$.

2.2.1. Lemma. Let $E$ be a Banach space and $\pi : G \to \text{GL}(E)$ a homomorphism such that $g \mapsto \pi(g)v$ is continuous function from $G$ into $E$ for all $v \in E$. Then $(\pi, E)$ is a representation of $G$ on $E$.

Proof. Assume that the function $g \mapsto \pi(g)v$ is continuous for any $v \in V$. Then the function $g \mapsto \|\pi(g)v\|$ is continuous on $G$. Since $G$ is compact, there exists $M$ such that $\|\pi(g)v\| < M$ for all $g \in G$. By Banach-Steinhaus theorem, we see that the function $g \mapsto \|\pi(g)\|$ is bounded on $G$.

Pick $C > 0$ such that $\|\pi(g)\| \leq C$ for all $g \in G$. Then we have

$$\|\pi(g)v - \pi(g')v'\| = \|\pi(g)v - \pi(g')v + \pi(g')(v - v')\|$$
$$\leq \|\pi(g)v - \pi(g)\| + \|\pi(g')\||(v - v')\| \leq \|\pi(g)v - \pi(g')v\| + C\|v - v'\|$$

for all $g, g' \in G$ and $v, v' \in E$. This clearly implies the continuity of the function $(g, v) \mapsto \pi(g)v$.

If $E$ is a finite-dimensional Hausdorff topological vector space, by 2.15, any linear automorphism of $E$ is automatically an automorphism of topological vector spaces. Therefore $\text{GL}(E)$ is just the group of all linear automorphisms of $E$ as before.

Moreover, since the topology of $E$ is described by the euclidean norm and $E$ is a Banach space with respect to it, by 2.2.1, the only additional condition for a representation of $G$ is the continuity of the function $g \mapsto \pi(g)v$ for any $v \in E$. This implies the following result.

2.2.2. Lemma. Let $E$ be a finite-dimensional Hausdorff topological vector space and $\pi$ a homomorphism of $G$ into $\text{GL}(E)$. Let $v_1, v_2, \ldots, v_n$ be a basis of $E$.

(i) $(\pi, E)$ is a representation of $G$ on $E$;

(ii) all matrix coefficients of $\pi(g)$ with respect to the basis $v_1, v_2, \ldots, v_n$ are continuous functions on $G$.

2.3. Algebra of matrix coefficients. Let $G$ be a compact group. The Banach space $\text{C}(G)$ is an commutative algebra with pointwise multiplication of functions, i.e., $(\psi, \phi) \mapsto \psi \cdot \phi$ where $(\psi \cdot \phi)(g) = \psi(g)\phi(g)$ for any $g \in G$.

First, we remark the following fact.

2.3.1. Lemma. $R$ and $L$ are representations of $G$ on $\text{C}(G)$.

Proof. Clearly, we have

$$\|R(g)\phi\| = \max_{h \in G} |(R(g)\phi)(h)| = \max_{h \in G} |\phi(hg)| = \max_{h \in G} |\phi(h)| = \|\phi\|.$$ 

Hence $R(g)$ is a continuous linear map on $\text{C}(G)$. Its inverse is $R(g^{-1})$, so $R(g) \in \text{GL}(\text{C}(G))$. 

By 2.2.1, it is enough to show that the function $g \mapsto R(g)\phi$ is continuous for any function $\phi \in C(G)$.

By 1.1.1, $\phi$ is uniformly continuous, i.e., there exists a neighborhood $U$ of 1 in $G$ such that $g^{-1} g' \in U$ implies $|\phi(hg) - \phi(hg')| < \epsilon$ for all $h \in G$. Hence, we have

$$||R(g)\phi - R(g')\phi|| = \max_{h \in G} |(R(g)\phi)(h) - (R(g')\phi)(h)| = \max_{h \in G} |\phi(hg) - \phi(hg')| < \epsilon$$

for $g' \in gU$. Hence, the function $g \mapsto R(g)\phi$ is continuous.

The proof for $L$ is analogous. \(\square\)

We say that the function $\phi \in C(G)$ is right (resp. left) $G$-finite if the vectors

$$\{R(g)\phi; g \in G\} \quad \text{(resp.} \quad \{L(g)\phi; g \in G\})$$

span a finite-dimensional subspace of $C(G)$.

2.3.2. LEMMA. Let $\phi \in C(G)$. The following conditions are equivalent.

(i) $\phi$ is left $G$-finite;
(ii) $\phi$ is right $G$-finite;
(iii) there exist $n$ and functions $a_i, b_i \in C(G)$, $1 \leq i \leq n$, such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

for all $g, h \in G$.

PROOF. Let $\phi$ be a right $G$-finite function. Then $\phi$ is in a finite-dimensional

subspace $F$ invariant for $R$. The restriction of the representation $R$ to $F$ is continuous. Let $a_1, a_2, \ldots, a_n$ be a basis of $F$. Then, by 2.2.2, there exist $b_1, b_2, \ldots, b_n \in C(G)$ such that $R(g)\phi = \sum_{i=1}^{n} b_i(g)a_i$. Therefore we have

$$\phi(gh) = \sum_{i=1}^{n} b_i(g)a_i(h) = \sum_{i=1}^{n} a_i(h)b_i(g)$$

for all $h, g \in G$. Therefore (iii) holds.

If (iii) holds,

$$R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i$$

and $\phi$ is right $G$-finite.

Since the condition (iii) is symmetric, the equivalence of (i) and (iii) follows by applying the above argument to the opposite group of $G$. \(\square\)

Therefore, we can call $\phi$ just a $G$-finite function in $C(G)$. Let $R(G)$ be the subset of all $G$-finite functions in $C(G)$.

2.3.3. PROPOSITION. The set $R(G)$ is a subalgebra of $C(G)$.

PROOF. Clearly, a multiple of a $G$-finite function is a $G$-finite function.

Let $\phi$ and $\psi$ be two $G$-finite functions. Then, by 2.3.2, there exists functions

$$a_i, b_i, c_i, d_i \in C(G)$$

such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) \quad \text{and} \quad \psi(gh) = \sum_{i=1}^{m} c_i(g)d_i(h)$$

for all $g, h \in G$. This implies that

$$(\phi + \psi)(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) + \sum_{i=1}^{m} c_i(g)d_i(h)$$

is in $R(G)$. \(\square\)
and
\[(\phi \cdot \psi)(gh) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(g)c_j(g)b_i(h)d_j(h) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot c_j)(g)(b_i \cdot d_j)(h)\]
for all \(g, h \in G\). Hence, \(\phi + \psi\) and \(\phi \cdot \psi\) are \(G\)-finite. \(\square\)

Clearly, \(R(G)\) is an invariant subspace for \(R\) and \(L\).

The main result of this section is the following observation. Let \(V\) be a finite-dimensional complex linear space and \(\pi\) a continuous homomorphism of \(G\) into \(\text{GL}(V)\), i.e., \((\pi, V)\) is a representation of \(G\). For \(v \in V\) and \(v^* \in V^*\) we call the continuous function \(g \mapsto c_{v,v^*}(g) = \langle \pi(g)v, v^* \rangle\) a matrix coefficient of \((\pi, V)\).

2.3.4. Theorem. Let \(\phi \in C(G)\). Then the following statements are equivalent:

(i) \(\phi\) is in \(R(G)\);
(ii) \(\phi\) is a matrix coefficient of a finite-dimensional representation of \(G\).

Proof. Let \((\pi, V)\) be a finite-dimensional representation of \(G\). Let \(v \in V\) and \(v^* \in V^*\). By scaling \(v^*\) if necessary, we can assume that \(v\) is a vector in a basis of \(V\) and \(v^*\) a vector in the dual basis of \(V^*\). Then, \(c_{v,v^*}(g)\) is a matrix coefficient of the matrix of \(\pi(g)\) in the basis of \(V\). The rule of matrix multiplication implies that (iii) from 2.3.2 holds for \(c_{v,v^*}\). Hence \(\phi\) is \(G\)-finite.

Assume that \(\phi\) is \(G\)-finite. Then, by 2.3.2, we have
\[
R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i(1) = \phi(g),
\]
where \(a_i, b_i \in C(G)\). We can also assume that \(b_i\) are linearly independent. Let \(V\) be the subspace of \(R(G)\) spanned by \(b_1, b_2, \ldots, b_n\). Then \(V\) is a \(G\)-invariant subspace. Let \(v = \phi\) and \(v^* \in V^*\) such that \(b_i(1) = \langle b_i, v^* \rangle\). Then
\[
\langle R(g)v, v^* \rangle = \sum_{i=1}^{n} a_i(g)(b_i(1)) = \sum_{i=1}^{n} a_i(g)b_i(1) = \phi(g),
\]
i.e., \(\phi\) is a matrix coefficient of the restriction of \(R\) to \(V\). \(\square\)

Therefore, we call \(R(G)\) the algebra of matrix coefficients of \(G\).

We also have the following stronger version of 2.3.2

2.3.5. Corollary. Let \(\phi \in R(G)\). Then there exist \(n\) and functions \(a_i, b_i \in R(G), 1 \leq i \leq n\), such that
\[
\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)
\]
for all \(g, h \in G\).

Proof. Since \(\phi\) is a matrix coefficient of a finite-dimensional representation by 2.3.4, the statement follows from the formula for the product of two matrices. \(\square\)

Moreover, \(R(G)\) has the following properties. For a function \(\phi \in C(G)\) we denote by \(\hat{\phi}\) the function \(g \mapsto \overline{f(g)}\) on \(G\); and by \(\hat{\phi}\) the function \(g \mapsto f(g^{-1})\).

2.3.6. Lemma. Let \(\phi \in R(G)\). Then

(i) the function \(\hat{\phi}\) is in \(R(G)\);
(ii) the function \(\hat{\phi}\) is in \(R(G)\).

Proof. Obvious by 2.3.2. \(\square\)
3. Some facts from functional analysis

3.1. Compact operators. Let $E$ be a Hilbert space and $T : E \rightarrow E$ a bounded linear operator.

We say that $T$ is a compact operator if $T$ is a bounded linear operator which maps the unit ball in $E$ into a relatively compact set.

3.1.1. Lemma. Compact operators for a two-sided ideal in the algebra of all bounded linear operators on $E$.

Proof. Let $S$ and $T$ be compact operators. Let $B$ be the unit ball in $E$. Then the images of $B$ in $E$ under $T$ and $S$ have compact closure. Hence, the image of $B \times B$ under $S \times T : E \times E \rightarrow E \times E$ has compact closure. Since the addition is a continuous map from $E \times E$ into $E$, the image of $B$ under $S + T$ also has compact closure. Therefore, $S + T$ is a compact operator.

If $S$ is a bounded linear operator and $T$ a compact operator, the image of $B$ under $T$ has compact closure. Since $S$ is continuous, the image of $B$ under $ST$ also has compact closure. Hence, $ST$ is compact.

Analogously, the image of $B$ under $S$ is a bounded set since $S$ is bounded. Therefore, the image of $B$ under $TS$ has compact closure and $TS$ is also compact. \qed

3.2. Compact selfadjoint operators. Let $E$ be a Hilbert space. Let $T : E \rightarrow E$ be a nonzero compact selfadjoint operator.

3.2.1. Theorem. Either $\|T\|$ or $-\|T\|$ is an eigenvalue of $T$.

First we recall a simple fact.

3.2.2. Lemma. Let $u$ and $v$ be two nonzero vectors in $E$ such that $\langle (u|v) \rangle = \|u\| \cdot \|v\|$. Then $u$ and $v$ are colinear.

Proof. Let $\lambda v$ be the orthogonal projection of $u$ to $v$. Then $u = \lambda v + w$ and $w$ is perpendicular to $v$. This implies that $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2$. On the other hand, we have $\|u\| \cdot \|v\| = \langle (u|v) \rangle = |\lambda| \|v\|^2$, i.e., $|\lambda| = \frac{\|u\|}{\|v\|}$. Hence, it follows that $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2 = \|u\|^2 + \|w\|^2$,

i.e., $\|w\|^2 = 0$ and $w = 0$. \qed

Now we can prove the theorem. By rescaling $T$, we can assume that $\|T\| = 1$. Let $B$ be the unit ball in $E$. By our assumption, we know that $1 = \|T\| = \sup_{v \in B} \|Tv\|$.

Therefore, there exists a sequence of vectors $v_n \in B$ such that $\lim_{n \to \infty} \|Tv_n\| = 1$. Since $T$ is compact, by going to a subsequence, we can also assume that $\lim_{n \to \infty} Tv_n = u$. This implies that $1 = \lim_{n \to \infty} \|Tv_n\| = \|u\|$.

Moreover, we have $\lim_{n \to \infty} T^2 v_n = Tu$. Hence, we have $1 = \|T\| \cdot \|u\| \geq \|Tu\| = \lim_{n \to \infty} \|T^2 v_n\| \geq \limsup_{n \to \infty} (\|T^2 v_n\| \cdot \|v_n\|) \geq \limsup_{n \to \infty} (T^2 v_n |v_n|) = \lim_{n \to \infty} (Tv_n |Tv_n|) = \lim_{n \to \infty} \|Tv_n\|^2 = 1.$
It follows that \( \| Tu \| = 1 \).

Moreover, we have
\[
1 = \| Tu \|^2 = (Tu(Tu) = (T^2u|u) \leq \| T^2u \| \| u \| \leq \| T^2 \| \| u \|^2 \leq \| T \|^2 \| u \|^2 = 1.
\]

This finally implies that
\[
(T^2u|u) = \| T^2u \| \| u \|.
\]

By 3.2.2, it follows that \( T^2u \) is proportional to \( u \), i.e. \( T^2u = \lambda u \). Moreover, we have
\[
\lambda = \lambda(u|u) = (T^2u|u) = \| Tu \|^2 = 1.
\]

It follows that \( T^2u = u \).

Therefore, the linear subspace \( F \) of \( E \) spanned by \( u \) and \( Tu \) is \( T \)-invariant.

Either \( Tu = u \) or \( v = \frac{1}{2}(u - Tu) \neq 0 \). In the second case, we have \( Tv = -v \).

This completes the proof of the existence of eigenvalues.

We need another fact.

3.2.3. Lemma. Let \( T \) be a compact selfadjoint operator. Let \( \lambda \) be an eigenvalue different from 0. Then the eigenspace of \( \lambda \) is finite-dimensional.

Proof. Assume that the corresponding eigenspace \( V \) is infinite-dimensional. Then there would exist an orthonormal sequence \( (e_n, n \in \mathbb{N}) \) in \( F \). Clearly, then the sequence \( (Te_n, n \in \mathbb{N}) \) would consist of mutually orthogonal vectors of length \( \| \lambda \| \), hence it could not have compact closure in \( V \), contradicting the compactness of \( T \). Therefore, \( V \) cannot be infinite-dimensional.

3.3. An example. Denote by \( \mu \) the Haar measure on \( G \). Let \( L^2(G) \) be the Hilbert space of square-integrable complex valued functions on \( G \) with respect to the Haar measure \( \mu \). We denote its norm by \( \| \cdot \|_2 \). Clearly, we have
\[
\| f \|_2^2 = \int_G |f(g)|^2 \, d\mu(g) \leq \| f \|^2
\]
for any \( f \in C(G) \). Hence the inclusion \( C(G) \to L^2(G) \) is a continuous map.

3.3.1. Lemma. The continuous linear map \( i : C(G) \to L^2(G) \) is injective.

Proof. Let \( f \in C(G) \) be such that \( i(f) = 0 \). This implies that \( \| f \|_2 = 0 \). On the other hand, the function \( g \mapsto |f(g)| \) is a nonnegative continuous function on \( G \). Assume that \( M \) is the maximum of this function on \( G \). If we would have \( M > 0 \), there would exist a nonempty open set \( U \subset G \) such that \( |f(g)| \geq \frac{M}{2} \) for \( g \in U \). Therefore, we would have
\[
\| f \|_2^2 = \int_G |f(g)|^2 \, d\mu(g) \geq \frac{M^2}{4} \mu(U) > 0,
\]
by 1.3.16. Therefore, we must have \( M = 0 \).

Since the measure of \( G \) is 1, by Cauchy-Schwartz inequality, we have
\[
\int_G |\phi(g)| \, d\mu(g) = \int_G 1 \cdot |\phi(g)| \, d\mu(g) \leq \| 1 \|_2 \cdot \| \phi \|_2 = \| \phi \|_2
\]
for any \( \phi \in L^2(\mu) \). Hence, \( L^2(G) \subset L^1(G) \), where \( L^1(G) \) is the Banach space of integrable functions on \( G \).
Let $f$ be a continuous function on $G$. For any $\phi \in L^2(G)$, we put

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)d\mu(h)$$

for $g \in G$.

By 1.1.1, $f$ is uniformly continuous on $G$. This implies that for any $\epsilon > 0$ there exists a neighborhood $U$ of 1 in $G$ such that $g'g^{-1} \in U$ implies $|f(g) - f(g')| < \epsilon$. Therefore, for arbitrary $h \in G$, we see that for $(g^{-1}h)(g^{-1}h)^{-1} = g^{-1}g \in U$ and we have

$$|f(g^{-1}h) - f(g'h)| < \epsilon.$$ 

This in turn implies that

$$|(R(f)\phi)(g) - (R(f)\phi)(g')| = \left| \int_G f(h)\phi(gh)d\mu(h) - \int_G f(h)\phi(g'h)d\mu(h) \right| = \left| \int_G (f(g^{-1}h) - f(g'h))\phi(h)d\mu(h) \right| = \int_G |f(g^{-1}h) - f(g'h)| |\phi(h)|d\mu(h) < \epsilon \cdot \int_G |\phi(h)|d\mu(h) \leq \epsilon \cdot \|\phi\|_2$$

for any $g' \in Ug$ and $\phi$ in $L^2(G)$. This proves that functions $R(f)\phi$ are in $C(G)$ for any $\phi \in L^2(G)$.

Moreover, by the invariance of Haar measure, we have

$$|(R(f)\phi)(g)| \leq \int_G |f(h)||\phi(gh)|d\mu(h) \leq \|f\| \int_G |\phi(gh)|d\mu(h) \leq \|f\| \int_G |\phi(h)|d\mu(h) \leq \|f\| \cdot \|\phi\|_2,$$

it follows that

$$\|R(f)\phi\| \leq \|f\| \cdot \|\phi\|_2$$

for any $\phi \in L^2(G)$. Hence, $R(f)$ is a bounded linear operator from $L^2(G)$ into $C(G)$.

Hence the set $\mathcal{S} = \{ R(f)\phi \mid \|\phi\|_2 \leq 1 \}$ is bounded in $C(G)$.

Clearly, the composition of $R(f)$ with the natural inclusion $i : C(G) \rightarrow L^2(G)$ is a continuous linear map from $L^2(G)$ into itself which will denote by the same symbol. Therefore, the following diagram of continuous maps

$$\begin{array}{ccc}
L^2(G) & \xrightarrow{R(f)} & L^2(G) \\
\downarrow{R(f)} & & \downarrow{i} \\
C(G) & & \\
\end{array}$$

is commutative.

We already remarked that $\mathcal{S}$ is a bounded set in $C(G)$. Hence, $\mathcal{S}$ is a pointwise bounded family of continuous functions. In addition, by the above formula

$$|(R(f)\phi)(g) - (R(f)\phi)(g')| < \epsilon,$$

for all $g' \in Ug$ and $\phi$ in the unit ball in $L^2(G)$. Hence, the set $\mathcal{S}$ is equicontinuous.

Hence we proved the following result.

3.3.2. Lemma. The set $\mathcal{S} \subset C(G)$ is pointwise bounded and equicontinuous.
By 1.2.1, the closure of the set $S$ in $C(G)$ is compact. Since $i : C(G) \rightarrow L^2(G)$ is continuous, $S$ has compact closure in $L^2(G)$. Therefore, we have the following result.

3.3.3. **Lemma.** The linear operator $R(f) : L^2(G) \rightarrow L^2(G)$ is compact.

Put $f^*(g) = f(g^{-1})$, $g \in G$. Then $f^* \in C(G)$.

3.3.4. **Lemma.** For any $f \in C(G)$ we have

$$R(f)^* = R(f^*).$$

**Proof.** For $\phi, \psi \in L^2(G)$, we have, by 1.3.17,

$$\langle R(f)\phi | \psi \rangle = \int_G (R(f)\phi)(g)\overline{\psi(g)} \, d\mu(g) = \int_G \left( \int_G f(h)\phi(gh) \, d\mu(h) \right) \overline{\psi(g)} \, d\mu(g)$$

$$= \int_G f(h) \left( \int_G \phi(gh) \overline{\psi(g)} \, d\mu(g) \right) \, d\mu(h) = \int_G f(h) \left( \int_G \phi(g) \overline{\psi(g)h^{-1}} \, d\mu(g) \right) \, d\mu(h)$$

$$= \int_G \phi(g) \left( \int_G \overline{f(h)\psi(gh^{-1})} \, d\mu(h) \right) \, d\mu(g)$$

$$= \int_G \phi(g) \left( \int_G f^*(h^{-1})\psi(gh^{-1}) \, d\mu(h) \right) \, d\mu(g)$$

$$= \int_G \phi(g) \left( \int_G f^*(h)\psi(gh) \, d\mu(h) \right) \, d\mu(g) = \langle \phi | R(f^*)\psi \rangle.$$

Thus, $R(f)^*$ is compact.

3.3.5. **Corollary.** The operator $R(f^*)R(f) = R(f^*)R(f)$ is a positive compact selfadjoint operator on $L^2(G)$.

### 4. Peter-Weyl theorem

4.1. **$L^2$ version.** Let $\phi \in L^2(G)$. Let $g \in G$. We put $(R(g)\phi)(h) = \phi(hg)$ for any $h \in G$. Then we have

$$\|R(g)\phi\|^2 = \int_G |(R(g)\phi)(h)|^2 \, d\mu(h) = \int_G |\phi(hg)|^2 \, d\mu(h) = \int_G |\phi(h)|^2 \, d\mu(h) = \|\phi\|^2.$$

Therefore, $R(g)$ is a continuous linear operator on $L^2(G)$. Clearly it is in GL($L^2(G)$). Moreover, $R(g)$ is unitary.

Clearly, for any $g \in G$, the following diagram

$$
\begin{array}{ccc}
C(G) & \xrightarrow{R(g)} & C(G) \\
\downarrow & & \downarrow \\
L^2(G) & \xrightarrow{R(g)} & L^2(G)
\end{array}
$$

is commutative.

Analogously, we define $(L(g)\phi)(h) = \phi(g^{-1}h)$ for $h \in G$. Then $L(g)$ is a unitary operator on $L^2(G)$ which extends from $C(G)$.

Clearly, $R(g)$ and $L(h)$ commute for any $g, h \in G$.

4.1.1. **Lemma.** $L$ and $R$ are unitary representations of $G$ on $L^2(G)$. 

Proof. It is enough to discuss \( R \). The proof for \( L \) is analogous.

Let \( g \in G \) and \( \phi \in L^2(G) \). We have to show that \( h \mapsto R(h)\phi \) is continuous at \( g \). Let \( \varepsilon > 0 \). Since \( C(G) \) is dense in \( L^2(G) \), there exists \( \psi \in C(G) \) such that 
\[
\|\phi - \psi\|_2 < \frac{\varepsilon}{3}.
\]
Since \( R \) is a representation on \( C(G) \), there exists a neighborhood \( U \) of \( g \) such that \( h \in U \) implies \( \|R(h)\psi - R(g)\psi\| < \frac{\varepsilon}{3} \). This in turn implies that
\[
\|R(h)\psi - R(g)\psi\|_2 < \frac{\varepsilon}{3}.
\]
Therefore we have
\[
\|R(h)\phi - R(g)\phi\|_2 \leq \|R(h)(\phi - \psi)\|_2 + \|R(h)\psi - R(g)\psi\|_2 + \|R(g)(\psi - \phi)\|_2 
\leq 2\|\phi - \psi\|_2 + \|R(h)\psi - R(g)\psi\|_2 < \varepsilon
\]
for any \( h \in U \).

Let \( f \) be a continuous function on \( G \). By 3.3.3, \( R(f) \) is a compact operator on \( L^2(G) \).

Let \( \phi \in L^2(G) \). Then
\[
(R(f)L(g)\phi)(h) = \int_G f(k) (L(g)\phi)(hk) \, d\mu(k) = \int_G f(k) \phi(g^{-1}hk) \, d\mu(k) = (R(f)\phi)(g^{-1}h) = (L(g)R(f)\phi)(h)
\]
for all \( g,h \in G \). Therefore, \( R(f) \) commutes with \( L(g) \) for any \( g \in G \).

Let \( F \) be the eigenspace of \( R(f^*)R(f) \) for eigenvalue \( \lambda > 0 \). Then \( F \) is finite-dimensional by 3.2.3.

4.1.2. Lemma. (i) Let \( \phi \in F \). Then \( \phi \) is a continuous function.
(ii) The vector subspace \( F \) of \( C(G) \) is in \( R(G) \).

Proof. (i) The function \( \phi \) is in the image of \( R(f^*) \). Hence it is a continuous function.

(ii) By (i), \( F \subset C(G) \). As we remarked above, the operator \( R(f^*)R(f) \) commutes with the representation \( L \). Therefore, the eigenspace \( F \) is invariant subspace for \( L \). Let \( \phi \) be a function in \( F \). Since \( F \) is invariant for \( L \), \( \phi \) is \( G \)-finite. Hence, \( \phi \in R(G) \).

4.1.3. Lemma. The subspace \( R(G) \) is invariant for \( R(f) \).

Proof. Let \( \phi \in R(G) \). By 2.3.5 we have
\[
(R(f)\phi)(g) = \int_G f(h)\phi(gh) \, d\mu(h) = \sum_{i=1}^n a_i(g) \int_G f(h)b_i(h) \, d\mu(h)
\]
for any \( g \in G \), i.e., \( R(f)\phi \) is a linear combination of \( a_i, 1 \leq i \leq n \).

Let \( E = R(G)^\perp \) in \( L^2(G) \). Then, by 4.1.3, \( R(G) \) is invariant for selfadjoint operator \( R(f^*)R(f) \). This in turn implies that \( E \) is also invariant for this operator. Therefore the restriction of this operator to \( E \) is a positive selfadjoint compact operator. Assume that its norm is greater than 0. Then, by 3.2.1, the norm is an eigenvalue of this operator, and there exists a nonzero eigenvector \( \phi \in E \) for that eigenvalue. Clearly, \( \phi \) is an eigenvector for \( R(f^*)R(f) \) too. By 4.1.2, \( \phi \) is also in \( R(G) \). Hence, we have \( \|\phi\|_2^2 = (\phi | \phi) = 0 \), and \( \phi = 0 \) in \( L^2(G) \). Hence, we have a contradiction.

Therefore, the operator \( R(f^*)R(f) \) is 0 when restricted to \( E \). Hence
\[
0 = (R(f^*)R(f)\psi|\psi) = \|R(f)\psi\|_2^2
\]
for any $\psi \in E$. It follows that $R(f)\psi = 0$. Since $R(f)\psi$ is a continuous function, we have

$$0 = (R(f)\psi)(1) = \int_{G} f(g)\psi(g) d\mu(g),$$

i.e., $\psi$ is orthogonal to $f$.

Since $f \in \mathcal{C}(G)$ was arbitrary and $\mathcal{C}(G)$ is dense in $L^2(G)$, it follows that $\psi = 0$. This implies that $E = \{0\}$.

This completes the proof of the following result.

4.1.4. Theorem (Peter-Weyl). The algebra $R(G)$ is dense in $L^2(G)$.

4.2. Continuous version. Let $g \in G$. Assume that $g \neq 1$. Then there exists an open neighborhood $U$ of $1$ such that $U$ and $Ug$ are disjoint. There exists positive function $\phi$ in $\mathcal{C}(G)$ such that $\phi|_U = 0$ and $\phi|_{Ug} = 1$. This implies that

$$\|R(g)\phi - \phi\|^2 = \int_{G} |\phi(hg) - \phi(h)|^2 d\mu(h)$$

$$= \int_{U} |\phi(hg) - \phi(h)|^2 d\mu(h) + \int_{G-U} |\phi(hg) - \phi(h)|^2 d\mu(h) \geq \mu(U).$$

Therefore $R(g) \neq I$. Since by 4.1.4, $R(G)$ is dense in $L^2(G)$, $R(g)|_{R(G)}$ is not the identity operator.

This implies the following result.

4.2.1. Lemma. Let $g, g' \in G$ and $g \neq g'$. Then there exists a function $\phi \in R(G)$ such that $\phi(g) \neq \phi(g')$.

Proof. Let $h = g^{-1}g' \neq 1$. Then there exists $\psi \in R(G)$ such that $R(h)\psi \neq \psi$. Hence, we have $R(g)\psi \neq R(g')\psi$. It follows that $\psi(hg) \neq \psi(hg')$ for some $h \in G$. Therefore, the function $\phi = L(h^{-1})\psi$ has the required property. \hfill \square

In other words, $R(G)$ separates points in $G$. By Stone-Weierstrass theorem, we have the following result which is a continuous version of Peter-Weyl theorem.

4.2.2. Theorem (Peter-Weyl). The algebra $R(G)$ is dense in $C(G)$.

Another consequence of 4.2.1 is the following result.

4.2.3. Lemma. Let $U$ be an open neighborhood of $1$ in $G$. Then there exists a finite-dimensional representation $(\pi, V)$ of $G$ such that $\ker \pi \subset U$.

Proof. The complement $G - U$ of $U$ is a compact set. Since $R(G)$ separates the points of $G$, for any $g \in G - U$ there exists a function $\phi_g \in \mathcal{C}(G)$ and an open neighborhood $U_g$ of $g$ such that $|\phi_g(h) - \phi_g(1)| > \epsilon$ for $h \in U_g$. Since $G - U$ is compact, there exists a finite set $g_1, g_2, \ldots, g_m$ in $G - U$ such that $U_{g_1}, U_{g_2}, \ldots, U_{g_m}$ form an open cover of $G - U$ and $|\phi_{g_i}(h) - \phi_{g_i}(1)| > \epsilon$ for $h \in U_{g_i}$. Let $\pi_i$ be a finite-dimensional representation of $G$ with matrix coefficient $\phi_{g_i}$. Then $\pi_i(h) \neq I$ for $h \in U_{g_i}$, $1 \leq i \leq n$. Let $\pi$ be the direct sum of $\pi_i$. Then $\pi(h) \neq I$ for $h \in G - U$, i.e., $\ker \pi \subset U$. \hfill \square

4.3. Matrix groups. Let $G$ be a topological group. We say that $G$ has no small subgroups if there exists a neighborhood $U$ of $1 \in G$ such that any subgroup of $G$ contained in $U$ is trivial.
4.3.1. Lemma. Let $V$ be a finite-dimensional complex vector space. Then the group $\text{GL}(V)$ has no small subgroups.

Proof. Let $\mathcal{L}(V)$ be the space of all linear endomorphisms of $V$. Then $\exp : \mathcal{L}(V) \to \text{GL}(V)$ given by

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

defines a holomorphic map. Its differential at 0 is the identity map $I$ on $\mathcal{L}(V)$. Hence, by the inverse function theorem, it is a local diffeomorphism.

Let $U$ be an open neighborhood of 1 in $\text{GL}(V)$ and $V$ the open ball around 0 in $\mathcal{L}(V)$ of radius $\epsilon$ (with respect to the linear operator norm) such that $\exp : V \to U$ is a diffeomorphism. Let $V'$ be the open ball of radius $\frac{\epsilon}{2}$ around 0 in $\mathcal{L}(V)$. Then $U' = \exp(V')$ is an open neighborhood of 1 in $\text{GL}(V)$. Let $H$ be a subgroup of $\text{GL}(V)$ contained in $U'$. Let $S \in H$. Then $S = \exp(T)$ for some $T \in V'$. Hence, we have $S^2 = \exp(T)^2 = \exp(2T) \in H$. Moreover, $S^2 \in H$ and $S^2 = \exp(T')$ for some $T' \in V'$. It follows that $\exp(T') = \exp(2T')$ for $2T, T' \in V$. Since $\exp$ is injective on $V$, we must have $2T = T'$. Hence, $T \in \frac{1}{2} V'$. It follows that $H \subset \exp \left( \frac{1}{2} V' \right)$. By induction we get that $H \subset \exp \left( \frac{1}{2^n} V' \right)$ for any $n \in \mathbb{N}$. This implies that $H = \{1\}$. □

A compact subgroup of $\text{GL}(V)$ we call a compact matrix group.

4.3.2. Theorem. Let $G$ be a compact group. Then the following conditions are equivalent:

(i) $G$ has no small subgroups;

(ii) $G$ is isomorphic to a compact matrix group.

Proof. (i)$\Rightarrow$(ii) Let $U$ be an open neighborhood of 1 in $G$ such that it contains no nontrivial subgroups of $G$. By 4.2.3, there exists a finite-dimensional representation $(\pi, V)$ of $G$ such that $\ker \pi \subset U$. This clearly implies that $\ker \pi = \{1\}$, and $\pi : G \to \text{GL}(V)$ is an injective homomorphism. Since $G$ is compact, $\pi$ is homeomorphism of $G$ onto $\pi(G)$. Therefore, $G$ is isomorphic to the compact subgroup $\pi(G)$ of $\text{GL}(V)$.

(ii)$\Rightarrow$(i) Assume that $G$ is a compact subgroup of $\text{GL}(V)$. By 4.3.1, there exists an open neighborhood $U$ of 1 in $\text{GL}(V)$ such that it contains no nontrivial subgroups. This implies that $G \cap U$ contains nontrivial subgroups of $G$. □

4.3.3. Remark. For a compact matrix group $G$, since matrix coefficients of the natural representation separate points in $G$, 4.2.1 obviously holds. Therefore, in this situation, Stone-Weierstrass theorem immediately implies the second version of Peter-Weyl theorem, which in turn implies the first one.

4.3.4. Remark. By Cartan’s theorem [1], any compact matrix group is a Lie group. On the other hand, by [1] any Lie group has no small subgroups. Hence, compact Lie groups have no small subgroups and therefore they are compact matrix groups.

4.3.5. Remark. Let $T = \mathbb{R}/\mathbb{Z}$. Then $T$ is a compact abelian group. Let $G$ be the product of infinite number of copies of $T$. Then $G$ is a compact abelian group. By the definition of product topology, any neighborhood of 1 contains a nontrivial subgroup.
Let $G$ be an arbitrary compact group. Let $(\pi, V)$ be a finite-dimensional representation. Put $N = \ker \pi$. Then $N$ is a compact normal subgroup of $G$ and $G/N$ equipped with the quotient topology is a compact group. Clearly, $G/N$ is a compact matrix group.

Let $\mathcal{S}$ be the family of all compact normal subgroups $N$ of $G$ such that $G/N$ is a compact matrix group. Clearly, $N, N'$ in $\mathcal{S}$ implies $N \cap N' \in \mathcal{S}$. Therefore, $\mathcal{S}$ ordered by inclusion is a directed set. One can show that $G$ is a projective limit of the system $G/N, N \in \mathcal{S}$. Therefore, any compact group is a projective limit of compact matrix groups. By the above remark, this implies that any compact group is a projective limit of compact Lie groups.