Lectures on Algebraic Theory of D-Modules

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Contents

CHAPTER 1

Modules over rings of differential operators with polynomial coefficients

1. Hilbert polynomials

Let $A = \bigoplus_{n\in\mathbb{Z}}^{\infty} A^n$ be a graded nötherian commutative ring with identity 1 contained in A^0. Then A^0 is a commutative ring with identity 1. Assume that $A^n = 0$ for $n < 0$.

1.1. Lemma. (i) *A^0 is a nötherian ring.*
(ii) *A is a finitely generated A^0-algebra.*

Proof. (i) Put $A_+ = \bigoplus_{n=1}^{\infty} A^n$. Then A_+ is an ideal in A and $A^0 = A/A_+$.

(ii) A_+ is finitely generated. Let $x_1, x_2, \dots, x_s$ be a set of homogeneous generators of A_+ and denote $d_i = \deg x_i$, $1 \le i \le s$. Let B be the A_0-subalgebra generated by $x_1, x_2, \dots, x_s$. We claim that $A^n \subseteq B$, $n \in \mathbb{Z}_+$. Clearly, $A^0 \subseteq B$. Assume that $n > 0$ and $y \in A^n$. Then $y \in A_+$ and therefore $y = \sum_{i=1}^{s} y_i x_i$ where $y_i \in A^{n-d_i}$. It follows that the induction assumption applies to y_i, $1 \le i \le s$. This implies that $y \in B$. □

The converse of 1.1 follows from Hilbert's theorem which states that the polynomial ring $A^0[X_1, X_2, \dots, X_n]$ is nötherian if the ring A^0 is nötherian.

Let $M = \bigoplus_{n\in\mathbb{Z}} M^n$ be a finitely generated graded A-module. Then each M^n, $n \in \mathbb{Z}$, is an A^0-module. Also, $M^n = 0$ for sufficiently negative $n \in \mathbb{Z}$.

1.2. Lemma. *The A^0-modules M^n, $n \in \mathbb{Z}$, are finitely generated.*

Proof. Let m_i, $1 \le i \le k$, be homogeneous generators of M and deg $m_i = r_i$, $1 \le i \le k$. For $j \in \mathbb{Z}_+$ denote by $z_i(j)$, $1 \le i \le \ell(j)$, all homogeneous monomials in $x_1, x_2, \dots, x_s$ of degree j. Let $m \in M^n$. Then $m = \sum_{i=1}^{k} y_i m_i$ where $y_i \in A^{n-r_i}$, $i \le i \le k$. By 1.1, $y_i = \sum_j a_{ij} z_j(n - r_i)$, with $a_{ij} \in A^0$. This implies that $m = \sum_{i,j} a_{ij} z_j(n-r_i)\, m_i$; hence M^n is generated by $(z_j(n-r_i)\, m_i;\ 1 \le j \le \ell(n-r_i)$, $1 \le i \le k)$. □

Let $\mathcal{M}_{fg}(A^0)$ be the category of finitely generated A^0-modules. Let λ be a function on $\mathcal{M}_{fg}(A^0)$ with values in $\mathbb{Z}$. The function λ is called *additive* if for any short exact sequence:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

Clearly, additivity implies that $\lambda(0) = 0$.

1.3. Lemma. *Let*

$$0 \to M_0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$$

be an exact sequence in $\mathcal{M}_{fg}(A^0)$. *Then*

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0.$$

Proof. Evident. □

Let $\mathbb{Z}[[t]]$ be the ring of formal power series in t with coefficients in $\mathbb{Z}$. Denote by $\mathbb{Z}((t))$ the localization of $\mathbb{Z}[[t]]$ with respect to the multiplicative system $\{t^n \mid n \in \mathbb{Z}_+\}$.

Let M be a finitely generated graded A-module. Then the *Poincaré series* $P(M,t)$ of M (with respect to λ) is

$$P(M,t) = \sum_{n \in \mathbb{Z}} \lambda(M^n)\, t^n \in \mathbb{Z}((t)).$$

For example, let $A = k[X_1, X_2, \dots, X_s]$ be the algebra of polynomials in s variables with coefficients in a field k graded by the total degree. Then, $A^0 = k$ and for every finitely generated graded A-module M, we have $\dim_k M_n < \infty$. Hence, we can define the Poincaré series for $\lambda = \dim_k$. In particular, for the A-module A itself, we have

$$P(A,t) = \sum_{n \in \mathbb{Z}} \dim_k A^n\, t^n = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} t^n = \frac{1}{(1-t)^s}.$$

The next result shows that Poincaré series in general have an analogous form.

1.4. Theorem (Hilbert, Serre). *For any finitely generated graded A-module M we have*

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1 - t^{d_i})}$$

where $f(t) \in \mathbb{Z}[t, t^{-1}]$.

Proof. We prove the theorem by induction in s. If $s = 0$, $A = A_0$ and M is a finitely generated A^0-module. This implies that $M^n = 0$ for sufficiently large n. Therefore, $\lambda(M^n) = 0$ except for finitely many $n \in \mathbb{Z}$ and $P(M,t)$ is in $\mathbb{Z}[t, t^{-1}]$.

Assume now that $s > 0$. The multiplication by x_s defines an A-module endomorphism f of M. Let $K = \ker f$, $I = \operatorname{im} f$ and $L = M/I$. Then K, I and L are graded A-modules and we have an exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} M \longrightarrow L \longrightarrow 0.$$

This implies that

$$0 \longrightarrow K^n \longrightarrow M^n \xrightarrow{x_s} M^{n+d_s} \longrightarrow L^{n+d_s} \longrightarrow 0$$

is an exact sequence of A^0-modules for all $n \in \mathbb{Z}$. In particular, by 1.3,

$$\lambda(K^n) - \lambda(M^n) + \lambda(M^{n+d_s}) - \lambda(L^{n+d_s}) = 0,$$

for all $n \in \mathbb{Z}$. This implies that

$$\begin{aligned}(1-t^{d_s})\,P(M,t) &= \sum_{n\in\mathbb{Z}} \lambda(M^n)\,t^n - \sum_{n\in\mathbb{Z}} \lambda(M^n)\,t^{n+d_s} \\ &= \sum_{n\in\mathbb{Z}} (\lambda(M^{n+d_s}) - \lambda(M^n))\,t^{n+d_s} \\ &= \sum_{n\in\mathbb{Z}} (\lambda(L^{n+d_s}) - \lambda(K^n))\,t^{n+d_s} \\ &= P(L,t) - P(K,t)\,t^{d_s},\end{aligned}$$

i.e.,

$$(1-t^{d_s})\,P(M,t) = P(L,t) - t^{d_s} P(K,t).$$

From the construction it follows that x_s act as multiplication by 0 on L and K, i.e., we can view them as $A/(x_s)$-modules. Hence, the induction assumption applies to them. This immediately implies the assertion. □

Since the Poincaré series $P(M,t)$ a rational function, we can talk about the order of its pole at a point. Let $d_\lambda(M)$ be the order of the pole of $P(M,t)$ at 1.

By the theorem, $f(t) = \sum_{k\in\mathbb{Z}} a_k t^k$ with $a_k \in \mathbb{Z}$ and $a_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Let p be the order of zero of f at 1. Assume that $p > 0$. Then $f(t) = (1-t)g(t)$ where $g(t) = \sum_{k\in\mathbb{Z}} b_k t^k$, with $b_k \in \mathbb{Q}$ and $b_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Moreover, we have $a_k = b_k - b_{k-1}$ for all $k \in \mathbb{Z}$. By induction in k this implies that $b_k \in \mathbb{Z}$. By repeating this procedure if necessary, we see that $f(t) = (1-t)^p g(t)$ where $g(t) = \sum_{k\in\mathbb{Z}} b_k t^k$, with $b_k \in \mathbb{Z}$ and $b_k = 0$ for all $k \in \mathbb{Z}$ except finitely many. Moreover, $g(1) \neq 0$.

1.5. Corollary. *If $d_i = 1$ for $1 \leq i \leq s$, the function $n \longmapsto \lambda(M^n)$ is equal to a polynomial with rational coefficients of degree $d_\lambda(M) - 1$ for sufficiently large $n \in \mathbb{Z}$.*

Proof. Let p be the order of zero of f at 1. Then we can write $f(t) = (1-t)^p g(t)$ with $g(1) \neq 0$. In addition, we put $d = d_\lambda(M) = s - p$, hence

$$P(M,t) = \frac{g(t)}{(1-t)^d}.$$

Now,

$$(1-t)^{-d} = \sum_{k=0}^{\infty} \frac{d(d+1)\dots(d+k-1)}{k!}\,t^k = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k,$$

and if we put $g(t) = \sum_{k=-N}^{N} a_k t^k$ we get

$$\lambda(M^n) = \sum_{k=-N}^{N} a_k \binom{d+n-k-1}{d-1}$$

for all $n \geq N$. This is equal to

$$\sum_{k=-N}^{N} a_k\,\frac{(d+n-k-1)!}{(d-1)!(n-k)!} = \sum_{k=-N}^{N} a_k\,\frac{(n-k+1)(n-k+2)\dots(n-k+d-1)}{(d-1)!},$$

hence $\lambda(M^n)$ is a polynomial in n with the leading term

$$\left(\sum_{k=-N}^{N} a_k\right)\frac{n^{d-1}}{(d-1)!} = g(1)\frac{n^{d-1}}{(d-1)!} \neq 0.$$

□

We call the polynomial which gives $\lambda(M^n)$ for large $n \in \mathbb{Z}$ the *Hilbert polynomial* of M (with respect to λ). From the proof we see that the leading coefficient of the Hilbert polynomial of M is equal to $\frac{g(1)}{(d-1)!}$.

Returning to our example of $A = k[X_1, X_2, \ldots, X_s]$, we see that

$$\dim_k A^n = \binom{s+n-1}{s-1} = \frac{n^{s-1}}{(s-1)!} + \ldots.$$

Hence, the degree of the Hilbert polynomial for $A = k[X_1, X_2, \ldots, X_s]$ is equal to $s-1$.

Now we are going to prove a characterization of polynomials (with coefficients in a field of characteristic 0) having integral values for large positive integers. First, we remark that, for any $s \in \mathbb{Z}_+$ and $q \geq s$, we have

$$q^s = s!\binom{q}{s} + Q(q)$$

where Q is a polynomial of degree $s-1$. Therefore any polynomial P of degree d, for large q, can be uniquely written as

$$P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \ldots + c_{d-1}\binom{q}{1} + c_d,$$

with suitable coefficients c_i, $0 \leq i \leq d$. Since binomial coefficients are integers, if c_i, $0 \leq i \leq d$, are integers, the polynomial P has integral values for integers $n \geq d$. The next result is a converse of this observation.

1.6. Lemma. *If the polynomial*

$$q \longmapsto P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \ldots + c_{d-1}\binom{q}{1} + c_d$$

takes integral values $P(n)$ for large $n \in \mathbb{Z}$, all its coefficients c_i, $0 \leq i \leq d$, are integers.

Proof. We prove the statement by induction in d. If $d = 0$ the assertion is obvious. Also

$$\begin{aligned} P(q+1) - P(q) &= \sum_{i=0}^{d} c_i\binom{q+1}{d-i} - \sum_{i=0}^{d} c_i\binom{q}{d-i} \\ &= \sum_{i=0}^{d} c_i\left(\binom{q+1}{d-i} - \binom{q}{d-i}\right) = \sum_{i=0}^{d-1} c_i\binom{q}{d-i-1}, \end{aligned}$$

using the identity

$$\binom{q+1}{s} = \binom{q}{s} + \binom{q}{s-1}$$

for $q \geq s \geq 1$. Therefore, $q \longmapsto P(q+1) - P(q)$ is a polynomial with coefficients $c_0, c_1, \ldots, c_{d-1}$, and $P(n) \in \mathbb{Z}$ for large $n \in \mathbb{Z}$. By the induction assumption all c_i, $0 \leq i \leq d-1$, are integers. This immediately implies that c_d is an integer too. □

We shall need another related remark. If F is a polynomial of degree d with the leading coefficient a_0,

$$\begin{aligned} G(n) &= F(n) - F(n-1) \\ &= (a_0 n^d + a_1 n^{d-1} + \dots) - (a_0(n-1)^d + a_1(n-1)^{d-1} + \dots) = a_0 d n^{d-1} + \dots \end{aligned}$$

is polynomial in n of degree $d-1$ with the leading coefficient da_0. The next result is a converse of this fact.

1.7. Lemma. *Let F be a function on $\mathbb{Z}$ such that*

$$G(n) = F(n) - F(n-1),$$

is equal to a polynomial in n of degree $d-1$ for large $n \in \mathbb{Z}$. Then F is equal to a polynomial in n of degree d for large $n \in \mathbb{Z}$.

Proof. Assume that $G(n) = P(n-1)$ for $n \geq N \geq d$, where P is a polynomial in n of degree $d-1$. Then by 1.6 we have

$$P(n) = \sum_{i=0}^{d-1} c_i \binom{n}{d-i-1}$$

Hence, for $n \geq N+1$,

$$F(n) = \sum_{k=N+1}^{n} (F(k) - F(k-1)) + F(N) = \sum_{k=N+1}^{n} G(k) + F(N) = \sum_{k=d}^{n} P(k-1) + C$$

where C is a constant. Also, by the identity used in the previous proof,

$$\binom{q}{s} = \sum_{j=s+1}^{q} \left(\binom{j}{s} - \binom{j-1}{s} \right) + 1 = \sum_{j=s+1}^{q} \binom{j-1}{s-1} + 1 = \sum_{j=s}^{q} \binom{j-1}{s-1}$$

for $q > s \geq 1$. This implies that

$$\begin{aligned} \sum_{k=d}^{n} P(k-1) &= \sum_{k=d}^{n} \sum_{i=0}^{d-1} c_i \binom{k-1}{d-i-1} = \sum_{i=0}^{d-1} c_i \left(\sum_{k=d}^{n} \binom{k-1}{d-i-1} \right) \\ &= \sum_{i=0}^{d-1} c_i \left(\sum_{k=d-i}^{n} \binom{k-1}{d-i-1} \right) - \sum_{i=1}^{d-1} c_i \left(\sum_{k=d-i}^{d-1} \binom{k-1}{d-i-1} \right) \\ &= \sum_{i=0}^{d-1} c_i \binom{n}{d-i} + C' \end{aligned}$$

for some constant C'. □

In particular, it follows that the sum $\sum_{n \leq N} \lambda(M^n)$ is equal to a polynomial of degree $d_\lambda(M)$ for large $N \in \mathbb{Z}$. In addition, if we put

$$\sum_{n \leq N} \lambda(M^n) = a_0 N^d + a_1 N^{d-1} + \dots + a_{d-1} N + a_d$$

for large $N \in \mathbb{Z}$, then $d!\, a_0$ is an integer.

For example, if $A = k[X_1, X_2, \dots, X_s]$, the dimension of the space of all polynomials of degree $\leq N$ is equal to

$$\sum_{n=0}^{N} \dim_k(A^n) = \sum_{n=0}^{N} \binom{s+n-1}{s-1} = \binom{s+N}{s} = \frac{N^s}{s!} + \dots .$$

2. Dimension of modules over local rings

2.1. Lemma (Nakayama). *Let A be a local ring with the maximal ideal $\mathbf{m}$. Let V be a finitely generated A-module such that $\mathbf{m}V = V$. Then $V = 0$.*

Proof. Assume that $V \neq 0$. Then we can find a minimal system of generators $v_1, , \dots, v_s$ of V as an A-module. By the assumption, $v_s = \sum_{i=1}^{s} m_i v_i$ for some $m_i \in \mathbf{m}$. Therefore, $(1 - m_s)v_s = \sum_{i=1}^{s-1} m_i v_i$. Since $1 - m_s$ is invertible, this implies that $v_1, \dots, v_{s-1}$ generate V, contrary to our assumption. □

In the following we assume that A is a nötherian local ring, $\mathbf{m}$ its maximal ideal and $k = A/\mathbf{m}$ the residue field of A.

2.2. Lemma. $\dim_k(\mathbf{m}/\mathbf{m}^2) < +\infty$.

Proof. By the nötherian assumption $\mathbf{m}$ is finitely generated. If $a_1, \dots, a_p$ are generators of $\mathbf{m}$, their images $\bar{a}_1, \dots, \bar{a}_p$ in $\mathbf{m}/\mathbf{m}^2$ span it as a vector space over k. □

Let $s = \dim_k(\mathbf{m}/\mathbf{m}^2)$. Then we can find $a_1, \dots, a_s \in \mathbf{m}$ such that $\bar{a}_1, \dots, \bar{a}_s$ form a basis of $\mathbf{m}/\mathbf{m}^2$. We claim that they generate $\mathbf{m}$. Let I be the ideal generated by $a_1, \dots, a_s$. Then $I + \mathbf{m}^2 = \mathbf{m}$ and $\mathbf{m}(\mathbf{m}/I) = \mathbf{m}/I$. Hence, by 2.1, we have $\mathbf{m}/I = 0$. Therefore, we proved:

2.3. Lemma. *The positive integer $\dim_k(\mathbf{m}/\mathbf{m}^2)$ is equal to the minimal number of generators of $\mathbf{m}$.*

Any s-tuple $(a_1, \dots, a_s)$ of elements from $\mathbf{m}$ such that $(\bar{a}_1, \dots, \bar{a}_s)$ form a basis of $\mathbf{m}/\mathbf{m}^2$ is called a *coordinate system* in A.

Clearly, $\mathbf{m}^p$, $p \in \mathbb{Z}_+$, is a decreasing filtration of A. Therefore, we can form $\operatorname{Gr} A = \bigoplus_{p=0}^{\infty} \mathbf{m}^p/\mathbf{m}^{p+1}$. We claim that $\operatorname{Gr} A$ is a finitely generated algebra over k and therefore a nötherian graded ring. Actually, the map $X_i \longmapsto \bar{a}_i \in \mathbf{m}/\mathbf{m}^2 \subset \operatorname{Gr} A$ extends to a surjective morphism of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$.

Let M be a finitely generated A-module. Then we can define a decreasing filtration of M by $\mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and consider the graded $\operatorname{Gr} A$-module $\operatorname{Gr} M = \bigoplus_{p=0}^{\infty} \mathbf{m}^p M/\mathbf{m}^{p+1} M$.

2.4. Lemma. *If M is a finitely generated A-module, $\operatorname{Gr} M$ is a finitely generated $\operatorname{Gr} A$-module.*

Proof. From the definition of the graded module $\operatorname{Gr} M$ we see that $\mathbf{m} \cdot \operatorname{Gr}^p M = \operatorname{Gr}^{p+1} M$ for all $p \in \mathbb{Z}_+$. Hence $\operatorname{Gr}_0 M = M/\mathbf{m}M$ generates $\operatorname{Gr} M$. On the other hand, $M/\mathbf{m}M$ is a finite dimensional linear space over k. □

This implies, by 1.2, that $\dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M) < +\infty$, in particular, the A-modules $\mathbf{m}^p M/\mathbf{m}^{p+1} M$ are of finite length. Since length is clearly an additive function, by 1.5 we see that $p \longmapsto \operatorname{length}_A(\mathbf{m}^p M/\mathbf{m}^{p+1} M) = \dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M)$

is equal to a polynomial in p with rational coefficients for large $p \in \mathbb{Z}_+$. Moreover, the function

$$p \longmapsto \operatorname{length}_A(M/\mathbf{m}^p M) = \sum_{q=0}^{p-1} \operatorname{length}_A(\mathbf{m}^q M/\mathbf{m}^{q+1} M)$$

is equal to a polynomial with rational coefficients for large $p \in \mathbb{Z}_+$, and its leading coefficient is of the form $e\frac{p^d}{d!}$, where $e, d \in \mathbb{Z}_+$. We put $d(M) = d$ and $e(M) = e$, and call these numbers the *dimension* and *multiplicity* of M.

Now we want to discuss some properties of the function $M \longmapsto d(M)$. The critical result in controlling the filtrations of A-modules is the Artin-Rees lemma.

2.5. Theorem (Artin, Rees). *Let M be a finitely generated A-module and N its submodule. Then there exists $m_0 \in \mathbb{Z}_+$ such that*

$$\mathbf{m}^{p+m_0} M \cap N = \mathbf{m}^p(\mathbf{m}^{m_0} M \cap N)$$

for all $p \in \mathbb{Z}_+$.

Proof. Put $A^* = \bigoplus_{n=0}^\infty \mathbf{m}^n$. Then A^* has a natural structure of a graded ring. Let $(a_1, \dots, a_s)$ be a coordinate system in A. Then we have a natural surjective morphism $A[a_1, \dots, a_s] \longrightarrow A^*$, and A^* is a graded nötherian ring. Let $M^* = \bigoplus_{n=0}^\infty \mathbf{m}^n M$. Then M^* is a graded A^*-module. It is clearly generated by $M_0^* = M$ as an A^*-module. Since M is a finitely generated A-module, we conclude that M^* is a finitely generated A^*-module.

In addition, put $N^* = \bigoplus_{n=0}^\infty (N \cap \mathbf{m}^n M) \subset M^*$. Then

$$\mathbf{m}^p(N \cap \mathbf{m}^n M) \subset \mathbf{m}^p N \cap \mathbf{m}^{n+p} M \subset N \cap \mathbf{m}^{n+p} M$$

implies that N^* is an A^*-submodule of M^*. Since A^* is a nötherian ring, N^* is finitely generated. There exists $m_0 \in \mathbb{Z}_+$ such that $\bigoplus_{n=0}^{m_0} (N \cap \mathbf{m}^n M)$ generates N^*. Then for any $p \in \mathbb{Z}_+$,

$$N \cap \mathbf{m}^{p+m_0} M = \sum_{s=0}^{m_0} \mathbf{m}^{p+m_0-s}(N \cap \mathbf{m}^s M) \subset \mathbf{m}^p(N \cap \mathbf{m}^{m_0} M) \subset N \cap \mathbf{m}^{p+m_0} M.$$

□

This result has the following consequence — the Krull intersection theorem.

2.6. Theorem (Krull). *Let M be a finitely generated A-module. Then*

$$\bigcap_{p=0}^\infty \mathbf{m}^p M = \{0\}.$$

Proof. Put $E = \bigcap_{p=0}^\infty \mathbf{m}^p M$. Then, by 2.5,

$$E = \mathbf{m}^{p+m_0} M \cap E = \mathbf{m}^p(\mathbf{m}^{m_0} M \cap E) = \mathbf{m}^p E,$$

in particular, $\mathbf{m}E = E$, and $E = 0$ by Nakayama lemma. □

2.7. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated A-modules. Then

(i) *$d(M) = \max(d(M'), d(M''))$;*
(ii) *if $d(M) = d(M') = d(M'')$, we have $e(M) = e(M') + e(M'')$.*

Proof. We can view M' as a submodule of M. If we equip M with the filtration $\mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and M' and M'' with the induced filtrations $M' \cap \mathbf{m}^p M$, $p \in \mathbb{Z}_+$, and $\mathbf{m}^p M''$, $p \in \mathbb{Z}_+$, we get the exact sequence

$$0 \longrightarrow \operatorname{Gr} M' \longrightarrow \operatorname{Gr} M \longrightarrow \operatorname{Gr} M'' \longrightarrow 0.$$

This implies that for any $p \in \mathbb{Z}_+$

$$\begin{aligned}\operatorname{length}_A(\mathbf{m}^p M/\mathbf{m}^{p+1}M) \\ = \operatorname{length}_A((M' \cap \mathbf{m}^p M)/(M' \cap \mathbf{m}^{p+1}M)) + \operatorname{length}_A(\mathbf{m}^p M''/\mathbf{m}^{p+1}M'')\end{aligned}$$

and, by summation,

$$\operatorname{length}_A(M/\mathbf{m}^p M) = \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M)) + \operatorname{length}_A(M''/\mathbf{m}^p M'').$$

Therefore the function $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ is equal to a polynomial in p for large $p \in \mathbb{Z}_+$. On the other hand, by 2.5,

$$\mathbf{m}^{p+m_0} M' \subset \mathbf{m}^{p+m_0} M \cap M' \subset \mathbf{m}^p M';$$

hence, for large $p \in \mathbb{Z}_+$, the functions $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ and $p \longmapsto \operatorname{length}_A(M'/\mathbf{m}^p M')$ are given by polynomials in p with equal leading terms. □

2.8. Corollary. *Let A be a nötherian local ring with $s = \dim_k(\mathbf{m}/\mathbf{m}^2)$. Then, for any finitely generated A-module M we have $d(M) \leq s$.*

Proof. By 2.7 it is enough to show that $d(A) \leq s$. This follows immediately from the existence of a surjective homomorphism of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$, and the fact that the dimension of the space of polynomials of degree $\leq n$ in s variables is a polynomial in n of degree s. □

A nötherian local ring is called *regular* if $d(A) = \dim_k(\mathbf{m}/\mathbf{m}^2)$.

2.9. Theorem. *Let A be a nötherian local ring and $(a_1, a_2, \dots, a_s)$ a coordinate system in A. Then the following conditions are equivalent:*

(i) *A is a regular local ring;*
(ii) *the canonical morphism of $k[X_1, X_2, \dots, X_s]$ into $\operatorname{Gr} A$ defined by $X_i \longmapsto \bar{a}_i$, $1 \leq i \leq s$, is an isomorphism.*

Proof. By definition, the canonical morphism of $k[X_1, \dots, X_s]$ into $\operatorname{Gr} A$ is surjective. Let I be the graded ideal which is the kernel of the natural surjection of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$. If $I \neq 0$, it contains a homogeneous polynomial P of degree $d > 0$. Let J be the ideal in $k[X_1, X_2, \dots, X_s]$ generated by P. Then its Poincaré series is $P(J, t) = \frac{t^d}{(1-t)^s}$. Clearly,

$$\begin{aligned}P(k[X_1, X_2, \dots, X_s]/J, t) &= P(k[X_1, X_2, \dots, X_s], t) - P(J, t) \\ &= \frac{1 - t^d}{(1-t)^s} = \frac{1 + t + \dots + t^{d-1}}{(1-t)^{s-1}}.\end{aligned}$$

The order of the pole of the Poincaré series $P(k[X_1, X_2, \dots, X_s]/J, t)$ at 1 is $s - 1$, and by 1.5 the function $\dim_k(k[X_1, X_2, \dots, X_s]/J)_n$ is given by a polynomial in n of degree $s - 2$ for large $n \in \mathbb{Z}_+$. It follows that the function $\dim_k(k[X_1, \dots, X_s]/I)_n = \dim_k \operatorname{Gr}^n A$ is given by a polynomial in n of degree $\leq s - 2$ for large $n \in \mathbb{Z}_+$. This implies that $d(A) \leq s - 1$. Therefore, $I = 0$ if and only if $d(A) = s$. □

2.10. Theorem. *Let A be a regular local ring. Then A is integral.*

PROOF. Let $a, b \in A$ and $a \neq 0$, $b \neq 0$. Then, by 2.6, we can find $p, q \in \mathbb{Z}_+$ such that $a \in \mathbf{m}^p$, $a \notin \mathbf{m}^{p+1}$, and $b \in \mathbf{m}^q$, $b \notin \mathbf{m}^{q+1}$. Then their images $\bar{a} \in \operatorname{Gr}^p A$ and $\bar{b} \in \operatorname{Gr}^q A$ are different form zero, and since $\operatorname{Gr} A$ is integral by 2.9, we see that $\bar{a}\bar{b} \neq 0$. Therefore, $ab \neq 0$. □

Finally we want to discuss an example which will play an important role later. Let k be a field, $A = k[X_1, X_2, \dots, X_n]$ be the ring of polynomials in n-variables with coefficients in k and $\hat{A} = k[[X_1, X_2, \dots, X_n]]$ the ring of formal power series in n-variables with coefficients in k. It is easy to check that $\hat{A}$ is a local ring with maximal ideal $\hat{\mathbf{m}}$ generated by $X_1, X_2, \dots X_n$. Also, the canonical morphism from $k[X_1, X_2, \dots, X_n]$ into $\operatorname{Gr} \hat{A}$ is clearly an isomorphism.

For any $x \in k^n$ we denote by $\mathbf{m}_x$ the maximal ideal in A generated by $X_i - x_i$, $1 \leq i \leq n$. Then its complement in A is a multiplicative system in A, and we denote by A_x the corresponding localization of A. It is isomorphic to the ring of all rational functions on k^n regular at x. This is clearly a nötherian local ring. The localization of $\mathbf{m}_x$ is the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ of all rational functions vanishing at x. The automorphism of A defined by $X_i \longmapsto X_i - x_i$, $1 \leq i \leq n$, gives an isomorphism of A_0 with A_x for any $x \in k^n$. On the other hand, the natural homomorphism of A into $\hat{A}$ extends to an injective homomorphism of A_0 into $\hat{A}$. This homomorphism preserves the filtrations on these local rings and induces a canonical isomorphism of $\operatorname{Gr} A_0$ onto $\operatorname{Gr} \hat{A}$. Therefore we have the following result.

2.11. PROPOSITION. *The rings A_x, $x \in k^n$, are n-dimensional regular local rings.*

3. Dimension of modules over filtered rings

Let D be a ring with identity and $(D_n\,;\, n \in \mathbb{Z})$ an increasing filtration of D by additive subgroups such that

(i) $D_n = \{0\}$ for $n < 0$;
(ii) $\bigcup_{n \in \mathbb{Z}} D_n = D$;
(iii) $1 \in D_0$;
(iv) $D_n \cdot D_m \subset D_{n+m}$, for any $n, m \in \mathbb{Z}$;
(v) $[D_n, D_m] \subset D_{n+m-1}$, for any $n, m \in \mathbb{Z}$.

Then $\operatorname{Gr} D = \bigoplus_{n \in \mathbb{Z}} \operatorname{Gr}^n D = \bigoplus_{n \in \mathbb{Z}} D_n / D_{n-1}$ is a graded ring with identity. The property (v) implies that it is commutative. In particular, $D_0 = \operatorname{Gr}^0 D$ is a commutative ring with identity. Therefore, we can view $\operatorname{Gr} D$ as an algebra over D_0. Let's assume in addition that D satisfies

(vi) $\operatorname{Gr} D$ is a nötherian ring;
(vii) $\operatorname{Gr}^1 D$ generates $\operatorname{Gr} D$ as a D_0-algebra.

Then, by 1.1, D_0 is a nötherian ring. Moreover, by (vi), (vii) and 1.2 we know that we can choose finitely many elements $x_1, x_2, \dots, x_s \in \operatorname{Gr}^1 D$ such that $\operatorname{Gr} D$ is generated by them as a D_0-algebra. Clearly, by (vii), we also have

$$\operatorname{Gr}^{n+1} D = \operatorname{Gr}^1 D \cdot \operatorname{Gr}^n D \text{ for } n \in \mathbb{Z}_+$$

and therefore

$$D_{n+1} = D_n \cdot D_1 \text{ for } n \in \mathbb{Z}_+.$$

Let D° be the opposite ring of D. Then the filtration $(D_n\,;\, n \in \mathbb{Z})$ has the same properties with respect to the multiplication of D°. Moreover, the identity map $D \longrightarrow D^\circ$ induces an isomorphism of graded rings $\operatorname{Gr} D$ and $\operatorname{Gr} D^\circ$.

Let M be a D-module. An increasing filtration $\mathrm{F}\, M = (\mathrm{F}_n\, M\,;\, n \in \mathbb{Z})$ of M by additive subgroups is a *D-module filtration* if $D_n \cdot \mathrm{F}_m\, M \subset \mathrm{F}_{m+n}\, M$, for $n, m \in \mathbb{Z}$. In particular, $\mathrm{F}_n\, M$ are D_0-modules.

A D-module filtration $\mathrm{F}\, M$ is *hausdorff* if $\bigcap_{n\in\mathbb{Z}} F_n M = \{0\}$. It is *exhaustive* if $\bigcup_{n\in\mathbb{Z}} \mathrm{F}_n\, M = M$. It is called *stable* if there exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot \mathrm{F}_m\, M = \mathrm{F}_{m+n}\, M$ for all $n \in \mathbb{Z}_+$ and $m \geq m_0$.

A D-module filtration is called *good* if

(i) $\mathrm{F}_n\, M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$;
(ii) the filtration $\mathrm{F}\, M$ is exhaustive;
(iii) $\mathrm{F}_n\, M$, $n \in \mathbb{Z}$, are finitely generated D_0-modules;
(iv) the filtration $\mathrm{F}\, M$ is stable.

In particular, a good filtration is hausdorff.

3.1. Lemma. *Let $\mathrm{F}\, M$ be an exhaustive hausdorff D-module filtration of M. Then the following statements are equivalent:*

(i) *$\mathrm{F}\, M$ is a good filtration;*
(ii) *$\operatorname{Gr} D$-module $\operatorname{Gr} M$ is finitely generated.*

Proof. First we prove (i)⇒(ii). There exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot \mathrm{F}_{m_0}\, M = \mathrm{F}_{n+m_0}\, M$ for all $n \in \mathbb{Z}_+$. Therefore $\operatorname{Gr}^n D \cdot \operatorname{Gr}^{m_0} M = \operatorname{Gr}^{n+m_0} M$ for all $n \in \mathbb{Z}_+$. It follows that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ generates $\operatorname{Gr} M$ as a $\operatorname{Gr} D$-module. Since $\mathrm{F}_n\, M$ are finitely generated D_0-modules, $\operatorname{Gr}^n M$ are finitely generated D_0-modules too. This implies, since $\mathrm{F}_n\, M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$, that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ is a finitely generated D_0-module.

(ii)⇒(i). Clearly, $\operatorname{Gr}^n M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$. Also, by 1.2, all $\operatorname{Gr}^n M$ are finitely generated D_0-modules. The exact sequence

$$0 \longrightarrow \mathrm{F}_{n-1}\, M \longrightarrow \mathrm{F}_n\, M \longrightarrow \operatorname{Gr}^n M \longrightarrow 0$$

implies that $\mathrm{F}_n\, M = \mathrm{F}_{n-1}\, M$ for sufficiently negative n, hence there exists $n_0 \in \mathbb{Z}$ such that $\bigcap_{n\in\mathbb{Z}} \mathrm{F}_n\, M = \mathrm{F}_{n_0}\, M$. Since the filtration $\mathrm{F}\, M$ is hausdorff, $\mathrm{F}_{n_0}\, M = \{0\}$. This implies, by induction in n, that all $\mathrm{F}_n\, M$ are finitely generated D_0-modules. Let $m_0 \in \mathbb{Z}$ be such that $\bigoplus_{n\leq m_0} \operatorname{Gr}^n M$ generates $\operatorname{Gr} M$ as $\operatorname{Gr} D$-module. Let $m \geq m_0$. Then

$$\begin{aligned}\operatorname{Gr}^{m+1} M &= \bigoplus_{k\leq m_0} \operatorname{Gr}^{m+1-k} D \cdot \operatorname{Gr}^k M \\ &= \bigoplus_{k\leq m_0} \operatorname{Gr}^1 D \cdot \operatorname{Gr}^{m-k} D \cdot \operatorname{Gr}^k M \subset \operatorname{Gr}^1 D \cdot \operatorname{Gr}^m M \subset \operatorname{Gr}^{m+1} M,\end{aligned}$$

i.e., $\operatorname{Gr}^1 D \cdot \operatorname{Gr}^m M = \operatorname{Gr}^{m+1} M$. This implies that

$$\mathrm{F}^{m+1}\, M = D_1 \cdot \mathrm{F}_m\, M + \mathrm{F}_m\, M = D_1 \cdot \mathrm{F}_m\, M$$

and by induction in n,

$$\mathrm{F}_{m+n}\, M = D_1 \cdot D_1 \cdot \ldots \cdot D_1 \cdot \mathrm{F}_m\, M \subset D_n \cdot \mathrm{F}_m\, M \subset \mathrm{F}_{m+n}\, M.$$

Therefore, $\mathrm{F}_{m+n}\, M = D_n \cdot \mathrm{F}_m\, M$ for all $n \in \mathbb{Z}_+$. Hence, $\mathrm{F}\, M$ is a good filtration.

□

In particular, $(D_n\,;\, n \in \mathbb{Z})$ is a good filtration of D considered as a D-module for left multiplication.

3.2. Remark. From the proof it follows that the stability condition in the definition of a good filtration can be replaced by an apparently weaker condition:

(iv)' There exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot F_{m_0} M = F_{m_0+n} M$ for all $n \in \mathbb{Z}_+$.

3.3. Lemma. *Let M be a D-module with a good filtration $\mathrm{F}\, M$. Then M is finitely generated.*

Proof. By definition, $\bigcup_{n\in\mathbb{Z}} \mathrm{F}_n\, M = M$ and $\mathrm{F}_{n+m_0}\, M = D_n \cdot F_{m_0} M$ for $n \in \mathbb{Z}_+$ and some sufficiently large $m_0 \in \mathbb{Z}$. Therefore, $\mathrm{F}_{m_0}\, M$ generates M as a D-module. Since $\mathrm{F}_{m_0}\, M$ is a finitely generated D_0-module, the assertion follows. □

3.4. Lemma. *Let M be a finitely generated D-module. Then M admits a good filtration.*

Proof. Let U be a finitely generated D_0-module which generates M as a D-module. Put $\mathrm{F}_n\, M = 0$ for $n < 0$ and $\mathrm{F}_n\, M = D_n \cdot U$ for $n \geq 0$. Then $U = \mathrm{Gr}^0\, M$, and

$$\mathrm{Gr}^n\, M = \mathrm{F}_n\, M / \mathrm{F}_{n-1}\, M = (D_n \cdot U)/(D_{n-1} \cdot U) \subset \mathrm{Gr}^n\, D \cdot \mathrm{Gr}^0\, M \subset \mathrm{Gr}^n\, M,$$

i.e., $\mathrm{Gr}^n\, M = \mathrm{Gr}^n\, D \cdot \mathrm{Gr}^0\, M$ for all $n \in \mathbb{Z}_+$. Hence, $\mathrm{Gr}\, M$ is finitely generated as a $\mathrm{Gr}\, D$-module. The statement follows from 3.1. □

The lemmas 3.1 and 3.3 imply that the D-modules admitting good filtrations are precisely the finitely generated D-modules.

3.5. Proposition. *The ring D is a left and right nötherian.*

Proof. Let L be a left ideal in D. The natural filtration of D induces a filtration $(L_n = L \cap D_n\,;\, n \in \mathbb{Z})$, on L. This is evidently a D-module filtration. The graded module $\mathrm{Gr}\, L$ is naturally an ideal in $\mathrm{Gr}\, D$, and since $\mathrm{Gr}\, D$ is a nötherian ring, it is finitely generated as $\mathrm{Gr}\, D$-module. Therefore, the filtration $(L_n\,;\, n \in \mathbb{Z})$ is good by 3.1, and L is finitely generated by 3.3. This proves that D is left nötherian.

To get the right nötherian property one has to replace D with its opposite ring D°. □

If we have two filtrations $\mathrm{F}\, M$ and $\mathrm{F}'\, M$ of a D-module M, we say that $\mathrm{F}\, M$ is *finer* than $\mathrm{F}'\, M$ if there exists a number $k \in \mathbb{Z}_+$ such that $\mathrm{F}_n\, M \subset \mathrm{F}'_{n+k}\, M$ for all $n \in \mathbb{Z}$. If $\mathrm{F}\, M$ is finer than $\mathrm{F}'\, M$ and $\mathrm{F}'\, M$ finer than $\mathrm{F}\, M$, we say that they are *equivalent*.

3.6. Lemma. *Let $\mathrm{F}\, M$ be a good filtration on a finitely generated D-module M. Then $\mathrm{F}\, M$ is finer than any other exhaustive D-module filtration on M.*

Proof. Fix $m_0 \in \mathbb{Z}_+$ such that $D_n \cdot \mathrm{F}_{m_0}\, M = \mathrm{F}_{n+m_0}\, M$ for all $n \in \mathbb{Z}_+$. Let $\mathrm{F}'\, M$ be another exhaustive D-module filtration on M. Then $\mathrm{F}_{m_0}\, M$ is finitely generated as a D_0-module. Since $\mathrm{F}'\, M$ is exhaustive, it follows that there exists $p \in \mathbb{Z}$ such that $\mathrm{F}_{m_0}\, M \subset \mathrm{F}'_p\, M$. Since $\mathrm{F}\, M$ is a good filtration, there exists n_0 such that $\mathrm{F}_{n_0}\, M = \{0\}$. Put $k = p + |n_0|$. Clearly, for $m \leq n_0$, we have $\mathrm{F}_m\, M = 0 \subset \mathrm{F}'_{m+k}\, M$. For $n_0 < m \leq m_0$, we have $-|n_0| \leq n_0 < m$ and $p = -|n_0| + k < m + k$. This yields

$$\mathrm{F}_m\, M \subset \mathrm{F}_{m_0}\, M \subset \mathrm{F}'_p\, M \subset \mathrm{F}'_{m+k}\, M.$$

Finally, for $m > m_0$, we have $m - m_0 \leq m$ since m_0 is positive, and $p \leq k$. It follows that

$$\mathrm{F}_m M = D_{m-m_0} \cdot \mathrm{F}_{m_0} M \subset D_m \cdot \mathrm{F}'_p M \subset \mathrm{F}'_{m+p} M \subset \mathrm{F}'_{m+k} M.$$

□

3.7. Corollary. *Any two good filtrations on a finitely generated D-module are equivalent.*

Let M be a finitely generated D-module and $\mathrm{F}\,M$ a good filtration on M. Then $\mathrm{Gr}\,M$ is a finitely generated $\mathrm{Gr}\,D$-module, hence we can apply the results on Hilbert polynomials from §1. Let λ be an additive function on finitely generated D_0-modules. Assume also that λ takes only nonnegative values on objects of the category $\mathcal{M}_{fg}(D_0)$ of finitely generated D_0-modules. Then, by 1.5,

$$\lambda(\mathrm{F}_n M) - \lambda(\mathrm{F}_{n-1} M) = \lambda(\mathrm{Gr}^n M)$$

is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. By 1.7 this implies that $\lambda(\mathrm{F}_n M)$ is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. If $\mathrm{F}'\,M$ is another good filtration on M, by 3.7 we know that $\mathrm{F}\,M$ an $\mathrm{F}'\,M$ are equivalent,i.e., there is a number $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}_n M \subset \mathrm{F}'_{n+k} M \subset \mathrm{F}_{n+2k} M$$

for all $n \in \mathbb{Z}$. Since λ is additive and takes nonnegative values only, we conclude that

$$\lambda(\mathrm{F}_n M) \leq \lambda(\mathrm{F}'_{n+k} M) \leq \lambda(\mathrm{F}_{n+2k} M)$$

for all $n \in \mathbb{Z}$. This implies that the polynomials representing $\lambda(\mathrm{F}_n M)$ and $\lambda(\mathrm{F}'_n M)$ for large n have equal leading terms. We denote the common degree of these polynomials by $d_\lambda(M)$ and call it the *dimension* of the D-module M (with respect to λ). By 1.6 the leading coefficient of these polynomials has the form $e_\lambda(M)/d_\lambda(M)!$ where $e_\lambda(M) \in \mathbb{N}$. We call $e_\lambda(M)$ the *multiplicity* of the D-module M (with respect to λ).

Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of D-modules. If M is equipped by a D-module filtration $\mathrm{F}\,M$, it induces filtrations $\mathrm{F}\,M' = (f^{-1}(f(M') \cap \mathrm{F}_n M)\,;\, n \in \mathbb{Z})$ on M' and $\mathrm{F}\,M'' = (\,g(F_n M)\,;\, n \in \mathbb{Z})$ on M''. Clearly, these filtrations are D-module filtrations.

Moreover, the sequence

$$0 \longrightarrow \mathrm{Gr}\,M' \xrightarrow{\mathrm{Gr}\,f} \mathrm{Gr}\,M \xrightarrow{\mathrm{Gr}\,g} \mathrm{Gr}\,M'' \longrightarrow 0$$

is exact. If the filtration $\mathrm{F}\,M$ is good, $\mathrm{Gr}\,M$ is a finitely generated $\mathrm{Gr}\,D$-module, hence both $\mathrm{Gr}\,M'$ and $\mathrm{Gr}\,M''$ are finitely generated $\mathrm{Gr}\,D$-modules. By 3.1, $\mathrm{F}\,M'$ and $\mathrm{F}\,M''$ are good filtrations. Therefore, we proved the following result.

3.8. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of D-modules. If $\mathrm{F}\,M$ is a good filtration on M, the induced filtrations $\mathrm{F}\,M'$ and $\mathrm{F}\,M''$ are good.

By the preceding discussion

$$\lambda(\operatorname{Gr}^n M) = \lambda(\operatorname{Gr}^n M') + \lambda(\operatorname{Gr}^n M'')$$

for all $n \in \mathbb{Z}$. This implies, by induction in n, that

$$\lambda(\operatorname{F}_n M) = \lambda(\operatorname{F}_n M') + \lambda(\operatorname{F}_n M'')$$

for all $n \in \mathbb{Z}$. This leads to the following result.

3.9. Proposition. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated D-modules. Then

(i) $d_\lambda(M) = \max(d_\lambda(M'), d_\lambda(M''))$;
(ii) *if* $d_\lambda(M) = d_\lambda(M') = d_\lambda(M'')$, *then* $e_\lambda(M) = e_\lambda(M') + e_\lambda(M'')$.

Finally, let ϕ be an automorphism of the ring D such that $\phi(D_0) = D_0$. We can define a functor $\tilde{\phi}$ from the category $\mathcal{M}(D)$ of D-modules into itself which attaches to a D-module M a D-module $\tilde{\phi}(M)$ with the same underlying additive group structure and with the action of D given by $(T, m) \longmapsto \phi(T)m$ for $T \in D$ and $m \in M$. Clearly, $\tilde{\phi}$ is an automorphism of the category $\mathcal{M}(D)$, and it preserves finitely generated D-modules.

3.10. Proposition. *Let M be a finitely generated D-module. Then*

$$d_\lambda(\tilde{\phi}(M)) = d_\lambda(M).$$

Proof. Let $T_1, T_2, \dots, T_s$ be the representatives in D_1 of classes in $\operatorname{Gr}^1 D$ generating $\operatorname{Gr} D$ as a D_0-algebra. Then there exists $d \in \mathbb{N}$ such that $\phi(T_i) \in D_d$ for $1 \le i \le s$. Since $T_1, T_2, \dots, T_s$ and 1 generate D_1 as a D_0-module, we conclude that $\phi(D_1) \subset D_d$.

Let $\operatorname{F} M$ be a good filtration of M. Define a filtration $\operatorname{F} \tilde{\phi}(M)$ by

$$\operatorname{F}_p \tilde{\phi}(M) = \operatorname{F}_{dp} M \text{ for } p \in \mathbb{Z}.$$

Clearly, $\operatorname{F} \tilde{\phi}(M)$ is an increasing filtration of $\tilde{\phi}(M)$ by finitely generated D_0-submodules. Also,

$$D_1 \cdot \operatorname{F}_m \tilde{\phi}(M) = \phi(D_1) \operatorname{F}_{dm} M \subset D_d \operatorname{F}_{dm} M \subset \operatorname{F}_{d(m+1)} M = \operatorname{F}_{m+1} \tilde{\phi}(M)$$

for $m \in \mathbb{Z}$. Hence, by induction, we have

$$D_n \cdot \operatorname{F}_m \tilde{\phi}(M) = D_1 \cdot D_{n-1} \cdot \operatorname{F}_m \tilde{\phi}(M) \subset D_1 \operatorname{F}_{m+n-1} \tilde{\phi}(M) \subset \operatorname{F}_{m+n} \tilde{\phi}(M)$$

for all $n, m \in \mathbb{Z}$, i.e., $\operatorname{F} \tilde{\phi}(M)$ is a D-module filtration. By 3.6, there exists a good filtration $\operatorname{F}' \tilde{\phi}(M)$ which is finer than this filtration, i.e, there exists $k \in \mathbb{Z}_+$ such that

$$\operatorname{F}'_n \tilde{\phi}(M) \subset \operatorname{F}_{n+k} \tilde{\phi}(M) = \operatorname{F}_{d(n+k)} M$$

for all $n \in \mathbb{Z}$. Therefore,

$$\lambda(\operatorname{F}'_n \tilde{\phi}(M)) \le \lambda(\operatorname{F}_{d(n+k)} M)$$

for $n \in \mathbb{Z}$. For large $n \in \mathbb{Z}$, $\lambda(\operatorname{F}_{d(n+k)} M)$ is equal to a polynomial in n with the leading term equal to

$$\frac{e_\lambda(M) d^{d_\lambda(M)}}{d_\lambda(M)!} n^{d_\lambda(M)}.$$

Since $\lambda(\mathrm{F}'_n\, \tilde{\phi}(M))$ is also given by a polynomial of degree $d_\lambda(\tilde{\phi}(M))$ for large $n \in \mathbb{Z}$, we conclude that $d_\lambda(\tilde{\phi}(M)) \leq d_\lambda(M)$. By applying the same reasoning to ϕ^{-1} we also conclude that

$$d_\lambda(M) = d_\lambda(\tilde{\phi}^{-1}(\tilde{\phi}(M))) \leq d_\lambda(\tilde{\phi}(M)).$$

□

4. Dimension of modules over polynomial rings

Let $A = k[X_1, \dots, X_n]$ where k is an algebraically closed field. We can filter A by degree of polynomials, i.e., we can put $A_m = \{\sum c_I x^I \mid c_I \in k, |I| \leq m\}$. Then $\operatorname{Gr} A = k[X_1, \dots, X_n]$, hence A satisfies properties (i)-(vii) from the preceding section.

Since $A_0 = k$ we can take for the additive function λ the function $\dim_k$. This leads to notions of dimension $d(M)$ and multiplicity $e(M)$ of a finitely generated A-module M.We know that for any $p \in \mathbb{Z}_+$, we have

$$\dim_k A_p = \binom{n+p}{n} = \frac{p^n}{n!} + \text{ lower order terms in } p,$$

i.e., $d(A) = n$ and $e(A) = 1$. In addition, for any finitely generated A-module M we have an exact sequence

$$0 \longrightarrow K \longrightarrow A^p \longrightarrow M \longrightarrow 0,$$

hence, by 3.9, $d(M) \leq n$. We shall give later a geometric interpretation of $d(M)$.

Let $x \in k^n$ and denote by $\mathbf{m}_x$ be the maximal ideal in $k[X_1, \dots, X_n]$ of all polynomials vanishing at x. We denote by A_x the localization of A at x, i.e., the ring of all rational k-valued functions on k^n regular at x. As we have seen in 2.11, A_x is an n-dimensional regular local ring with the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ consisting of all rational k-valued functions on k^n vanishing at x. Let M be an A-module. Its localization M_x at x is an A_x-module. We define the *support* of M by $\operatorname{supp}(M) = \{x \in k^n \mid M_x \neq 0\}$.

4.1. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then

$$\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'').$$

Proof. By exactness of localization we see that

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0$$

is an exact sequence of A_x-modules. This immediately implies our statement. □

For an ideal $I \subset k[X_1, \dots, X_n]$ we denote $V(I) = \{x \in k^n \mid f(x) = 0 \text{ for } f \in I\}$.

4.2. Proposition. *Let M be a finitely generated A-module and I its annihilator in A. Then* $\operatorname{supp}(M) = V(I)$.

Proof. We prove the statement by induction in the number of generators of M.

Assume first that M has one generator, i.e., $M = A/I$. Then $M_x = (A/I)_x = A_x/I_x$. Let $x \in V(I)$. Then $I \subset \mathbf{m}_x$ and $I_x \subset \mathbf{n}_x$. Hence $I_x \neq A_x$. It follows that $(A/I)_x \neq 0$ and $x \in \operatorname{supp}(M)$. Conversely, if $x \notin V(I)$, there exists $f \in I$ such that $f(x) \neq 0$, i.e., $f \notin \mathbf{m}_x$. Therefore, f is invertible in the local ring A_x and $f \in I_x$ implies that $I_x = A_x$. Hence $(A/I)_x = 0$ and $x \notin \operatorname{supp}(A/I)$. Therefore, $\operatorname{supp}(A/I) = V(I)$.

Now we consider the general situation. Let $m_1, \dots, m_p$ be a set of generators of M. Denote by M' the submodule generated by $m_1, \dots, m_{p-1}$. Then we have the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and M'' is cyclic. Moreover, by 4.1, $\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'')$. Hence, by the induction assumption, $\operatorname{supp}(M) = V(I') \cup V(I'')$ where I' and I'' are the annihilators of M' and M'' respectively.

Clearly, $I' \cdot I''$ is in the annihilator I of M. On the other hand, I annihilates M' and M'', hence $I \subset I' \cap I''$. It follows that

$$I' \cdot I'' \subset I \subset I' \cap I''.$$

This implies that

$$V(I') \cup V(I'') \subset V(I' \cap I'') \subset V(I) \subset V(I' \cdot I'').$$

Let $x \notin V(I') \cup V(I'')$. Then there exist $f \in I'$ and $g \in I''$ such that $f(x) \neq 0$ and $g(x) \neq 0$. It follows that $(f \cdot g)(x) = f(x) \cdot g(x) \neq 0$ and $x \notin V(I' \cdot I'')$. Hence, $V(I' \cdot I'') \subset V(I') \cup V(I'')$ and all inclusions above are equalities. Hence, we have $V(I) = V(I') \cup V(I'')$ and $\operatorname{supp}(M) = V(I)$. □

This immediately implies the following consequence.

4.3. Corollary. *Let M be a finitely generated A-module. Then its support* $\operatorname{supp}(M)$ *is a Zariski closed subset in k^n.*

The next lemma is useful in some reduction arguments.

4.4. Lemma. *Let B be a nötherian commutative ring and $M \neq 0$ be a finitely generated B-module. Then there exist a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ of M by B-submodules, and prime ideals J_i of B such that $M_i/M_{i-1} \cong B/J_i$, for $1 \leq i \leq n$.*

Proof. For any $x \in M$ we put $\operatorname{Ann}(x) = \{a \in B \mid ax = 0\}$. Let $\mathcal{A}$ be the family of all such ideals $\operatorname{Ann}(x)$, $x \in M$, $x \neq 0$. Because B is a nötherian ring, $\mathcal{A}$ has maximal elements. Let I be a maximal element in $\mathcal{A}$. We claim that I is prime. Let $x \in M$ be such that $I = \operatorname{Ann}(x)$. Then $ab \in I$ implies $abx = 0$. Assume tha $b \notin I$, i.e., $bx \neq 0$. Then $I \subset \operatorname{Ann}(bx)$ and $a \in \operatorname{Ann}(bx)$. By the maximality of I, $a \in \operatorname{Ann}(bx) = I$, and I is prime. Therefore, there exists $x \in M$ such that $J_1 = \operatorname{Ann}(x)$ is prime. If we put $M_1 = Bx$, $M_1 \cong B/J_1$. Now, denote by $\mathcal{F}$ the family of all B-submodules of M having filtrations $0 = N_0 \subset N_1 \subset \cdots \subset N_k = N$ such that $N_i/N_{i-1} \cong B/J_i$ for some prime ideals J_i. Since M is a nötherian module, $\mathcal{F}$ contains a maximal element L. Assume that $L \neq M$. Then we would have the exact sequence:

$$0 \longrightarrow L \longrightarrow M \longrightarrow L' \longrightarrow 0,$$

and by the first part of the proof, L' would have a submodule N' of the form B/J' for some prime ideal J', contradicting the maximality of L. Hence, $L = M$. This proves the existence of the filtration with required properties. □

4.5. Theorem. *Let M be a finitely generated A-module and* $\operatorname{supp}(M)$ *its support. Then $d(M) = \dim \operatorname{supp}(M)$.*

This result has the following companion local version. The localization A_x of A at $x \in k^n$ is a nötherian local ring. Moreover, its maximal ideal $\mathbf{n}_x$ is the ideal generated by the polynomials $X_i - x_i$, $1 \leq i \leq n$, and their images in $\mathbf{n}_x/\mathbf{n}_x^2$ span it as a vector space over k. Therefore, $X_i - x_i$, $1 \leq i \leq n$, form a coordinate system in A_x. For any finitely generated A-module M, its localization M_x at x is a finitely generated A_x-module, hence we can consider its dimension $d(M_x)$.

For any algebraic variety V over k and $x \in V$ we denote by $\dim_x V$ the local dimension of V at x.

4.6. Theorem. *Let M be a finitely generated A-module and $x \in \operatorname{supp}(M)$. Then $d(M_x) = \dim_x(\operatorname{supp}(M))$.*

We shall simultaneously prove 4.5 and 4.6. First we observe that if we have an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and 4.5 and 4.6 hold for M' and M'', we have, by 3.9 and 4.1, that

$$\begin{aligned} d(M) &= \max(d(M'), d(M'')) = \max(\dim \operatorname{supp}(M'), \dim \operatorname{supp}(M'')) \\ &= \dim(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim \operatorname{supp}(M). \end{aligned}$$

Also, for any $x \in \operatorname{supp}(M)$, by the exactness of localization we have the exact sequence:

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0;$$

hence, by 2.7 and 4.1,

$$\begin{aligned} d(M_x) &= \max(d(M'_x), d(M''_x)) = \max(\dim_x \operatorname{supp}(M'), \dim_x \operatorname{supp}(M'')) \\ &= \dim_x(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim_x \operatorname{supp}(M). \end{aligned}$$

Assume that 4.5 and 4.6 hold for all $M = A/J$ where J is a prime ideal. Then the preceding remark, 4.4 and an induction in the length of the filtration would prove the statements in general.

Hence we can assume that $M = A/J$ with J prime. Assume first that J is such that A/J is a finite-dimensional vector space over k. Then A/J is an integral ring and it is integral over k. Hence it is a field which is an algebraic extension of k. Since k is algebraically closed, $A/J = k$ and J is a maximal ideal. In this case, by Hilbert Nulstellenatz, $\operatorname{supp}(M) = V(J)$ is a point x in k^n, i.e., $\dim \operatorname{supp}(M) = 0$. On the other hand, since M_x is one-dimensional linear space, $d(M_x) = 0$, and the assertion is evident. It follows that we can assume that J is not of finite codimension in A, in particular it is not a maximal ideal. Let $J_1 \supset J$ be a prime ideal different form J. Then there exists $f \in J_1$ such that $f \notin J$. It follows that $J \subset (f) + J \subset J_1$ and $J \neq (f) + J$. Therefore, A/J_1 is a quotient of $A/((f) + J)$, and $A/((f) + J)$ is a quotient of A/J. In addition, $A/((f) + J) = M/fM$. Consider the endomorphism of M given by multiplication by f. Then, if $g + J$ is in the kernel of this map, $0 = f(g + J) = fg + J$ and $fg \in J$. Since J is prime and $f \notin J$ it follows that $g \in J$, $g + J = 0$ and the map is injective. Therefore, we have an exact sequence of A-modules:

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

This implies, by 3.9, that $d(M/fM) \le d(M)$. If $d(M/fM) = d(M)$, we would have in addition that $e(M) = e(M) + e(M/fM)$, hence $e(M/fM) = 0$. This is possible only if $d(M/fM) = 0$, and in this case it would also imply that $d(M) = 0$ and M is finite-dimensional, which is impossible by our assumption. Therefore, $d(M/fM) < d(M)$. Since A/J_1 is a quotient of M/fM, this implies that $d(A/J_1) < d(A/J)$.

Let $x \in V(J_1)$. Then, by localization, we get the exact sequence:

$$0 \longrightarrow M_x \xrightarrow{f} M_x \longrightarrow M_x/fM_x \longrightarrow 0$$

of A_x-modules. This implies, by 2.7, that $d(M_x/fM_x) \le d(M_x)$. If $d(M_x/fM_x) = d(M_x)$, we would have in addition that $e(M_x) = e(M_x) + e(M_x/fM_x)$, hence $e(M_x/fM_x) = 0$. This is possible only if $d(M_x/fM_x) = 0$, and in this case it would imply that $\mathbf{m}_x(M_x/fM_x) = M_x/fM_x$ and, by Nakayama lemma, $M_x/fM_x = 0$. It would follow that the multiplication by f is surjective on M_x, and, since $f \in \mathbf{m}_x$, by Nakayama lemma this would imply that $M_x = 0$ contrary to our assumptions. Therefore, $d(M_x/fM_x) < d(M_x)$. Since A/J_1 is a quotient of M/fM this implies that $d((A/J_1)_x) < d((A/J)_x)$.

Let

$$Z_0 = \{x\} \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = k^n$$

be a maximal chain of nonempty irreducible closed subsets of k^n. Then

$$I(Z_0) = \mathbf{m}_x \supset I(Z_1) \supset \cdots \supset I(Z_{n-1}) \supset I(Z_n) = \{0\}$$

is a maximal chain of prime ideals in A. By the preceding arguments we have the following sequences of strict inequalities

$$0 \le d(A/I(Z_0)) < d(A/I(Z_1)) < \cdots < d(A/I(Z_n)) = d(A) = n,$$

and

$$0 \le d((A/I(Z_0))_x) < d((A/I(Z_1))_x) < \cdots < d((A/I(Z_n))_x) = d(A_x) = n,$$

by 2.11. It follows that

$$d((A/I(Z_j))_x) = d(A/I(Z_j)) = j = \dim Z_j$$

for $0 \le j \le n$. Since every closed irreducible subset Z can be put in a maximal chain, it follows that $d((A/I(Z))_x) = d(A/I(Z)) = \dim Z$ for any closed irreducible subset $Z \subset k^n$ and any $x \in Z$. On the other hand, this implies that $d((A/J)_x) = d(A/J) = \dim V(J)$ for any prime ideal J in A and $x \in V(J)$. By 4.2, this ends the proof of 4.5 and 4.6.

Next result follows immediately from 4.5 and 4.6.

4.7. Corollary. *Let M be a finitely generated A-module. Then*

$$d(M) = \sup_{x \in \operatorname{supp}(M)} d(M_x).$$

Finally, we prove a result we will need later.

4.8. Lemma. *Let be I an ideal in A. Then* $\dim V(I) = \dim V(\operatorname{Gr} I)$.

Proof. The short exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

where the modules are equipped with the filtrations induced by the natural filtration of A leads to the short exact sequence

$$0 \longrightarrow \operatorname{Gr} I \longrightarrow A \longrightarrow \operatorname{Gr}(A/I) \longrightarrow 0$$

of graded A-modules. Hence, we have

$$\begin{aligned}\dim_k \mathrm{F}_p(A/I) &= \sum_{q=0}^{p} (\dim_k \mathrm{F}_q(A/I) - \dim_k \mathrm{F}_{q-1}(A/I)) \\ &= \sum_{q=0}^{p} \dim_k \mathrm{Gr}^p(A/I) = \sum_{q=0}^{p} (\dim_k \mathrm{Gr}^p A - \dim \mathrm{Gr}^p I) \\ &= \dim_k F_p A - \dim_k \mathrm{F}_p \mathrm{Gr}\, I = \dim_k F_p(A/\mathrm{Gr}\, I).\end{aligned}$$

Therefore, $d(A/I) = d(A/\mathrm{Gr}\, I)$. The assertion follows from 4.4 and 4.5. □

5. Rings of differential operators with polynomial coefficients

Let k be a field of characteristic zero. Let A be a commutative algebra over k. Let $\mathrm{End}_k(A)$ be the algebra of all k-linear endomorphisms of A. It is a Lie algebra with the commutator $[S,T] = ST - TS$ for any $S, T \in \mathrm{End}_k(A)$. Clearly, $\mathrm{End}_k(A)$ contains, as a subalgebra, the set $\mathrm{End}_A(A)$ of all A-linear endomorphisms of A. To any element $a \in A$ we can attach the A-linear endomorphism of A defined by $b \longmapsto ab$ for $b \in A$. Since this endomorphism takes the value a on 1, this map is clearly an injective morphism of algebras.

On the other hand, if $T \in \mathrm{End}_A(A)$, we have

$$T(b) = bT(1) = T(1)b$$

for any $b \in A$, i.e., T is given by multiplication by $T(1)$. This implies the folowing result.

5.1. Lemma. *The algebra homomorphism which attaches to an element $a \in A$ the A-linear endomorphism $b \longmapsto ab$, $b \in A$, is an isomorphism of A onto* $\mathrm{End}_A(A)$.

In the following, we identify A with the subalgebra $\mathrm{End}_A(A)$ of $\mathrm{End}_k(A)$.

A *k-derivation* of A is a $T \in \mathrm{End}_k(A)$ such that

$$T(ab) = T(a)b + aT(b)$$

for any $a, b \in A$. In particular, $[T,a](b) = T(ab) - aT(b) = T(a)b$, i.e., $[T,a] = T(a) \in A$ for any $a \in A$. This implies that $[[T,a_0],a_1] = 0$ for any $a_0, a_1 \in A$.

This leads to the following definition. Let $n \in \mathbb{Z}_+$. We say that an element $T \in \mathrm{End}_k(A)$ is a (*k-linear*) *differential operator* on A of *order* $\leq n$ if

$$[\dots[[T,a_0],a_1],\dots,a_n] = 0$$

for any $a_0, a_1, \dots, a_n \in A$. We denote by $\mathrm{Diff}_k(A)$ the space of all differential operators on A.

5.2. Lemma. *Let T, S be two differential operators of order $\leq n$, $\leq m$ respectively. Then $T \circ S$ is a differential operator of order $\leq n + m$.*

Proof. We prove the statement by induction in $n+m$. If $n = m = 0$, $T, S \in \mathrm{End}_A(A)$, hence $T \circ S \in \mathrm{End}_a(A)$ and it is a differential operator of order 0.

Assume now that $n + m \leq p$. Then

$$[T \circ S, a] = TSa - aTS = T[S,a] + [T,a]S,$$

and $[T,a]$ and $[S,a]$ are differential operators of order $\leq n-1$ and $\leq m-1$ respectively. By the induction assumption, this differential operator is of order $\leq n+m-1$. Therefore $T \circ S$ is of order $\leq n + m$. □

Therefore $\mathrm{Diff}_k(A)$ is a subalgebra of $\mathrm{End}_k(A)$. We call it the *algebra of all k-linear differential operators* on A. Also, we put $\mathrm{F}_n \mathrm{Diff}_k(A) = \{0\}$ for $n < 0$ and

$$\mathrm{F}_n \mathrm{Diff}_k(A) = \{T \in \mathrm{Diff}_k(A) \mid \mathrm{order}(T) \leq n\}$$

for $n \geq 0$. clearly, this is an increasing exhaustive filtration of $\mathrm{Diff}_k(A)$ by vector subspaces over k. This filtration is compatible with the ring structure of $\mathrm{Diff}_k(A)$, i.e., it satisfies

$$\mathrm{F}_n \mathrm{Diff}_k(A) \circ \mathrm{F}_m \mathrm{Diff}_k(A) \subset \mathrm{F}_{n+m} \mathrm{Diff}_k(A)$$

for any $n, m \in \mathbb{Z}$.

5.3. Lemma. (i) $\mathrm{F}_0 \mathrm{Diff}_k(A) = A$.
(ii) $\mathrm{F}_1 \mathrm{Diff}_k(A) = \mathrm{Der}_k(A) \oplus A$.
(iii) $[\mathrm{F}_n \mathrm{Diff}_k(A), \mathrm{F}_m \mathrm{Diff}_k(A)] \subset \mathrm{F}_{n+m-1} \mathrm{Diff}_k(A)$ *for any* $n, m \in \mathbb{Z}_+$.

Proof. (i) is evident.

(ii) As we remarked before, $\mathrm{Der}_k(A) \subset \mathrm{F}_1 \mathrm{Diff}_k(A)$. Also, for any $T \in \mathrm{Der}_k(A)$, we have $T(1) = T(1 \cdot 1) = 2T(1)$, hence $T(1) = 0$. This implies that $\mathrm{Der}_k(A) \cap A = 0$.

Let $S \in \mathrm{F}_1 \mathrm{Diff}_k(A)$ and $T = S - S(1)$. Then $T(1) = 0$, hence $T(a) = [T, a](1)$, and

$$\begin{aligned} T(ab) = [T, ab](1) = ([T, a]b)(1) + (a[T, b])(1) \\ = (b[T, a])(1) + (a[T, b])(1) = T(a)b + aT(b), \end{aligned}$$

i.e., $T \in \mathrm{Der}_k(A)$.

(iv) Let T, S be of order $\leq n, \leq m$ respectively. We claim that $[T, S]$ is of order $\leq n + m - 1$. We prove it by induction on $n + m$. If $n = m = 0$, there is nothing to prove. In general, by Jacobi identity, we have

$$[[T, S], a] = [[T, a], S] + [T, [S, a]]$$

where $[T, a]$ and $[S, a]$ are of order $\leq n - 1$ and $\leq m - 1$ respectively. Hence, by the induction assumption, $[[T, S], a]$ is of order $\leq n + m - 2$ and $[T, S]$ is of order $\leq n + m - 1$. □

This implies that the graded ring $\mathrm{Gr}\,\mathrm{Diff}_k(A)$ is a commutative A-algebra. In addition, $\mathrm{Diff}_k(A)$ satisfies properties (i)-(v) from §3.

Let $n \geq 1$. Let T be a differential operator on A of order $\leq n$. Then we can define a map from A^n into $\mathrm{Diff}_k(A)$ by

$$\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n) = [[\ldots[[T, a_1], a_2], \ldots, a_{n-1}], a_n].$$

Since $\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n)$ is of order ≤ 0, we can consider this map as a map from A^n into A.

5.4. Lemma. *Let T be a differential operator on A of order $\leq n$. Then:*
(i) *the map $\sigma_n(T) : A^n \longrightarrow A$ is a symmetric k-multilinear map;*
(ii) *the operator T is of order $\leq n - 1$ if and only if $\sigma_n(T) = 0$.*

Proof. (i) We have to check the symmetry property only. To show this, we observe that, by the Jacobi identity, we have

$$[[S, a], b] = [[S, b], a]$$

for any $S \in \mathrm{Diff}_k(A)$ and $a, b \in A$. This implies that

$$\begin{aligned}\sigma_n(T)(a_1, a_2, \dots, a_i, a_{i+1}, \dots, a_{n-1}, a_n) \\ = [[\dots[[\dots[[T, a_1], a_2], \dots, a_i], a_{i+1}]\dots, a_{n-1}], a_n] \\ = [[\dots[[\dots[[T, a_1], a_2], \dots, a_{i+1}], a_i]\dots, a_{n-1}], a_n] \\ = \sigma_n(T)(a_1, a_2, \dots, a_{i+1}, a_i, \dots, a_{n-1}, a_n),\end{aligned}$$

hence $\sigma(T)$ is symmetric.

(ii) is obvious. □

Now we want to discuss a special case. Let $A = k[X_1, X_2, \dots, X_n]$. Then we put $D(n) = \mathrm{Diff}_k(A)$. We call $D(n)$ the algebra of all differential operators on k^n. Let $\partial_1, \partial_2, \dots, \partial_n$ be the standard derivations of $k[X_1, X_2, \dots, X_n]$. For $I, J \in \mathbb{Z}_+^n$ we put

$$X^I = X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$$

and

$$\partial^J = \partial_1^{j_1} \partial_2^{j_2} \dots \partial_n^{j_n}.$$

Then $X^I \partial^J \in D(n)$, and it is a differential operator of order $\leq |J| = j_1 + j_2 + \dots + j_n$. Moreover, if T is a differential operator given by

$$T = \sum_{|I| \leq p} P_I(X_1, X_2, \dots, X_n) \partial^I,$$

with polynomials $P_I \in k[X_1, X_2, \dots, X_n]$, we see that T is of order $\leq p$.

5.5. Lemma. *The derivations* $(\partial_i \,;\, 1 \leq i \leq n)$ *form a basis of the free* $k[X_1, \dots, X_n]$-*module* $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$.

Proof. Let $T \in \mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. Put $P_i = T(X_i)$ for $1 \leq i \leq n$, and define $S = \sum_{i=1}^n P_i \partial_i$. Clearly,

$$S(X_i) = \sum_{j=1}^n P_j \, \partial_j(X_i) = P_i = T(X_i)$$

for all $1 \leq i \leq n$. Since $X_1, X_2, \dots, X_n$ generate $k[X_1, X_2, \dots, X_n]$ as a k-algebra it follows that $T = S$. Therefore, $(\partial_i \,;\, 1 \leq i \leq n)$ generate the $k[X_1, X_2, \dots, X_n]$-module $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. Assume that $\sum_{i=1}^n Q_i \partial_i = 0$ for some $Q_i \in k[X_1, X_2, \dots, X_n]$. Then $0 = (\sum_{j=1}^n Q_j \partial_j)(X_i) = Q_i$ for all $1 \leq i \leq n$. This implies that ∂_i, $1 \leq i \leq n$, are free generators of $\mathrm{Der}_k(k[X_1, X_2, \dots, X_n])$. □

Let T be a differential operator of order $\leq p$ on $k[X_1, X_2, \dots, X_n]$. If $p < 0$, $T = 0$ and we put $\mathrm{Symb}_p(T) = 0$. If $p = 0$, $T \in A$, and we put $\mathrm{Symb}_0(T) = T$. For $p \geq 1$, we define a polynomial $\mathrm{Symb}_p(T)$ in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ in the following way. Let $(\xi_1, \xi_2, \dots, \xi_n) \in k^n$. Then we can define a linear polynomial $\ell_\xi = \sum_{i=1}^n \xi_i X_i \in k[X_1, X_2, \dots, X_n]$ and the function

$$(\xi_1, \xi_2, \dots, \xi_n) \longmapsto \frac{1}{p!} \sigma_p(T)(\ell_\xi, \ell_\xi, \dots, \ell_\xi)$$

on k^n with values in $k[X_1, X_2, \dots, X_n]$. Clearly, one can view this function as a polynomial in $X_1, X_2, \dots, X_n$ and $\xi_1, \xi_2, \dots, \xi_n$ homogeneous of degree p in $\xi_1, \xi_2, \dots, \xi_n$, and denote it by $\mathrm{Symb}_p(T)$. The polynomial $\mathrm{Symb}_p(T)$ is called the *p-symbol* of the differential operator T. By its definition, $\mathrm{Symb}_p(T)$ vanishes for T of order $< p$. Therefore, for $p \geq 0$, it induces a k-linear map of $\operatorname{Gr}_p D(n)$ into $k[X_1, X_2, \ldots, X_n, \xi_1, \xi_2, \ldots, \xi_n]$. We denote by Symb the corresponding k-linear map of $\operatorname{Gr} D(n)$ into $k[X_1, X_2, \ldots, X_n, \xi_1, \xi_2, \ldots, \xi_n]$.

5.6. Theorem. *The map* $\operatorname{Symb} : \operatorname{Gr} D(n) \longrightarrow k[X_1, X_2, \ldots, X_n, \xi_1, \xi_2, \ldots, \xi_n]$ *is a k-algebra isomorphism.*

The proof of this result consists of several steps. First we prove the symbol map is an algebra morphism.

5.7. Lemma. *Let $T, S \in D(n)$ of order $\leq p$ and $\leq q$ respectively. Then*

$$\operatorname{Symb}_{p+q}(TS) = \operatorname{Symb}_p(T) \operatorname{Symb}_q(S).$$

Proof. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \longrightarrow D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then

$$\tau_\xi(TS) = [TS, \ell_\xi] = TS\ell_\xi - \ell_\xi TS = [T, \ell_\xi]S + T[S, \ell_\xi] = \tau_\xi(T)S + T\tau_\xi(S).$$

Therefore, for any $k \in \mathbb{Z}_+$, we have

$$\tau_\xi^k(TS) = \sum_{i=0}^{k} \binom{k}{i} \tau_\xi^{k-i}(T)\, \tau_\xi^i(S).$$

This implies that

$$\begin{aligned} \operatorname{Symb}_{p+q}(TS) &= \frac{1}{(p+q)!}\sigma_{p+q}(T)(\ell_\xi, \ell_\xi, \ldots, \ell_\xi) = \frac{1}{(p+q)!}\tau_\xi^{p+q}(TS) \\ &= \frac{1}{p!q!}\tau_\xi^p(T)\, \tau_\xi^q(S) = \operatorname{Symb}_p(T) \operatorname{Symb}_q(S). \end{aligned}$$

□

Since $\operatorname{Symb}_0(X_i) = X_i$ and $\operatorname{Symb}_1(\partial_i) = \xi_i$, $1 \leq i \leq n$, we see that for $X^I \partial^J$ with $p = |J|$ we have

$$\operatorname{Symb}_p(X^I \partial^J) = X^I \xi^J.$$

In particular, for

$$T = \sum_{|J| \leq p} P_I(X_1, X_2, \ldots, X_n)\partial^I,$$

with polynomials $P_I \in k[X_1, X_2, \ldots, X_n]$, we see that

$$\operatorname{Symb}_p(T) = \sum_{|I|=p} P_I(X_1, X_2, \ldots, X_n)\xi^I.$$

Hence, the symbol morphism is surjective. It remains to show that the symbol map is injective.

5.8. Lemma. *Let $T \in \operatorname{F}_p D(n)$. Then* $\operatorname{Symb}_p(T) = 0$ *if and only if T is of order $\leq p - 1$.*

Proof. We prove the statement by induction in p. It is evident if $p = 0$.

Therefore we can assume that $p > 0$. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \longrightarrow D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then, for any $\lambda \in k$ and $\eta \in k^n$, we have

$$\tau_{\xi+\lambda\eta}(T) = [T, \ell_{\xi+\lambda\eta}] = [T, \ell_\xi] + \lambda[T, \ell_\eta] = \tau_\xi(T) + \lambda\tau_\eta(T).$$

Since τ_ξ and τ_η commute we see that, for any $k \in \mathbb{Z}_+$, we have

$$\tau^k_{\xi+\lambda\eta}(T) = \sum_{i=0}^{k} \binom{k}{i} \lambda^i \, \tau_\xi^{k-i}(\tau^i_\eta(T)).$$

By our assumption, $\tau^p_{\xi+\lambda\eta}(T) = 0$ for arbitrary $\lambda \in k$. Therefore, since the field k is infinite, $\tau_\xi^{p-i}(\tau^i_\eta(T)) = 0$ for $0 \le i \le p$. In particular, we see that $\tau_\xi^{p-1}(\tau_\eta(T)) = 0$ for any $\xi, \eta \in k^n$. This implies that $\operatorname{Symb}_{p-1}([T, \ell_\eta]) = 0$ for any $\eta \in k^n$, in particular

$$\operatorname{Symb}_{p-1}([T, X_i]) = 0$$

for $1 \le i \le n$, and by the induction assumption, $[T, X_i]$, $1 \le i \le n$, are of order $\le p-2$. Let $P, Q \in k[X_1, X_2, \dots, X_n]$. Then

$$[T, PQ] = TPQ - PQT = [T, P]Q + P[T, Q],$$

hence the order of $[T, PQ]$ is less than or equal to the maximum of the orders of $[T, P]$ and $[T, Q]$. Since X_i, $1 \le i \le n$, generate $k[X_1, X_2, \dots, X_n]$ we conclude that the order of $[T, P]$ is $\le p-2$ for any polynomial P. This implies that the order of T is $\le p-1$. ☐

This also ends the proof of 4.6. In particular, we see that $D(n)$ satisfies properties (i)-(vii) from §3. From 3.5 we immediately deduce the following result.

5.9. Theorem. *The ring $D(n)$ is right and left nötherian.*

5.10. Corollary. *$(X^I \partial^J \,;\, I, J \in \mathbb{Z}^n_+)$ is a basis of $D(n)$ as a vector space over k.*

Proof. If $|J| = p$, the p-symbol of $X^I \partial^J$ is equal to $X^I \xi^J$ and $(X^I \xi^J \,;\, I, J \in \mathbb{Z}^n_+)$ form a basis of $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ as a vector space over k. ☐

The following caracterization of $D(n)$ is frequently useful.

5.11. Theorem. *The k-algebra $D(n)$ is the k-algebra generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ satisfying the defining relations $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, X_j] = \delta_{ij}$ for all $1 \le i, j \le n$.*

Proof. Let B be the k-algebra generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ satisfying the defining relations $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, X_j] = \delta_{ij}$ for all $1 \le i, j \le n$. Since these relations hold in $D(n)$ and it is generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ we conclude that there is a unique surjective morphism of B onto $D(n)$ which maps generators into the corresponding generators. Clearly, B is spanned by $(X^I \partial^J \,;\, I, J \in \mathbb{Z}^n_+)$. Therefore, by 5.10, this morphism is also injective. ☐

5.12. Proposition. *The center of $D(n)$ is equal to $k \cdot 1$.*

Proof. Let T be a central element of $D(n)$. Then, $[T, P] = 0$ for any polynomial P, and T is of order ≤ 0. Therefore, by 5.3, $T \in k[X_1, X_2, \dots, X_n]$. On the other hand, $0 = [\partial_i, T] = \partial_i(T)$ for $1 \le i \le n$. This implies that T is a constant polynomial. ☐

Let $D(n)^\circ$ be the opposite algebra of $D(n)$. Then, by 5.11, there exists a unique isomorphism $\phi : D(n)^\circ \longrightarrow D(n)$ which is defined by $\phi(X_i) = X_i$ and $\phi(\partial_i) = -\partial_i$ for $1 \le i \le n$. The morphism ϕ is called the *principal antiautomorphism* of $D(n)$. This proves the following result.

5.13. Proposition. *The algebra $D(n)^\circ$ is isomorphic to $D(n)$.*

Moreover, by 5.11, we can define an automorphism $\mathcal{F}$ of $D(n)$ by $\mathcal{F}(X_i) = \partial_i$ and $\mathcal{F}(\partial_i) = -X_i$ for $1 \le i \le n$. This automorphism is called the *Fourier automorphism* of $D(n)$. The square $\mathcal{F}^2$ of $\mathcal{F}$ is an automorphism ι of $D(n)$ which acts as $\iota(X_i) = -X_i$ and $\iota(\partial_i) = -\partial_i$ for $1 \le i \le n$. Clearly, $\iota^2 = 1$.

In contrast to the filtration by the order of differential operators, $D(n)$ has another filtration compatible with its ring structure which is not defined on more general rings of differential operators. We put

$$D_p(n) = \left\{ \sum a_{IJ} X^I \partial^J \;\middle|\; |I| + |J| \le p \right\}$$

for $p \in \mathbb{Z}$. Clearly, $(D_p(n) \mid p \in \mathbb{Z})$ is an increasing exhaustive filtration of $D(n)$ by finite-dimensional vector spaces over k.

5.14. Lemma. *For any $p, q \in \mathbb{Z}$ we have*

(i) $D_p(n) \circ D_q(n) \subset D_{p+q}(n)$;
(ii) $[D_p(n), D_q(n)] \subset D_{p+q-2}(n)$.

Proof. By 5.10 and the definition of the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$, it is enough to check that

$$[\partial^I, X^J] \in D_{|I|+|J|-2}(n).$$

We prove this statement by an induction in $|I|$. If $|I| = 1$, we have $\partial^I = \partial_i$ for some $1 \le i \le n$ and $[\partial_i, X^J] = \partial_i(X^J) \in D_{|J|-1}(n)$. If $|I| > 1$, we can write $\partial^I = \partial^{I'} \partial_i$ for some $I' \in \mathbb{Z}_+^n$ and $1 \le i \le n$. This leads to

$$\begin{aligned}[\partial^I, X^J] &= [\partial^{I'}\partial_i, X^J] = \partial^{I'}\partial_i X^J - X^J \partial^{I'}\partial_i \\ &= \partial^{I'}[\partial_i, X^J] + [\partial^{I'}, X^J]\partial_i = [\partial^{I'}, [\partial_i, X^J]] + [\partial_i, X^J]\partial^{I'} + [\partial^{I'}, X^J]\partial_i,\end{aligned}$$

hence, by the induction assumption, $[\partial^I, X^J] \in D_{|I|+|J|-2}(n)$. □

This implies that $(D_p(n)\,;\, p \in \mathbb{Z})$ is a filtration compatible with the ring structure on $D(n)$. In addition, the graded ring $\operatorname{Gr} D(n)$ is a commutative k-algebra. If we define the linear map Ψ_p from $D_p(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ by

$$\Psi_p\left(\sum_{|I|+|J| \le p} a_{IJ} X^I \partial^J \right) = \sum_{|I|+|J|=p} a_{IJ} X^I \xi^J$$

we see that it is a linear isomorphism of $\operatorname{Gr}_p D(n)$ into the homogeneous polynomials of degree p. Therefore, it extends to a linear isomorphism

$$\Psi : \operatorname{Gr} D(n) \longrightarrow k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n].$$

By 5.14 we see that this map is an isomorphism of k-algebras. Therefore, the ring $D(n)$ equipped with the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ satisfies the properties (i)-(vii) from §3. The filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ is called the *Bernstein filtration* of $D(n)$.

Evidently, the principal antiautomorphism and the Fourier automorphism of $D(n)$ preserve the Bernstein filtration.

6. Modules over rings of differential operators with polynomial coefficients

In this section we study the category of modules over the rings $D(n)$ of differential operators with polynomial coefficients. Denote by $\mathcal{M}^L(D(n))$, resp. $\mathcal{M}^R(D(n))$ the categories of left, resp. right, $D(n)$-modules. These are abelian categories. The principal antiautomorphism ϕ of $D(n)$ defines then an exact functor from the category $\mathcal{M}^R(D(n))$ into the category $\mathcal{M}^L(D(n))$ which maps the module M into its *transpose* M^t, which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \phi(T)m$ for $T \in D(n)$ and $m \in M$. An analogous functor is defined from $\mathcal{M}^L(D(n))$ into $\mathcal{M}^R(D(n))$. Clearly these functors are mutually inverse isomorphisms of categories. If we denote by $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ the corresponding full subcategories of finitely generated modules, we see that these functors also induce their equivalence. Therefore in the following we can restrict ourselves to the discussion of left modules and drop the superscript L from our notation (except in the cases when we want to stress that we deal with right modules). Since $D(n)$ is a nötherian ring, the full subcategory $\mathcal{M}_{fg}(D(n))$ of $\mathcal{M}(D(n))$ is closed under taking submodules, quotient modules and extensions.

First we consider $D(n)$ as a ring equipped with the Bernstein filtration. Since in this case $D_0(n) = k$ we can define the dimension of modules from $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ using the additive function $\dim_k$ on the category of finite-dimensional vector spaces over k. This dimension $d(M)$ and the corresponding multiplicity $e(M)$ of a module M are called the *Bernstein dimension* and the *Bernstein multiplicity* respectively. Since the principal antiautomorphism preserves the Bernstein filtration we see that $d(M) = d(M^t)$ for any finitely generated $D(n)$-module M.

For any finitely generated $D(n)$-module M we have an exact sequence $D(n)^p \longrightarrow M \longrightarrow 0$, hence $d(M) \leq d(D(n))$. In addition, from 5.6 we conclude the following result.

6.1. Lemma. *For any finitely generated $D(n)$-module M we have $d(M) \leq 2n$.*

6.2. Example. Consider the algebra $D(1)$ of polynomial differential operators in one variable. Let M be a finitely generated $D(1)$-module different from 0. Then its Bernstein dimension $d(M)$ can be 0, 1 or 2. Clearly, $d(M) = 0$ would imply that for any good filtration $\mathrm{F}\, M$ of M, the function $p \longmapsto \dim F_p M$ is constant for large $p \in \mathbb{Z}$. Since $\mathrm{F}\, M$ is exhaustive, this would mean that M is finite dimensional. Denote by $\pi(x)$ and $\pi(\partial)$ the linear transformations on M induced my the action of x and ∂ respectively. Then we have $[\pi(x), \pi(\partial)] = 1_M$. Taking the trace of both sides of this equality we would get $\dim_k M = 0$, i.e., contradicting our assumption that $M \neq 0$. It follows that $d(M)$ is either 1 or 2.

The main result of the dimension theory of $D(n)$ is the following statement with generalizes the above example.

6.3. Theorem (Bernstein). *Let M be a finitely generated $D(n)$-module and $M \neq 0$. Then $d(M) \geq n$.*

Proof. Since M is a finitely generated $D(n)$-module, by 3.4, we can equip it with a good filtration. Also, by shift in indices, we can clearly assume that $\mathrm{F}_n\, M = 0$ for $n < 0$ and $\mathrm{F}_0\, M \neq 0$.

For any $p \in \mathbb{Z}_+$ we can consider the linear map $D_p(n) \longrightarrow \mathrm{Hom}_k(\mathrm{F}_p\, M, \mathrm{F}_{2p}\, M)$ which attaches to $T \in D_p(n)$ the linear map $m \longmapsto Tm$. We claim that this map is injective. For $p \leq 0$ this is evident. Assume that it holds for $p-1$ and that $T \in D_p(n)$ satisfies $Tm = 0$ for all $m \in F_p M$. Then, for any $v \in \mathrm{F}_{p-1} M$ and $1 \leq i \leq n$ we have $X_i v \in \mathrm{F}_p M$ and $\partial_i v \in \mathrm{F}_p M$, hence

$$[X_i, T]v = X_i T v - T X_i v = 0$$

and

$$[\partial_i, T]v = \partial_i T v - T \partial_i v = 0$$

and $[X_i, T], [\partial_i, T] \in D_{p-1}(n)$ by 5.14. By the induction assumption this implies that $[X_i, T] = 0$ and $[\partial_i, T] = 0$ for $1 \leq i \leq n$, and T is in the center of $D(n)$. Since the center of $D(n)$ is equal to k by 5.12, we conclude that $T = 0$. Therefore,

$$\dim_k(D_p(n)) \leq \dim_k(\mathrm{Hom}_k(\mathrm{F}_p M, \mathrm{F}_{2p} M)) = \dim_k(\mathrm{F}_p M) \cdot \dim_k(\mathrm{F}_{2p} M)$$

for any $p \in \mathbb{Z}$. On the other hand, for large $p \in \mathbb{Z}_+$ the left side is equal to a polynomial in p of degree $2n$ with positive leading coefficient and the right side is equal to a polynomial in p of degree $2d(M)$ with positive leading coefficient. This is possible only if $d(M) \geq n$. □

In the next section we are going to give a geometric interpretation of the Bernstein dimension.

Finally, if M is a $D(n)$-module, we can define its *Fourier transform* $\mathcal{F}(M)$ as the module which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \mathcal{F}(T)m$ for $T \in D(n)$ and $m \in M$. Clearly the Fourier transform is an automorphism of the category $\mathcal{M}(D(n))$. It also induces an automorphism of the category $\mathcal{M}_{fg}(D(n))$. From the fact that the Fourier automorphism $\mathcal{F}$ preserves the Bernstein filtration (or 3.9) we conclude that the following result holds.

6.4. Lemma. *Let M be a finitely generated $D(n)$-module. Then $d(\mathcal{F}(M)) = d(M)$.*

7. Characteristic variety

Now we want to study an invariant of finitely generated $D(n)$-modules which has a more geometric flavor. In particular, it will be constructed using the filtration $\mathrm{F}\,D(n)$ of $D(n)$ by the degree of differential operators instead of the Bernstein filtration. In contrast to the Bernstein filtration, the degree filtration makes sense for rings of differential operators on arbitrary smooth affine varieties.

First, since any $D(n)$-module M can be viewed as a $k[X_1, X_2, \dots, X_n]$-module, we can consider its support $\mathrm{supp}(M) \subset k^n$.

7.1. Proposition. *Let M be a finitely generated $D(n)$-module. Then* $\mathrm{supp}(M)$ *is a closed subvariety of k^n.*

Proof. Fix a good filtration $\mathrm{F}\,M$ on M. Then, for $x \in k^n$, $M_x = 0$ is equivalent to $(\mathrm{F}_p M)_x = 0$ for all $p \in \mathbb{Z}$. Therefore, by the exactness of localization, it is equivalent to $(\mathrm{Gr}\, M)_x = 0$. Let I_p be the annihilator of the $k[X_1, X_2, \dots, X_n]$-module $\mathrm{Gr}_p M$, $p \in \mathbb{Z}$. Since $\mathrm{Gr}_p M$ are finitely generated $k[X_1, X_2, \dots, X_n]$-modules, by 4.2 their supports $\mathrm{supp}(\mathrm{Gr}_p M)$ are equal to $V(I_p)$. This implies that $\mathrm{supp}(M) = \bigcup_{p \in \mathbb{Z}} V(I_p)$. Let $m_1, m_2, \dots, m_s$ be a set of homogeneous generators of $\mathrm{Gr}\, D(n)$-module $\mathrm{Gr}\, M$. Then the annihilator I of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n]$ annihilates whole $\mathrm{Gr}\, M$. Therefore, there is a finite subset $\mathcal{S}$ of $\mathbb{Z}$ such that $\cap_{p \in \mathcal{S}} I_p = I \subset I_q$ for all $q \in \mathbb{Z}$. This implies that $\cup_{p \in \mathcal{S}} V(I_p) = V(I) \supset V(I_q)$ for all $q \in \mathbb{Z}$, and $\mathrm{supp}(M) = V(I)$. □

Let D be a filtered ring with a filtration $\mathrm{F}\,D$ satisfying the properties (i)-(vii) from the beginning of 3. Let M be a finitely generated D-module and $\mathrm{F}\,M$ a good filtration of M. Then $\mathrm{Gr}\,M$ is a graded $\mathrm{Gr}\,D$-module. Let I be the annihilator of $\mathrm{Gr}\,M$ in $\mathrm{Gr}\,D$. This is clearly a graded ideal in $\mathrm{Gr}\,D$. Hence, its radical $r(I)$ is also a graded ideal. In general, I depends on the choice of the good filtration on M, but we also have the following result.

7.2. Lemma. *Let M be a finitely generated D-module and $\mathrm{F}\,M$ and $\mathrm{F}'\,M$ two good filtrations on M. Let I, resp. I' be the annihilators of the corresponding graded $\mathrm{Gr}\,D$-modules $\mathrm{Gr}\,M$ and $\mathrm{Gr}'\,M$. Then $r(I) = r(I')$.*

Proof. Let $T \in r(I) \cap \mathrm{Gr}^p\,D$. Then there exists $s \in \mathbb{Z}_+$ such that $T^s \in I$. If we take $Y \in \mathrm{F}_p\,D$ such that $Y + \mathrm{F}_{p-1}\,D = T$, we get $Y^s\,\mathrm{F}_q\,M \subset \mathrm{F}_{q+sp-1}\,M$ for all $q \in \mathbb{Z}$. Hence, by induction we get

$$Y^{ms}\,\mathrm{F}_q\,M \subset \mathrm{F}_{q+msp-m}\,M$$

for all $m \in \mathbb{N}$ and $q \in \mathbb{Z}$. On the other hand, by 3.7, we know that $\mathrm{F}\,M$ and $\mathrm{F}'\,M$ are equivalent. Hence there exists $l \in \mathbb{Z}_+$ such that $\mathrm{F}_q\,M \subset \mathrm{F}'_{q+l}\,M \subset \mathrm{F}_{q+2l}\,M$ for all $q \in \mathbb{Z}$. This leads to

$$Y^{ms}\,\mathrm{F}'_q\,M \subset Y^{ms}\,\mathrm{F}_{q+l}\,M \subset \mathrm{F}_{q+l+msp-m}\,M \subset \mathrm{F}'_{q+2l+msp-m}\,M$$

for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$. If we take $m > 2l$, it follows that $Y^{ms}\,\mathrm{F}'_q\,M \subset \mathrm{F}'_{q+msp-1}\,M$ for any $q \in \mathbb{Z}$, i.e., $T^{ms} \in I'$. Therefore, $T \in r(I')$ and we have $r(I) \subset r(I')$. Since the roles of I and I' are symmetric we conclude that $r(I) = r(I')$. □

Therefore the radical of the annihilator of $\mathrm{Gr}\,M$ is independent of the choice of a good filtration on $\mathrm{Gr}\,M$. We call it the *characteristic ideal* of M and denote by $J(M)$.

Now we can apply this construction to $D(n)$. Since $\mathrm{Gr}\,D(n) = k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ by 5.6, we can define the closed algebraic set

$$Ch(M) = V(J(M)) \subset k^{2n}$$

which we call the *characteristic variety* of M.

Since $J(M)$ is a homogeneous ideal in last n variables, we immediately obtain the following result.

7.3. Lemma. *The characteristic variety $Ch(M)$ of a finitely generated $D(n)$-module M has the following property: if $(x, \xi) \in Ch(M)$ then $(x, \lambda\xi) \in Ch(M)$ for any $\lambda \in k$.*

We say that $Ch(M)$ is a *conical* variety.

7.4. Proposition. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated $D(n)$-modules. Then

$$Ch(M) = Ch(M') \cup Ch(M'').$$

Proof. Let $\mathrm{F}\,M$ be a good filtration on M. Then it induces a filtration $\mathrm{F}\,M'$ on M' and $\mathrm{F}\,M''$ on M''. By 3.8 we know that these filtrations are also good. Moreover, we have the exact sequence

$$0 \longrightarrow \mathrm{Gr}\,M' \longrightarrow \mathrm{Gr}\,M \longrightarrow \mathrm{Gr}\,M'' \longrightarrow 0$$

of finitely generated $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$-modules, and their supports are, by 4.2, the characteristic varieties of $D(n)$-modules M, M' and M'' respectively. Therefore the assertion follows from 4.1. □

The next two results shed some light on the relationship between the characteristic variety and the support of a finitely generated $D(n)$-module.

Let $\pi : k^{2n} \longrightarrow k^n$ be the map defined by $\pi(x, \xi) = x$ for any $x, \xi \in k^n$.

7.5. PROPOSITION. *Let M be a finitely generated $D(n)$-module. Then* $\operatorname{supp}(M) = \pi(Ch(M))$.

PROOF. Denote by $m_1, m_2, \dots, m_s$ a set of homogeneous generators of $\operatorname{Gr} M$. Then, as in the proof of 7.1, the annihilator I of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n]$ satisfies $\operatorname{supp}(M) = V(I)$. On the other hand, if J is the annihilator of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$, it is a homogeneous ideal in $\xi_1, \xi_2, \dots, \xi_n$ which satisfies $I = k[X_1, X_2, \dots, X_n] \cap J$, and $Ch(M) = V(J)$. This implies that $x \in V(I) = \operatorname{supp}(M)$ is equivalent with $(x, 0) \in V(J) = Ch(M)$. Since $Ch(M)$ is conical this implies the assertion. □

Let M be a finitely generated $D(n)$-module. Define the *singular support* of M as

$$\operatorname{sing\,supp}(M) = \{x \in k^n \mid (x, \xi) \in Ch(M) \text{ for some } \xi \neq 0\}.$$

Clearly, we have $\operatorname{sing\,supp}(M) \subset \operatorname{supp}(M)$.

7.6. LEMMA. *Let M be a finitely generated $D(n)$-module. Then* $\operatorname{sing\,supp}(M)$ *is a closed subvariety of* $\operatorname{supp}(M)$.

PROOF. Let $p : k^n - \{0\} \longrightarrow \mathbb{P}^{n-1}(k)$ be the natural projection. Then

$$1 \times p : k^n \times (k^n - \{0\}) \longrightarrow k^n \times \mathbb{P}^{n-1}(k)$$

projects $Ch(M) - (k^n \times \{0\})$ onto the closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$ corresponding to the ideal $J(M)$ which is homogeneous in $\xi_1, \xi_2, \dots, \xi_n$. Finally, the projection to the first factor $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ maps it onto $\operatorname{sing\,supp}(M)$. Since $\mathbb{P}^{n-1}(k)$ is a complete variety, the projection $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ is a closed map. Therefore, $\operatorname{sing\,supp}(M)$ is closed. □

The fundamental result about characteristic varieties is the following theorem. It also gives a geometric description of the Bernstein dimension.

7.7. THEOREM. *Let M be a finitely generated $D(n)$-module. Then*

$$\dim Ch(M) = d(M).$$

To prove the theorem we need some preparation.

Let $A = k[X_1, X_2, \dots, X_n]$. Let $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}^n$. We define the grading $\operatorname{Gr}^{(\mathbf{t})} A$ by putting $\operatorname{Gr}^{(\mathbf{t})}_m A$ to be the linear span of X^I such that $\sum_{j=1}^n t_j i_j = m$. Clearly, in this way A becomes a graded ring. Moreover, we can define the corresponding filtration $\mathrm{F}^{(\mathbf{t})} A$ by $\mathrm{F}^{(\mathbf{t})}_p A = \sum_{m \leq p} \operatorname{Gr}^{(\mathbf{t})}_m A$. Clearly, if we denote by $\mathrm{F} A$ the natural filtration of A by degree of polynomials and put $t = \max_{1 \leq i \leq n} t_i$, we have

$$\mathrm{F}^{(\mathbf{t})}_p A \subset \mathrm{F}_p A \text{ and } \mathrm{F}_p A \subset \mathrm{F}^{(\mathbf{t})}_{tp} A$$

for any $p \in \mathbb{Z}$.

Let I be an ideal in A. Then we can consider the exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

of A-modules equipped with the filtrations induced by the filtrations on A. Then we have

$$\mathrm{F}_p^{(\mathbf{t})}(A/I) \subset \mathrm{F}_p(A/I) \text{ and } \mathrm{F}_p(A/I) \subset \mathrm{F}_{tp}^{(\mathbf{t})}(A/I)$$

for any $p \in \mathbb{Z}$. This in turn implies the following lemma.

7.8. Lemma. *For any $p \in \mathbb{Z}$, we have*

$$\dim_k \mathrm{F}_p^{(\mathbf{t})}(A/I) \leq \dim_k \mathrm{F}_p(A/I) \textit{ and } \dim_k \mathrm{F}_p(A/I) \leq \dim_k \mathrm{F}_{tp}^{(\mathbf{t})}(A/I).$$

Let $s \in \mathbb{N}$. Then we define a filtration $\mathrm{F}^{(s)} D(n)$ of the algebra $D(n)$ by

$$\mathrm{F}_m^{(s)} D(n) = \Big\{ T \in D(n) \ \Big| \ T = \sum_{|I|+s|J| \leq m} c_{I,J} X^I \partial^J, \ c_{I,J} \in k \Big\}.$$

Clearly, $\mathrm{F}^{(1)} D(n)$ is the Bernstein filtration of $D(n)$. The filtrations $\mathrm{F}^{(s)} D(n)$ have the properties (i)-(iii) of the ring filtrations considered in §3. Moreover, $T \in \mathrm{F}_m^{(s)} D(n)$ if and only if $T \in \mathrm{F}_p^{(1)} D(n)$ and the order of T is $\leq q$ for some p and q satisfying $m = p + (s-1)q$. Therefore, if $T \in \mathrm{F}_m^{(s)} D(n)$ and $S \in \mathrm{F}_{m'}^{(s)} D(n)$, there exist p, p' and q, q' such that $m = p + (s-1)q$ and $m' = p' + (s-1)q'$, $T \in \mathrm{F}_p^{(1)} D(n)$, $S \in \mathrm{F}_{p'}^{(1)} D(n)$, and the orders of T and S are $\leq q$ and $\leq q'$ respectively. This implies that the order of TS is $\leq q + q'$ and $TS \in \mathrm{F}_{p+p'}^{(1)} D(n)$. It follows that $TS \in \mathrm{F}_{m+m'}^{(s)} D(n)$. Hence, the filtration $\mathrm{F}^{(s)} D(n)$ satisfies also (iv), i.e., it is a ring filtration. In the same way we can check that (v) holds, i.e., the graded ring $\mathrm{Gr}^{(s)} D(n)$ is commutative. Moreover, the graded ring $\mathrm{Gr}^{(s)} D(n)$ is isomorphic to the graded ring $A = k[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$ with the graded structure corresponding to $\mathbf{s} = (1, \ldots, 1, s, \ldots, s)$. We denote that graded module by $\mathrm{Gr}^{(s)} A$ and its associated filtration by $\mathrm{F}^{(s)} A$.

Moreover, we have

$$\mathrm{F}_p^{(s)} D(n) \subset \mathrm{F}_p^{(1)} D(n) \text{ and } \mathrm{F}_p^{(1)} D(n) \subset \mathrm{F}_{sp}^{(s)} D(n)$$

for any $p \in \mathbb{Z}$.

Let L be an ideal in $D(n)$. Then we can consider the exact sequence

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow D(n)/L \longrightarrow 0$$

of $D(n)$-modules equipped with the filtrations induced by the filtrations on $D(n)$. Then we have

$$\mathrm{F}_p^{(s)}(D(n)/L) \subset \mathrm{F}_p^{(1)}(D(n)/L) \text{ and } \mathrm{F}_p^{(1)}(D(n)/L) \subset \mathrm{F}_{sp}^{(s)}(D(n)/L)$$

for any $p \in \mathbb{Z}$. This in turn implies the following lemma analogous to 7.8.

7.9. Lemma. *For any $p \in \mathbb{Z}$, we have*

$$\dim_k \mathrm{F}_p^{(s)}(D(n)/L) \leq \dim_k \mathrm{F}_p^{(1)}(D(n)/L)$$

and

$$\dim_k \mathrm{F}_p^{(1)}(D(n)/L) \leq \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L).$$

7.10. LEMMA. *Let L be a left ideal in $D(n)$. Then*

$$d(D(n)/L) = \dim V(\operatorname{Gr}^{(s)} L)$$

for any $s \in \mathbb{N}$.

PROOF. The exact sequence

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow D(n)/L \longrightarrow 0,$$

where $D(n)$ is equipped with the filtration $\mathrm{F}^{(s)} D(n)$ and L and $D(n)/L$ with the induced filtrations $\mathrm{F}^{(s)} L$ and $\mathrm{F}^{(s)}(D(n)/L)$ respectively, leads to the exact sequence

$$0 \longrightarrow \operatorname{Gr}^{(s)} L \longrightarrow \operatorname{Gr}^{(s)} D(n) \longrightarrow \operatorname{Gr}^{(s)}(D(n)/L) \longrightarrow 0.$$

This implies that

$$\begin{aligned}\dim_k \mathrm{F}_p^{(s)}(D(n)/L) &= \sum_{q=0}^{p} (\dim_k \mathrm{F}_q^{(s)}(D(n)/L) - \dim_k \mathrm{F}_{q-1}^{(s)}(D(n)/L)) \\ &= \sum_{q=0}^{p} \dim \operatorname{Gr}_q^{(s)}(D(n)/L) = \sum_{q=0}^{q} (\dim_k \operatorname{Gr}_q^{(s)} D(n) - \dim_k \operatorname{Gr}_q^{(s)} L) \\ &= \sum_{q=0}^{p} (\dim \operatorname{Gr}_q^{(\mathbf{s})} A - \dim \operatorname{Gr}_q^{(s)} L) = \sum_{q=0}^{p} \dim \operatorname{Gr}_q^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) \\ &= \dim_k \mathrm{F}_p^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L)\end{aligned}$$

for any $p \in \mathbb{Z}$. This in turn implies, using 7.8 and 7.9 that

$$\begin{aligned}\dim_k \mathrm{F}_p^{(1)}(D(n)/L) \leq \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L) &= \dim_k \mathrm{F}_{sp}^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) \\ &\leq \dim_k \mathrm{F}_{sp}(A/\operatorname{Gr}^{(s)} L)\end{aligned}$$

and

$$\begin{aligned}\dim_k \mathrm{F}_p(A/\operatorname{Gr}^{(s)} L) \leq \dim_k \mathrm{F}_{sp}^{(\mathbf{s})}(A/\operatorname{Gr}^{(s)} L) &= \dim_k \mathrm{F}_{sp}^{(s)}(D(n)/L) \\ &\leq \dim_k \mathrm{F}_{sp}^{(1)}(D(n)/L).\end{aligned}$$

Since the functions $p \longmapsto \dim_k \mathrm{F}_p^{(1)}(D(n)/L)$ and $p \longmapsto \dim_k \mathrm{F}_p(A/\operatorname{Gr}^{(s)} L)$ are represented by polynomials for large $p \in \mathbb{Z}$, these polynomials have to have equal degrees. This in turn implies that $d(D(n)/L) = d(A/\operatorname{Gr}^{(s)} L)$. □

For any $s \in \mathbb{N}$, we denote by $\sigma_p^{(s)}(T)$ the projection of $T \in F_p^{(s)} D(n)$ in $\operatorname{Gr}_p^{(s)} D(n) = A$. Also, for the natural filtration on A given by the degree of the polynomials, we denote by σ_p the map which attaches to a polynomial of degree p its homogeneous component of degree p.

7.11. EXAMPLE. Let $D = D(1)$ and $T \in D$ given by $T = x^3\partial + \partial^2$. Then the degree of T is equal to 2 and $\operatorname{Symb}_2(T) = \xi^2$. Hence, $\sigma_2(\operatorname{Symb}_2(T)) = \xi^2$.

On the other hand, we have $\sigma_4^{(1)}(T) = x^3\xi$; $\sigma_5^{(2)}(T) = x^3\xi$, $\sigma_6^{(3)}(T) = x^3\xi + \xi^2$ and $\sigma_{2s}^{(s)}(T) = \xi^2$ for $s > 3$.

Hence, for large s, the $\sigma^{(s)}(T)$ becomes equal to $\sigma(\operatorname{Symb}(T))$. This holds in general, more precisely we have the following result.

7.12. LEMMA. *Let T be a differential operator in $D(n)$ of order $\leq m$ such that its symbol $\operatorname{Symb}_m(T)$ is a polynomial of degree p. Then there exists s_0 such that*

$$\sigma_p(\operatorname{Symb}_m(T)) = \sigma^{(s)}_{p+(s-1)m}(T)$$

for $s \geq s_0$.

PROOF. By our assumption

$$T = \sum_{|J|\leq m} c_{I,J} X^I \partial^J.$$

Also, we can fix q_0 such that $c_{I,J} \neq 0$ implies that $|I| \leq q_0$. Then we have

$$\operatorname{Symb}_m(T) = \sum_{|J|=m} c_{I,J} X^I \xi^J$$

is a polynomial of degree p and its leading term is

$$\sigma_p(\operatorname{Symb}_m(T)) = \sum_{|I|=p-m, |J|=m} c_{I,J} X^I \xi^J.$$

On the other hand, the terms $X^I \partial^J$ are in $\mathrm{F}^{(s)}_{|I|+s|J|}\, D(n)$. Assume that $c_{I,J} \neq 0$. Then we have the following possibilities:

(i) $|J| = m$ and $|I| = p - m$: $X^I \partial^J$ is in $\mathrm{F}^{(s)}_{p+(s-1)m}\, D(n)$.

(ii) $|J| = m$ and $|I| < p - m$: $X^I \partial^J$ in $\mathrm{F}^{(s)}_{p+(s-1)m-1}\, D(n)$.

(iii) $m \geq 1$, $|J| < m$ and $|I| \leq q_0$: $X^I \partial^J$ is in $\mathrm{F}^{(s)}_{q_0+s(m-1)}\, D(n)$. Moreover,

$$q_0 + s(m-1) = q_0 + sm - s = q_0 + m - s + (s-1)m.$$

Hence, if $s \geq s_0 = q_0 + m - p + 1$, we have $q_0 + s(m-1) \leq p + (s-1)m - 1$. It follows that in this case the differential operator $X^I \partial^J$ is also in $\mathrm{F}^{(s)}_{p+(s-1)m-1}\, D(n)$.

This implies that for $s \geq s_0$ we have

$$\begin{aligned}\sigma^{(s)}_{p+(s-1)m}(T) = \sigma^{(s)}_{p+(s-1)m} &\left(\sum_{|I|=p-m, |J|=m} c_{I,J} X^I \partial^J \right) \\ &= \sum_{|I|=p-m, |J|=m} c_{I,J} X^I \xi^J = \sigma_p(\operatorname{Symb}_m(T)).\end{aligned}$$

□

In particular, if L is a left ideal in $D(n)$, we have the following consequence.

7.13. COROLLARY. *Let L be a left ideal in $D(n)$. Then there exists $s_0 \in \mathbb{Z}_+$ such that $\operatorname{Gr}(\operatorname{Gr} L) = \operatorname{Gr}^{(s)} L$ for $s \geq s_0$.*

PROOF. Since L is finitely generated, there exist $T_1, T_2, \dots, T_q \in L$ which generate L. This implies that the symbols $\operatorname{Symb}(T_1), \operatorname{Symb}(T_2), \dots, \operatorname{Symb}(T_q)$ generate $\operatorname{Gr} L$ and $\sigma^{(s)}(T_1), \sigma^{(s)}(T_2), \dots, \sigma^{(s)}(T_q)$ generate $\operatorname{Gr}^{(s)} L$. In addition, we see that $\sigma(\operatorname{Symb}(T_1)), \sigma(\operatorname{Symb}(T_2)), \dots, \sigma(\operatorname{Symb}(T_q))$ generate $\operatorname{Gr}(\operatorname{Gr} L)$. Hence the assertion follows from 7.12. □

Now we can prove 7.7. Assume first that $M = D(n)/L$ where L is a left ideal in $D(n)$. Then, by 7.10, we have $d(M) = \dim V(\operatorname{Gr}^{(s)} L)$ for any $s \in \mathbb{N}$. In addition, if s is large enough, by 7.13, we have $d(M) = \dim V(\operatorname{Gr}(\operatorname{Gr} L))$. finally, by 4.8, we have $\dim V(\operatorname{Gr}(\operatorname{Gr} L)) = \dim V(\operatorname{Gr} L)$. On the other hand, the exact sequence of $D(n)$-modules

$$0 \longrightarrow L \longrightarrow D(n) \longrightarrow M \longrightarrow 0$$

leads to the exact sequence

$$0 \longrightarrow \operatorname{Gr} L \longrightarrow \operatorname{Gr} D(n) \longrightarrow \operatorname{Gr} M \longrightarrow 0$$

of A-modules, where $A = \operatorname{Gr} D(n) = k[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$. Therefore, $\operatorname{Gr} M$ is the quotient $A/\operatorname{Gr} L$ and the annihilator of $\operatorname{Gr} M$ is equal to $\operatorname{Gr} L$. Hence, by definition, $V(\operatorname{Gr} L)$ is the characteristic variety of M. This proves the equality in this case.

To prove the general result we consider the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where M has q generators, M' has $q-1$ generators and M'' is cyclic. Therefore, M' is isomorphic to $D(n)/L$ for some left ideal L.

By the first part of the proof, we have $d(M'') = \dim Ch(M'')$. In addition, by the induction assumption, we have $d(M') = \dim Ch(M')$. From 3.9 and 7.4 we see that

$$\begin{aligned} d(M) = \max(d(M'), d(M'')) &= \max(\dim Ch(M'), \dim Ch(M'')) \\ &= \dim(Ch(M') \cup Ch(M'')) = \dim Ch(M). \end{aligned}$$

This completes the proof of 7.7.

In particular, by combining 6.3 and 7.7, we get the following result.

7.14. THEOREM. *Let M be a finitely generated $D(n)$-module, $M \neq 0$, and $Ch(M)$ its characteristic variety. Then* $\dim Ch(M) \geq n$.

8. Holonomic modules

Let M be a nontrivial finitely generated $D(n)$-module. Then, by 7.14, the dimension of its characteristic variety $Ch(M)$ is $\geq n$.

We say that a finitely generated $D(n)$-module is *holonomic* if the dimension of its characteristic variety $Ch(M)$ is $\leq n$. Therefore, M is holonomic if either $M = 0$ or $\dim Ch(M) = n$.

Roughly speaking, holonomic modules are the modules with smallest possible characteristic varieties.

The following result is the fundamental observation about holonomic modules.

8.1. THEOREM. (i) *Holonomic modules are of finite length.*
(ii) *Submodules, quotient modules and extensions of holonomic modules are holonomic.*

PROOF. (ii) follows immediately from 3.9.

(i) Let M be a holonomic $D(n)$-module different from zero. Then, by definition, its the dimension of its characteristic variety $Ch(M)$ is equal to n. By 7.7, its Bernstein dimension $d(M)$ is also equal to n. Since M is finitely generated and

$D(n)$ is a nötherian ring, there exists a maximal $D(n)$-submodule M' of M different from M. Therefore we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0.$$

By (ii), M' and M/M' are holonomic and M/M' is an irreducible $D(n)$-module. If $M' \neq 0$, we conclude from 3.9 that $e(M') < e(M)$. Therefore, by induction in $e(M)$, it follows that M has finite length. □

Therefore, the full subcategory $\mathcal{H}ol(D(n))$ of the category $\mathcal{M}_{fg}(D(n))$ is closed under taking submodules, quotient modules and extensions. Moreover, if we denote by $\mathcal{M}_{fl}(D(n))$, the full subcategory of $\mathcal{M}_{fg}(D(n))$ consisting of $D(n)$-modules of finite length, we see that $\mathcal{H}ol(D(n))$ is a subcategory of $\mathcal{M}_{fl}(D(n))$. One can show that $\mathcal{H}ol(D(n))$ is strictly smaller than $\mathcal{M}_{fl}(D(n))$ for $n > 1$.

In addition, the transpose functor and the Fourier functor map holonomic modules into holonomic modules.

Now we are going to discuss some examples of holonomic modules.

8.2. Example. Let $O_n = k[X_1, X_2, \dots, X_n]$. Then $O_n = D(n)/(D(n)(\partial_1, \partial_2, \dots, \partial_n))$ is a finitely generated $D(n)$-module. Moreover, if we put $\mathrm{F}_p\, O_n = 0$ for $p < 0$ and $\mathrm{F}_p\, O_n = O_n$ for $p \geq 0$, the filtration $\mathrm{F}\, O_n$ is a good filtration for the degree filtration of $D(n)$. The corresponding graded module $\mathrm{Gr}\, O_n$ is such that $\mathrm{Gr}^p\, O_n = 0$ for $p \neq 0$ and $\mathrm{Gr}^0\, O_n = k[X_1, X_2, \dots, X_n]$. It follows that the annihilator of $\mathrm{Gr}\, O_n$ is equal to the ideal in $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ generated by $\xi_1, \xi_2, \dots, \xi_n$. This implies that $Ch(O_n) = k^n \times \{0\} \subset k^{2n}$. In particular, $\dim Ch(O_n) = n$ and O_n is holonomic. Moreover, $\mathrm{supp}(O_n) = k^n$ and the projection $\pi : k^{2n} \longrightarrow k^n$ is an bijection of $Ch(O_n)$ onto O.

By differentiation, we see that any submodule of O_n has to contain contants. Therefore, O_n is irreducible.

8.3. Example. Consider now $\Delta_n = \mathcal{F}(O_n)$. Then, we have $\Delta_n = D(n)/(D(n)(X_1, X_2, \dots, X_n))$. Clearly, Δ_n is holonomic and irreducible. Let δ be the vector corresponding to $1 \in O_1$. Then $X_i \delta = 0$ for any $1 \leq i \leq n$. Clearly, Δ_n is spanned by $\delta^{(I)} = \partial^I \delta$, $I \in \mathbb{Z}_+^I$. Let $\mathrm{F}\, \Delta_n$ be a filtration of Δ_n such that: $\mathrm{F}_p\, \Delta_n = \{0\}$ for $p < 0$ and $\mathrm{F}_p\, \Delta_n$ is spanned by $\delta^{(I)}$, $|I| \leq p$, for $p \geq 0$. Denote be ϵ_i the multiindex $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 ant i-th position. Then, by the definition of the Fourier transform, we have

$$\partial_j \delta^{(I)} = \delta^{(I+\epsilon_j)} \text{ and } X_j \delta^{(I)} = -i_j \delta^{(I-\epsilon_j)}$$

for all $1 \leq j \leq n$ and $I \in \mathbb{Z}_+^I$. This implies that $\mathrm{F}_p\, \Delta_n$ are $k[X_1, X_2, \dots, X_n]$-submodules of Δ_n. Moreover, $\partial_i\, \mathrm{F}_p\, \Delta_n \subset \mathrm{F}_{p+1}\, \Delta_n$ for all $1 \leq i \leq n$ and $p \in \mathbb{Z}$. Hence, $\mathrm{F}\, \Delta_n$ is an exhaustive $D(n)$-module filtration for $D(n)$ filtered by the order of differential operators. Let $\bar{\delta}^{(I)}$ be the cosets represented by $\delta^{(I)}$ in $\mathrm{Gr}^{|I|}\, \Delta_n$. Then $\mathrm{Gr}^p\, \Delta_n$ is spanned by $\bar{\delta}^{(I)}$ for $I \in \mathbb{Z}_+$ such that $|I| = p$. Clearly, X_i act as 0 on $\mathrm{Gr}\, \Delta_n$, and ξ_i map $\bar{\delta}^{(I)}$ into $\bar{\delta}^{(I+\epsilon_i)}$. Therefore, $\bar{\delta}$ generates $\mathrm{Gr}\, \Delta_n$ as a $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$-module, and $\mathrm{F}\, \Delta_n$ is a good filtration. Moreover, the annihilator of $\mathrm{Gr}\, \Delta_n$ is the ideal generated by X_i, $1 \leq i \leq n$. Hence the characteristic variety of Δ_n is $Ch(\Delta_n) = \{0\} \times k^n \subset k^{2n}$. The support $\mathrm{supp}(\Delta_n)$ of Δ_n is $\{0\} \subset k^n$.

Now we want to construct more holonomic modules. We start with a simple criterion for holonomicity.

8.4. Lemma. *Let $D(n)$ be equipped with the Bernstein filtration. Let M be a $D(n)$-module and $\mathrm{F}\, M$ an exhaustive $D(n)$-module filtration on M. If*

$$\dim_k \mathrm{F}_p M \leq \frac{c}{n!}\, p^n + (\text{lower order terms in } p)$$

for all $p \in \mathbb{Z}_+$, M is a holonomic $D(n)$-module and its length is $\leq c$.

In particular, M is a finitely generated $D(n)$-module.

Proof. Let N be a finitely generated $D(n)$-submodule of M. Then $\mathrm{F}\, M$ induces an exhaustive $D(n)$-module filtration on N. By 3.6 there exists a good filtration $\mathrm{F}'\, N$ of N and $s \in \mathbb{Z}_+$ such that $\mathrm{F}'_p N \subset \mathrm{F}_{p+s} N$ for any $p \in \mathbb{Z}$. It follows that

$$\dim_k \mathrm{F}'_p N \leq \dim_k \mathrm{F}_{p+s} N \leq \dim_k \mathrm{F}_{p+s} M \leq \frac{c}{n!}\, p^n + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$. Therefore, $d(N) \leq n$ and N is holonomic. If $N \neq 0$, we have $e(N) \leq c$. Clearly this implies that the length of N is $\leq e(N) \leq c$. It follows that any increasing sequence of finitely generated $D(n)$-submodules of M stabilizes, and that M itself is finitely generated. □

8.5. Example. Let $n = 1$ and put $D = D(1)$. Consider the D-modules $M_\alpha = D/D(z\partial - \alpha)$ for any $\alpha \in k$.

Let $E = z\partial$. As in the proof of 5.10, we see that the operators $(z^p E^q, \partial^p E^q\,;\, p, q \in \mathbb{Z}_+)$ form a basis of D as a linear space over k. Moreover, the ideal $D(z\partial - \alpha)$ is spanned by the elements $(z^p E^q(E - \alpha), \partial^p E^q(E - \alpha)\,;\, p, q \in \mathbb{Z}_+)$. Hence, M_α is spanned by the cosets corresponding to $(z^p, \partial^p\,;\, p \in \mathbb{Z}_+)$.

Clearly,

$$[E, z] = z\partial z - z^2\partial = z$$

and

$$[E, \partial] = z\partial^2 - \partial z\partial = -\partial.$$

Therefore, we have

$$Ez = z(E + 1) \text{ and } E\partial = \partial(E - 1).$$

This immediately implies that the coset of z^n is an eigenvector of E with eigenvalue $\alpha + n$ for any $n \in \mathbb{Z}_+$. On the other hand, the coset of ∂^n is an eigenvector of E with eigenvalue $\alpha - n$ for any $n \in \mathbb{Z}_+$. Therefore, the spectrum of E on M_α is equal to $\{\alpha + n\,;\, n \in \mathbb{Z}\}$, and the multiplicity of each eigenvalue is equal to 1.

The Fourier transform of M_α is isomorphic to

$$D/D(-\partial z - \alpha) = D/D(z\partial + \alpha + 1) = M_{-\alpha-1}.$$

Assume first that $\alpha \notin \mathbb{Z}$. Then, E is a linear isomorphism and z must be surjective. Since z maps the eigenspace for the eigenvalue $\alpha + n$ onto the eigenspace for the eigenvalue $\alpha + n + 1$, z is also injective. Therefore, we can construct inductively a family of vectors $z^{\alpha+n}$, $n \in \mathbb{Z}$, such that $Ez^{\alpha+n} = (\alpha + n)z^{\alpha+n}$ and $zz^{\alpha+n} = z^{\alpha+n+1}$. Clearly, these vectors form a basis of M_α. Moreover,

$$\partial z^{\alpha+n} = \partial z z^{\alpha+n-1} = [\partial, z]z^{\alpha+n-1} + Ez^{\alpha+n-1} = (\alpha + n)z^{\alpha+n-1}$$

for any $n \in \mathbb{Z}$. This immediately implies that $M_\alpha \cong M_{\alpha+p}$ for any integer $p \in \mathbb{Z}$.

Moreover, any nonzero D-submodule of M is invariant under E, so it contains an eigenvector of E. This in turn implies that it contains $z^{\alpha+p}$ for some $p \in \mathbb{Z}$. It follows that it contains all $z^{\alpha+n}$, $n \in \mathbb{Z}$, i.e., it is equal to M_α. Hence, M_α are irreducible D-modules.

We define a filtration $\mathrm{F}\, M_\alpha$ of M_α by: $\mathrm{F}_p\, M_\alpha = \{0\}$ for $p < 0$; and $\mathrm{F}_p\, M_\alpha$ is the span of $\{z^{\alpha+n}\,;\, |n| \leq p\}$ for $p \geq 0$. Clearly, $\mathrm{F}\, M_\alpha$ is an increasing exhaustive filtration of M_α by linear subspaces. Moroever, by the above remarks, $z\, \mathrm{F}_p\, M_\alpha \subset \mathrm{F}_{p+1}\, M_\alpha$ and $\partial\, \mathrm{F}_p\, M_\alpha \subset \mathrm{F}_{p+1}\, M_\alpha$ for any $p \in \mathbb{Z}$. Therefore, $\mathrm{F}\, M_\alpha$ is a D-module filtration for D equipped by Bernstein filtration. Since $\dim_k \mathrm{F}_p\, M_\alpha = 2p + 1$ for $p \geq 0$, by 8.4, we see that M_α is holonomic.

To calculate its characteristic variety, consider the another filtration $\mathrm{F}\, M_\alpha$ such that $\mathrm{F}_n\, M_\alpha = \{0\}$ for $n < 0$ and $\mathrm{F}_n\, M_\alpha$ is spanned by $\{z^{\alpha+p}\,;\, p \geq -n\}$ for $n \geq 0$. Clearly, this is an exhaustive fitration of $\mathrm{F}\, M_\alpha$ by modules over the ring of polynomials in z. Moreover, $\partial\, \mathrm{F}_p\, M_\alpha = \mathrm{F}_{p+1}\, M_\alpha$, for any $p \in \mathbb{Z}_+$, and this a a good D-module filtration for the filtration of D by the order of differential operators. The graded module $\mathrm{Gr}\, M_\alpha$ is a direct sum of $\mathrm{Gr}^p\, M_\alpha$, where $\mathrm{Gr}^n\, M_\alpha = 0$ for $n < 0$; $\mathrm{Gr}^0\, M_\alpha$ is equal to the span of $z^{\alpha+p}$ for $p \geq 0$; and $\mathrm{Gr}^p\, M_\alpha$ is spanned by the coset of $z^{\alpha-p}$ modulo $\{z^{\alpha+q}\,;\, q > -p\}$. Therefore, z annihilates $\mathrm{Gr}^p\, M_\alpha$ for $p \neq 0$, and the symbol ξ of ∂ annihilates $\mathrm{Gr}^0\, M_\alpha$ and maps $\mathrm{Gr}^p\, M_\alpha$ onto $\mathrm{Gr}^{p+1}\, M_\alpha$ for $p > 0$. It follows that the annihilator of $\mathrm{Gr}\, M_\alpha$ is the ideal generated by $z\xi$ in $k[z, \xi]$. Hence, the characteristic variety $Ch(M_\alpha)$ is the union of lines $\{z = 0\}$ and $\{\xi = 0\}$ in k^2.

Assume now that $\alpha \in \mathbb{Z}$. Then the eigenvalues of E are integers. If v is a nonzero eigenvector of E for an eigenvalue $m \neq 0$, ∂v is an eigenvector of E for eigenvalue $m - 1$ and $z\partial v = mv \neq 0$. Therefore, z maps all eigenspaces of E with eigenvalues $q \neq -1$ onto the eigenspaces for the eigenvalue $q + 1$.

Assume first that $n = -\alpha > 0$. Then the coset of z^{n-1}, is an eigenvector of E for the eigenvalue -1. Therefore, z maps the eigenspace of E for eigenvalue -1 onto the eigenspace for the eigenvalue 0. Hence, in this case, we can select basis vectors v^m for the eigenspaces for the eigenvalues $m \in \mathbb{Z}$, such that $zv^m = v^{m+1}$ for $m \in \mathbb{Z}$. We have

$$\partial v^m = \partial z v^{m-1} = [\partial, z] v^{m-1} + E v^{m-1} = m v^{m-1}$$

for all $m \in \mathbb{Z}$. This implies that all M_{-n}, $n > 0$, are mutually isomorphic.

Moreover, by an inspection of the action of z and ∂, we see that the vectors v^m, $m \in \mathbb{Z}_+$, span a D-submodule N_{-n} isomorphic to O_1 from 8.2. In particular M_{-n} is reducible.

Clearly, $zv_{-1} \in N_{-n}$. It follows that the coset $\delta \in M_{-n}/N_{-n}$ of v_{-1} satisfies $z\delta = 0$.

The spectrum of E on $L_{-n} = M_{-n}/N_{-n}$ consists of all strictly negative integers. Therefore, ∂ is injective on L_{-n}. Hence $\delta^{(m)} = \partial^m \delta$ are nonzero eigenvectors of E for eigenvalues $-(m + 1)$, $m \in \mathbb{Z}_+$, i.e., they are proportional to the cosets of $v_{-(m+1)}$. Clearly, we have

$$z\delta^{(m)} = z\partial\delta^{(m-1)} = -m\delta^{(m-1)}$$

for all $m > 0$. It follows immediately that L_{-n} is isomorphic to the D-module Δ_1 described in 8.3. Hence we have the exact sequence

$$0 \longrightarrow O_1 \longrightarrow M_{-n} \longrightarrow \Delta_1 \longrightarrow 0$$

and this exact sequence doesn't split. In particular, all these D-modules are isomorphic to M_{-1}.

By Fourier transform, we see that D-modules M_n, $n \geq 0$, are isomorphic to M_0. Moreover, we have the exact sequence

$$0 \longrightarrow \Delta_1 \longrightarrow M_{-n} \longrightarrow O_1 \longrightarrow 0$$

which also doesn't split.

Since O_1 and Δ_1 are holonomic by 8.2 and 8.3, by 8.1 we see that M_n, $n \in \mathbb{Z}$, are holonomic. Moreover, from 7.4 we conclude that we have

$$Ch(M_n) = Ch(O_1) \cup Ch(\Delta_1)$$

for all $n \in \mathbb{Z}$. Hence, by 8.2 and 8.3, they are equal to the union of lines $\{z = 0\}$ and $\{\xi = 0\}$ in k^2.

From the above example we see that the characteristic varieties do not determine the corresponding D-modules. Moreover, the characteristic variety of an irreducible holonomic $D(n)$-module can be reducible.

Now we are going to generalize the construction of the module M_{-1} from the above example.

Let M be a $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then on the localization M_P of M we can define k-linear maps $\partial_i : M_P \longrightarrow M_P$ by

$$\partial_i(\frac{m}{P^p}) = -p\partial_i(P)\frac{m}{P^{p+1}} + \frac{\partial_i m}{P^p}$$

for any $m \in M$ and $p \in \mathbb{Z}_+$. By direct calculation we can check that

$$[\partial_i, \partial_j](\frac{m}{P^p}) = 0$$

and

$$[\partial_i, x_j](\frac{m}{P^p}) = \delta_{ij}\frac{m}{P^p}$$

for any $1 \leq i, j \leq n$ and $p \in \mathbb{Z}_+$. By 5.11 this defines a structure of $D(n)$-module on M_P.

8.6. Proposition. *Let M be a holonomic $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then M_P is a holonomic $D(n)$-module.*

Proof. We can clearly assume that $P \neq 0$. Let $\mathrm{F}\, M$ be a good filtration on M such that $\mathrm{F}_p\, M = 0$ for $p \leq 0$ and $m = \deg P$. Define $\mathrm{F}_p\, M_P = 0$ for $p < 0$ and

$$\mathrm{F}_p\, M_P = \left\{ \frac{v}{P^p} \;\middle|\; v \in F_{(m+1)p}M \right\}$$

for $p \in \mathbb{Z}_+$. Clearly $\mathrm{F}_p\, M_P$, $p \in \mathbb{Z}$, are vector subspaces of M_P.

Let $w \in \mathrm{F}_p\, M_P$, $p \geq 0$. Then $w = \frac{v}{P^p} = \frac{Pv}{P^{p+1}}$ for some $v \in \mathrm{F}_{(m+1)p}\, M$. Since $Pv \in \mathrm{F}_{(m+1)p+m}\, M \subset \mathrm{F}_{(m+1)(p+1)}\, M$, we see that $w \in \mathrm{F}_{p+1}\, M_P$. This proves that the filtration $\mathrm{F}\, M_P$ is increasing.

Let $v \in \mathrm{F}_q\, M$. Then $\frac{v}{P^p} = \frac{P^s v}{P^{p+s}}$ for any $s \in \mathbb{Z}_+$. Also, $P^s v \in \mathrm{F}_{q+sm}\, M$ for any $s \in \mathbb{Z}_+$. Moreover, $(m+1)(p+s) - (q+sm) = s + (m+1)p - q \geq 0$ for $s \geq q - (m+1)p$. Hence

$$P^s v \in \mathrm{F}_{q+sm}\, M \subset \mathrm{F}_{(m+1)(p+s)}\, M$$

and $\frac{v}{P^p} \in \mathrm{F}_{p+s}\, M_P$. Therefore, the filtration $\mathrm{F}\, M_P$ is exhaustive.

It remains to show that it is a $D(n)$-module filtration. First, for $v \in \mathrm{F}_{(m+1)p}\, M$, $x_i P v \in \mathrm{F}_{(m+1)(p+1)}\, M$, hence $x_i \frac{v}{P^p} = \frac{x_i P v}{P^{p+1}} \in \mathrm{F}_{p+1}\, M_P$. Also,

$$\partial_i\left(\frac{v}{P^p}\right) = \frac{-p\partial_i(P)v + P\partial_i v}{P^{p+1}}$$

and $-p\partial_i(P)v + P\partial_i v \in \mathrm{F}_{(m+1)(p+1)} M$; hence $\partial_i\left(\frac{v}{P^p}\right) \in \mathrm{F}_{p+1} M_P$.

Therefore, we constructed an exhaustive $D(n)$-module filtration on M_P. Since

$$\dim_k \mathrm{F}_p M_P \leq \dim_k \mathrm{F}_{(m+1)p} M \leq e(M)\frac{((m+1)p)^n}{n!} + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$, M_P is holonomic by 8.4. □

8.7. Corollary. *Let $P \in k[X_1, X_2, \dots, X_n]$. Then $k[X_1, X_2, \dots, X_n]_P$ is a holonomic $D(n)$-module.*

9. Exterior tensor products

Let $X = k^n$ and $Y = k^m$ in the following, and denote by D_X and D_Y the corresponding algebras of differential operators with polynomial coefficients. Then we can consider the algebra $D_X \boxtimes D_Y$ which is equal to $D_X \otimes_k D_Y$ as a vector space over k, and the multiplication is defined by $(T \otimes S)(T' \otimes S') = TT' \otimes SS'$ for $T, T' \in D_X$ and $S, S' \in D_Y$. We call $D_X \boxtimes D_Y$ the *exterior tensor product* of D_X and D_Y.

The following result is evident.

9.1. Lemma. $D_X \boxtimes D_Y = D_{X\times Y}$.

If M and N are D_X-, resp. D_Y-modules, we can define $D_{X\times Y}$-module $M \boxtimes N$ which is equal to $M \otimes_k N$ as a vector space over k, and the action of $D_X \boxtimes D_Y = D_{X\times Y}$ is given by $(T \otimes S)(m \otimes n) = Tm \otimes Sn$ for any $T \in D_X$, $S \in D_Y$, $m \in M$ and $n \in N$.

9.2. Lemma. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $M \boxtimes N$ is a finitely generated $D_{X\times Y}$-module.*

Proof. Let $e_1, e_2, \dots, e_p$ and $f_1, f_2, \dots, f_q$ be generators of M and N respectively. Then for any $m \in M$ and $n \in N$, we have $m = \sum T_i e_i$, $T_i \in D_X$, and $n = \sum S_j f_j$, $S_j \in D_Y$. This implies that $m \otimes n = \sum\sum T_i e_i \otimes S_j f_j = \sum\sum (T_i \otimes S_j)(e_i \otimes f_j)$, and $e_i \otimes f_j$, $1 \leq i \leq p$, $1 \leq j \leq q$, generate $M \boxtimes N$. □

Our main goal in this section is to prove the following result.

9.3. Theorem. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $d(M \boxtimes N) = d(M) + d(N)$.*

This result has the following important consequence.

9.4. Corollary. *Let M be a holonomic D_X-module and N a holonomic D_Y-module. Then $M \boxtimes N$ is a holonomic $D_{X\times Y}$-module.*

Let D_X and D_Y be equipped with the Bernstein filtration. Let M and N be finitely generated D_X-, resp. D_Y-modules with good filtrations $\mathrm{F} M$ and $\mathrm{F} N$ respectively. Define the *product filtration* on $M \boxtimes N$ by

$$\mathrm{F}_j(M \boxtimes N) = \sum_{p+q=j} \mathrm{F}_p M \otimes_k \mathrm{F}_q N$$

for any $j \in \mathbb{Z}$. Clearly the product filtration on $D_X \boxtimes D_Y = D_{X\times Y}$ agrees with the Bernstein filtration. Therefore, $\mathrm{F}(M \boxtimes N)$ is an exhaustive hausdorff $D_{X\times Y}$-module filtration.

To prove that this filtration is good we need some preparation in linear algebra. We start with the following lemma.

9.5. Lemma. *Let M, M', N and N' be linear spaces over k, and $\phi : M \longrightarrow M'$ and $\psi : N \longrightarrow N'$ linear maps. Then they define a linear map $\phi \otimes \psi : M \otimes_k N \longrightarrow M' \otimes_k N'$. We have*

(i)
$$\operatorname{im}(\phi \otimes \psi) = \operatorname{im} \phi \otimes \operatorname{im} \psi;$$

(ii)
$$\ker(\phi \otimes \psi) = \ker \phi \otimes_k N + M \otimes_k \ker \psi.$$

Proof. (i) This is obvious from the definition.

(ii) By (i), to prove (ii) we can assume that ϕ and ψ are surjective. In this case, we have short exact sequences

$$0 \longrightarrow M'' \longrightarrow M \xrightarrow{\phi} M' \longrightarrow 0$$

where $M'' = \ker \phi$, and

$$0 \longrightarrow N'' \longrightarrow N \xrightarrow{\psi} N' \longrightarrow 0$$

where $N'' = \ker \psi$.

Clearly, we have $\phi \otimes \psi = (\phi \otimes id_{N'}) \circ (id_M \otimes \psi)$. Since the tensoring with N' is exact, the first exact sequence implies that the sequence

$$0 \longrightarrow M'' \otimes_k N' \longrightarrow M \otimes_k N' \xrightarrow{\phi \otimes id_{N'}} M' \otimes_k N' \longrightarrow 0$$

is exact. Hence, $\ker(\phi \otimes id_{N'}) = M'' \otimes_k N' = \ker \phi \otimes_k N'$. Therefore, an element z in $M \otimes_k N$ is in the kernel of $\phi \otimes \psi$ if and only if $(id_M \otimes \psi)(z)$ is in $\ker \phi \otimes_k N'$.

Since

$$0 \longrightarrow M \otimes_k N'' \longrightarrow M \otimes_k N \xrightarrow{id_M \otimes \psi} M \otimes_k N' \longrightarrow 0$$

is also exact, $\ker \phi \otimes_k N$ maps surjectively onto $\ker \phi \otimes_k N'$ and $\ker(id_M \otimes \psi) = M \otimes_k N'' = M \otimes_k \ker \psi$. Therefore, z is in the kernel of $\phi \otimes \psi$ if and only if $z \in \ker \phi \otimes_k N + M \otimes_k \ker \psi$. □

9.6. Lemma. *Let $X_1, X_2, \ldots, X_n$ be linear subspaces which span a linear space X. If*

$$X_i \cap \sum_{j \neq i} X_j = \{0\}$$

for $1 \leq i \leq n$, the linear space X is the direct sum of $X_1, X_2, \ldots, X_n$.

Proof. Let $x_i \in X_i$, $1 \leq i \leq n$, be such that $x_1 + x_2 + \cdots + x_n = 0$. Then $x_i = -\sum_{j \neq i} x_j \in X_i \cap \sum_{j \neq i} X_j$, and by our assumption is equal to 0, for any $1 \leq i \leq n$. □

Now we want to describe $\operatorname{Gr}(M \boxtimes N)$. Let $j \in \mathbb{Z}$. If $p + q = j$ we have a well-defined k-linear map $\mathrm{F}_p\, M \otimes_k \mathrm{F}_q\, N \longrightarrow \mathrm{F}_j(M \boxtimes N)$. Hence, we have a well-defined k-linear map $\mathrm{F}_p\, M \otimes_k \mathrm{F}_q\, N \longrightarrow \operatorname{Gr}^j(M \boxtimes N)$. By 9.5, the kernel of the natural map $\mathrm{F}_p\, M \otimes_k \mathrm{F}_q\, N \longrightarrow \operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ is $\mathrm{F}_{p-1}\, M \otimes_k \mathrm{F}_q\, N + \mathrm{F}_p\, M \otimes_k \mathrm{F}_{q-1}\, N$, i.e., it is contained in $\mathrm{F}_{j-1}(M \boxtimes N)$. Hence, the linear map $\mathrm{F}_p\, M \otimes_k \mathrm{F}_q\, N \longrightarrow \operatorname{Gr}^j(M \boxtimes N)$ factors through $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$. This leads to the linear map

$$\pi : \bigoplus_{p+q=j} \operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N \longrightarrow \operatorname{Gr}^j(M \boxtimes N).$$

Clearly, by its construction, this map is surjective. Moreover, its restriction to each summand $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ in the direct sum is injective. Let $X_{p,q}$ be the image of $\operatorname{Gr}^p M \otimes_k \operatorname{Gr}^q N$ in $\operatorname{Gr}^j(M \boxtimes N)$. Since we have

$$\mathrm{F}_{p-1} M \otimes_k \mathrm{F}_q N + \mathrm{F}_p M \otimes_k \mathrm{F}_{q-1} N = (\mathrm{F}_p M \otimes_k \mathrm{F}_q N) \cap \left(\sum_{p'+q'=j,\ p' \neq p,\ q' \neq q} \mathrm{F}_{p'} M \otimes_k \mathrm{F}_{q'} N \right),$$

we see that

$$X_{p,q} \cap \left(\sum_{p'+q'=j,\ p' \neq p,\ q' \neq q} X_{p',q'} \right) = \{0\}.$$

Hence, by 9.6 the map is an isomorphism. This implies that $\operatorname{Gr}_j(M \boxtimes N) = \bigoplus_{p+q=j} \operatorname{Gr}_p M \otimes_k \operatorname{Gr}_q N$ for any $j \in \mathbb{Z}$.

If we define analogously the algebra $\operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y$ with grading given by the total degree, we see that $\operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y = \operatorname{Gr} D_{X \times Y}$. In addition, $\operatorname{Gr} M \boxtimes \operatorname{Gr} N$ becomes a graded $\operatorname{Gr} D_{X\times Y}$-module isomorphic to $\operatorname{Gr}(M \boxtimes N)$ by the preceding discussion. Since the filtrations $\mathrm{F} M$ and $\mathrm{F} N$ are good, $\operatorname{Gr} M$ and $\operatorname{Gr} N$ are finitely generated $\operatorname{Gr} D_X$-, resp. $\operatorname{Gr} D_Y$-modules by 3.1. By an analogue of 9.2, $\operatorname{Gr}(M \boxtimes N)$ is a finitely generated $\operatorname{Gr} D_{X\times Y}$-module. This implies that the product filtration is a good filtration on $M \boxtimes N$.

Let

$$P(M,t) = \sum_{p \in \mathbb{Z}} \dim_k(\operatorname{Gr}_p M)\, t^p$$

and

$$P(N,t) = \sum_{q \in \mathbb{Z}} \dim_k(\operatorname{Gr}_q N)\, t^q$$

be the Poincaré series of $\operatorname{Gr} M$ and $\operatorname{Gr} N$. Then

$$\begin{aligned} P(M,t)\, P(N,t) &= \sum_{p\in\mathbb{Z}} \sum_{q \in \mathbb{Z}} \dim_k(\operatorname{Gr}_p M) \dim_k(\operatorname{Gr}_q N)\, t^{p+q} \\ &= \sum_{j \in \mathbb{Z}} \left(\sum_{p+q=j} \dim_k(\operatorname{Gr}_p M)\ \dim_k(\operatorname{Gr}_q N) \right) t^j \\ &= \sum_{j \in \mathbb{Z}} \left(\sum_{p+q=j} \dim_k(\operatorname{Gr}_p M \otimes_k \operatorname{Gr}_q N) \right) t^j \\ &= \sum_{j \in \mathbb{Z}} \dim_k \operatorname{Gr}_j(M \boxtimes N)\, t^j = P(M \boxtimes N, t) \end{aligned}$$

is the Poincaré series of $M \boxtimes N$. Therefore, the order of the pole at 1 of $P(M \boxtimes N, t)$ is the sum of the orders of poles of $P(M,t)$ and $P(N,t)$. From 1.5, we see that this immediately implies 9.3.

We can deduce 9.3 also by considering characteristic varieties. Consider D_X, D_Y and $D_{X\times Y}$ as rings filtered by the order of differential operators. Let M and N be finitely generated D_X-, resp. D_Y-modules, equipped with good filtrations $\mathrm{F} M$ and $\mathrm{F} N$. As above, we define a $D_{X\times Y}$-module filtration $\mathrm{F}(M \boxtimes N)$ on $M \boxtimes N$. Then, as in the above argument, we see that $\mathrm{F}(M \boxtimes N)$ is a good filtration of $M \boxtimes N$. Let I be the annihilator of $\operatorname{Gr} M$ in $\operatorname{Gr} D_X$ and J the annihilator of $\operatorname{Gr} N$ in $\operatorname{Gr} D_Y$. Then, by 9.5, we see that the annihilator of $\operatorname{Gr}(M \boxtimes N)$ is equal to the ideal $I \otimes_k \operatorname{Gr} D_Y + \operatorname{Gr} D_X \otimes_k J$ in $\operatorname{Gr} D_{X\times Y} = \operatorname{Gr} D_X \boxtimes \operatorname{Gr} D_Y$.

We can identify $\operatorname{Gr} D_X$ with the polynomial ring $k[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$ and $\operatorname{Gr} D_Y$ with the polynomial ring $k[y_1,\dots,y_m,\eta_1,\dots,\eta_m]$. Moroever, we can identify $\operatorname{Gr} D_{X\times Y}$ with the polynomial ring $k[x_1,\dots,x_n,y_1,\dots,y_m,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m]$. Then the annihilator of $\operatorname{Gr}(M\boxtimes N)$ corresponds to the ideal in $k[x_1,\dots,x_n,y_1,\dots,y_m,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m]$ generated by the images of I and J in that ring. If we define the map $q : k^{2n}\times k^{2m}\longrightarrow k^{2(n+m)}$ by

$$q(x_1,\dots,x_n,\xi_1,\dots,\xi_n,y_1,\dots,y_m,\eta_1,\dots,\eta_m) = (x_1,\dots,x_n,y_1,\dots,y_m,\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_m),$$

we have the following result.

9.7. Theorem. *Let M and N be finitely generated D_X-, resp. D_Y-modules. Then we have*

$$Ch(M\boxtimes N) = q(Ch(M)\times Ch(N)).$$

This in turn implies that $\dim Ch(M\boxtimes N) = \dim Ch(M) + \dim Ch(N)$, and by 7.7, we get another proof of 9.3.

Either by using (the proof of) 7.1 and arguing like in the above proof, or by using 7.5 we also see that the following result holds.

9.8. Proposition. *Let M and N be finitely generated D_X-, resp. D_Y-modules. Then we have*

$$\operatorname{supp}(M\boxtimes N) = \operatorname{supp}(M)\times\operatorname{supp}(N).$$

10. Inverse images

Let $X = k^n$ and $Y = k^m$ and denote by $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_m$ the canonical coordinate functions on X and Y respectively. Let $R(X) = k[x_1, x_2, \dots, x_n]$ and $R(Y) = k[y_1, y_2, \dots, y_m]$ denote the rings of regular functions on X and Y respectively.

Let $F : X \longrightarrow Y$ be a polynomial map, i.e.,

$$F(x_1,x_2,\dots,x_n) = (F_1(x_1,x_2,\dots,x_n), F_2(x_1,x_2,\dots,x_n),\dots,F_m(x_1,x_2,\dots,x_n))$$

with $F_i \in R(X)$. Then F defines a ring homomorphism $\phi_F : R(Y)\longrightarrow R(X)$ by $\phi_F(P) = P\circ F$ for $P\in R(Y)$. Therefore we can view $R(X)$ as an $R(Y)$-module. Hence, we can define functor F^* from the category $\mathcal{M}(R(Y))$ of $R(Y)$-modules into the category $\mathcal{M}(R(X))$ of $R(X)$-modules given by the following formula

$$F^*(N) = R(X)\otimes_{R(Y)} N$$

for any $R(Y)$-module N. Clearly $F^* : \mathcal{M}(R(Y))\longrightarrow\mathcal{M}(R(X))$ is a right exact functor. We call it the *inverse image* functor from the category $\mathcal{M}(R(Y))$ into the category $\mathcal{M}(R(X))$.

Now we want to extend this functor to D-modules. Denote now by D_X and D_Y the algebras of differential operators with polynomial coefficients on X and Y respectively. If N is a left D_Y-module, we want to define a D_X-module structure on the inverse image $F^*(N)$. (As we remarked at the beginning of §6, the transposition functor is an equivalence of the category of left D-modules with the category of right D-modules, hence we can analogously treat right modules.) First we consider the bilinear map

$$(P,v)\longmapsto \frac{\partial P}{\partial x_i}\otimes v + \sum_{j=1}^{n} P\frac{\partial F_j}{\partial x_i}\otimes\frac{\partial}{\partial y_j}v,$$

from $R(X) \times N$ into $R(X) \otimes_{R(Y)} N$. Since

$$\begin{aligned}\frac{\partial P(Q \circ F)}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q \circ F) \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\ = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \left(\frac{\partial Q}{\partial y_j} \circ F \right) \frac{\partial F_j}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q \circ F) \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\ = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \frac{\partial F_j}{\partial x_i} \otimes (\frac{\partial Q}{\partial y_j} v + Q \frac{\partial}{\partial y_j} v) \\ = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} (Qv)\end{aligned}$$

for any $Q \in R(Y)$, this map factors through a linear endomorphism of $F^*(N)$ which we denote by $\frac{\partial}{\partial x_i}$. By direct calculation we get

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] (P \otimes v) = 0$$

and

$$\left[\frac{\partial}{\partial x_i}, x_j \right] (P \otimes v) = \delta_{ij} (P \otimes v),$$

hence, by 5.11, we see that $F^*(N)$ has a natural structure of a left D_X-module.

Its structure can be described in another way. Let

$$D_{X \to Y} = F^*(D_Y) = R(X) \otimes_{R(Y)} D_Y.$$

Then, as we just described, $D_{X \to Y}$ has the structure of a left D_X-module. But it also has a structure of a right D_Y-module given by the right multiplication on D_Y. These two actions clearly commute, hence $D_{X \to Y}$ is a (left D_X, right D_Y)-bimodule. Moreover, for any D_Y-module N we have

$$F^*(N) = R(X) \otimes_{R(Y)} N = (R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} N = D_{X \to Y} \otimes_{D_Y} N$$

and the action of D_X on $F^*(N)$ is given by the action on the first factor in the last expression.

We denote this D_X-module by $F^+(N)$ and call it the *inverse image* of the D_Y-module N.

It is evident that the inverse image functor F^+ is a right exact functor from $\mathcal{M}^L(D_Y)$ into $\mathcal{M}^L(D_X)$. Its left derived functors $L^i F^+$ are given by

$$L^i F^+(N) = \operatorname{Tor}_{-i}^{D_Y} (D_{X \to Y}, N)$$

for a left D_Y-module N.

Let For denote the forgetful functor from the category of D_X-modules (resp. D_Y-modules) into the category of $R(X)$-modules (resp. $R(Y)$-modules). Then the following diagram of functors commutes

$$\begin{array}{ccc} \mathcal{M}(D_Y) & \xrightarrow{F^+} & \mathcal{M}(D_X) \\ {\scriptstyle \text{For}} \downarrow & & \downarrow {\scriptstyle \text{For}} \\ \mathcal{M}(R(Y)) & \xrightarrow{F^*} & \mathcal{M}(R(X)) \end{array} .$$

We claim that analogous statement holds for the left derived functors, i.e., we have the following statement.

10.1. Proposition. *The following diagram of functors commutes*

$$\begin{array}{ccc} \mathcal{M}(D_Y) & \xrightarrow{L^iF^+} & \mathcal{M}(D_X) \\ {\scriptstyle\text{For}}\downarrow & & \downarrow{\scriptstyle\text{For}} \\ \mathcal{M}(R(Y)) & \xrightarrow{L^iF^*} & \mathcal{M}(R(X)) \end{array} .$$

for any $i \in \mathbb{Z}$.

Proof. Let $F^{\cdot}$ be a left resolution of a D_Y-module N by free D_Y-modules. Since a free D_Y-module is also a free $R(Y)$-module by 5.10, by the above remark, we have

$$\begin{aligned} \operatorname{For}(L^iF^+(N)) = \operatorname{For}(H^i(F^+(F^{\cdot}))) &= H^i(\operatorname{For}(F^+(F^{\cdot}))) \\ &= H^i(F^*(\operatorname{For} F^i)) = L^iF^*(\operatorname{For} N) \end{aligned}$$

for any $i \in \mathbb{Z}$. □

Now we want to study the behavior of derived inverse images for compositions of morphisms. First we need an acyclicity result.

10.2. Lemma. *Let P be a projective left D_Y-module. Then $F^*(P)$ is a projective $R(X)$-module.*

Proof. Let P be a projective D_Y-module. Then it is a direct summand of a free D_Y-module $(D_Y)^{(I)}$. This implies that $F^+(P)$ is a direct summand of $F^+(D_Y^{(I)})$. Since D_Y is a free $R(Y)$-module, $\operatorname{For}(F^+(D_Y^{(I)})) = R(X) \otimes_{R(Y)} D_Y^{(I)}$ is a free $R(X)$-module. □

10.3. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the the inverse image functor $(G \circ F)^+$ from $\mathcal{M}^L(D_Z)$ into $\mathcal{M}^L(D_X)$ is isomorphic to $F^+ \circ G^+$;*
(ii) *for any left D_Z-module N there exist a spectral sequence with E_2-term $E_2^{pq} = L^pF^+(L^qG^+(N))$ which converges to $L^{p+q}(G \circ F)^+(N)$.*

Proof. (i) We consider first the polynomial ring structures. In this case

$$(G \circ F)^*(N) = R(X) \otimes_{R(Z)} N = R(X) \otimes_{R(Y)} (R(Y) \otimes_{R(Z)} N) = F^*(G^*(N))$$

for any D_Z-module N.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial x_i}(P \otimes v) &= \frac{\partial}{\partial x_i}(P \otimes (1 \otimes v)) \\ &= \frac{\partial P}{\partial x_i} \otimes (1 \otimes v) + \sum_{j=1}^{m} P \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j}(1 \otimes v) \\ &= \frac{\partial P}{\partial x_i} \otimes v + \sum_{j=1}^{m} P \frac{\partial F_j}{\partial x_i} \otimes \left(\sum_{k=1}^{p} \frac{\partial G_k}{\partial y_j} \otimes \frac{\partial}{\partial z_k} v \right) \\ &= \frac{\partial P}{\partial x_i} \otimes v + \sum_{k=1}^{p} P \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i} \left(\frac{\partial G_k}{\partial y_j} \circ F \right) \otimes \frac{\partial}{\partial z_k} v \\ &= \frac{\partial P}{\partial x_i} \otimes v + \sum_{k=1}^{p} P \frac{\partial (G_k \circ F)}{\partial x_i} \otimes \frac{\partial}{\partial z_k} v\end{aligned}$$

for any $P \in R(X)$ and $v \in N$. Hence the D_X-actions agree.

(ii) By 10.1 and 10.2, for any projective D_Z-module P, the inverse image $G^+(P)$ is F^+-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

This result has the immediate following consequence.

10.4. Corollary. *Let* $X = k^n$, $Y = k^m$ *and* $Z = k^p$, *and* $F : X \longrightarrow Y$ *and* $G : Y \longrightarrow Z$ *polynomial maps. Then*

(i) $D_{X \to Z} = D_{X \to Y} \otimes_{D_Y} D_{Y \to Z}$;
(ii) $\operatorname{Tor}_j^{D_Y}(D_{X \to Y}, D_{Y \to Z}) = 0$ *for* $j \in \mathbb{N}$.

Proof. (i) By 10.3.(i) we have

$$D_{X \to Z} = (G \circ F)^+(D_Z) = F^+(G^+(D_Z)) = F^+(D_{Y \to Z}) = D_{X \to Y} \otimes_{D_Y} D_{Y \to Z}.$$

(ii) As we remarked in the proof of 10.3.(ii), by 10.1 and 10.2, we see that $D_{Y \to Z} = G^+(D_Z)$ is F^+-acyclic. Hence, for $j > 0$, we have

$$0 = L^{-j} F^+(G^+(D_Z)) = \operatorname{Tor}_j^{D_Y}(D_{X \to Y}, G^+(D_Z)) = \operatorname{Tor}_j^{D_Y}(D_{X \to Y}, D_{Y \to Z}).$$

□

Now we consider two simple examples. First, let p be the projection of $X \times Y$ defined by $p(x, y) = y$ for $x \in X$, $y \in Y$. Then, as it is well known, $R(X \times Y) = R(X) \boxtimes R(Y)$. Therefore, for a $R(Y)$-module N we have

$$p^*(N) = R(X \times Y) \otimes_{R(Y)} N = (R(X) \boxtimes R(Y)) \otimes_{R(Y)} N = R(X) \boxtimes N$$

as a module over $R(X \times Y) = R(X) \boxtimes R(Y)$. On the other hand, if N is a D_Y-module, it follows immediately that the actions $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_j}$ also agree, i.e., $p^+(N) = R(X) \boxtimes N$. From 9.3 and 9.4 we immediately get the following result.

10.5. Proposition. *Let* $p : X \times Y \longrightarrow Y$ *be the canonical projection. Then,*

(i) p^+ *is an exact functor from* $\mathcal{M}^L(D_Y)$ *into* $\mathcal{M}^L(D_{X \times Y})$;
(ii) $p^+(N) = R(X) \boxtimes N$ *for any left* D_Y*-module* N;
(iii) $p^+(N)$ *is a finitely generated* $D_{X \times Y}$*-module if* N *is a finitely generated;*
(iv) $d(p^+(N)) = d(M) + n$ *for any finitely generated left* D_Y*-module* N.

In particular, a finitely generated D_Y-module N is holonomic if and only if $p^+(N)$ is holonomic.

Now we consider another example. Let i be the canonical injection of X into $X \times Y$ given by $i(x) = (x, 0)$ for any $x \in X$. Then

$$D_{X \to X \times Y} = i^*(D_{X \times Y}) = R(X) \otimes_{R(X) \boxtimes R(Y)} (D_X \boxtimes D_Y) = D_X \boxtimes D_Y / ((y_1, y_2, \ldots, y_m) D_Y)$$

with the obvious actions of D_X by left multiplication in the first factor and $D_{X \times Y} = D_X \boxtimes D_Y$ by the right multiplication.

Assume in addition that $m = 1$. Then we have the exact sequence

$$0 \longrightarrow D_Y \xrightarrow{y_1} D_Y \longrightarrow D_Y / y_1 D_Y \longrightarrow 0$$

where the second arrow is given by left multiplication by y_1. By tensoring with D_X, we get the short exact sequence

$$0 \longrightarrow D_{X \times Y} \xrightarrow{y_1} D_{X \times Y} \longrightarrow D_{X \to X \times Y} \longrightarrow 0$$

of left D_X-modules for left multiplication and right $D_{X \times Y}$-modules for right multplication. Therefore, we can consider the first two terms of this exact sequence as a left resolution of $D_{X \to X \times Y}$ by (left D_X, right $D_{X \times Y}$)-bimodules which are free as $D_{X \times Y}$-modules. Therefore, for a $D_{X \times Y}$-module N, the cohomology of the complex

$$0 \longrightarrow N \xrightarrow{y_1} N \longrightarrow 0$$

computes the derived inverse images. In particular, we have the following lemma.

10.6. Lemma. *Let $\dim Y = 1$. Let i be the canonical injection of X into $X \times Y$. Then, for any $D_{X \times Y}$-module N we have*

(i) *$i^+(N) = \operatorname{coker} y_1$;*
(ii) *$L^{-1} i^+(N) = \ker y_1$;*
(iii) *$L^p i^+(N) = 0$ for p different from 0 or -1.*

In particular, the left cohomological dimension of i^+ is ≤ 1.

The last statement has an obvious generalization for arbitrary Y.

10.7. Lemma. *Let i be the canonical injection of X into $X \times Y$. Then, the left cohomological dimension of i^+ is $\leq \dim Y$.*

Proof. The proof is by induction in $\dim Y$. We already established the result for $\dim Y = 1$. We can represent $Y = Y' \times Y''$ where $Y' = k^{m-1}$ and $Y'' = k$. Denote by i' the canonical inclusion of X into $X \times Y'$ and by j the canonical inclusion of $X \times Y'$ into $X \times Y' \times Y'' = X \times Y$. Then $i = j \circ i'$. Moreover, by 10.6, the left cohomological dimension of j^+ is ≤ 1, and by the induction assumption the left cohomological dimension of i'^+ is $\leq \dim Y'$. Therefore, from the Grothendieck spectral sequence in 10.3.(ii) we conclude that derived inverse images $L^{-p} i^+$ vanish for $p \geq \dim Y' + 1 = \dim Y$. □

Let $F : X \longrightarrow X$ be an isomorphism of X and G its inverse. Then the map $\alpha : R(X) \longrightarrow R(X)$ defined by $\alpha(f) = f \circ F$ is an automorphism of the ring $R(X)$. Its inverse is β given by $\beta(f) = f \circ G$ for $f \in R(X)$. If M is a $R(X)$-module, $F^*(M)$ is isomorphic to M as a linear space over k via the map $\phi : m \longmapsto 1 \otimes m$. On the other hand, for $f \in R(X)$, we have

$$f \phi(m) = f \otimes m = f \circ G \circ F \otimes m = 1 \otimes (f \circ G) m = \phi(\beta(f) m)$$

for any $m \in M$, i.e., the $R(X)$-module $F^*(M)$ is isomorphic to M with the $R(X)$-module structure given by $(f, m) \longmapsto \beta(f)m$. for $f \in R(X)$ and $m \in M$.

Now we want to give an analogous description of $F^+(M)$. First we want to extend the automorphism β to D_X.

Let T be a differential operator on X, and put $\tilde{\beta}(T)(f) = \beta(T\alpha(f))$ for any $f \in R(X)$. Clearly, $\tilde{\beta}(T)$ is a k-linear endomorphism of $R(X)$. Moreover, $T \longmapsto \tilde{\beta}(T)$ is a linear map. In addition, for two differential operators T and S in D_X, we have

$$\begin{aligned}\tilde{\beta}(TS)(f) &= \beta(TS\alpha(f)) = \beta(T\alpha(\beta(S\alpha(f)))) \\ &= \beta(T\alpha(\beta(S\alpha(f)))) = \beta(T\alpha(\tilde{\beta}(S)(f))) = \tilde{\beta}(T)(\tilde{\beta}(S)(f))\end{aligned}$$

for all $f \in R(X)$, i.e., $\tilde{\beta}$ is a homomorphism of the k-algebra D_X into the algebra of k-linear endomorphisms of $R(X)$. Since for $g \in R(X)$ we have

$$\tilde{\beta}(g)f = \beta(g\alpha(f)) = \beta(g)f$$

for all $f \in R(X)$, we see that $\tilde{\beta}$ extends the automorphism β of $R(X)$. This in turn implies that $\omega(T) \in D_X$ for $T \in D_X$, i.e., $\tilde{\beta}$ is an automorphism of D_X which extends the automorphism β of $R(X)$. Therefore, we can denote it simply by β.

Let $1 \leq i \leq n$. Then we have

$$\begin{aligned}\beta(\partial_i)(f) &= \beta(\partial_i\alpha(f)) = \beta(\partial_i(f \circ F)) \\ &= \beta\left(\sum_{j=1}^{n}((\partial_j f) \circ F)\,\partial_i F_j\right) = \sum_{j=1}^{n}((\partial_i F_j) \circ G)\,\partial_j f = \left(\sum_{j=1}^{n}\beta(\partial_i F_j)\,\partial_j\right)(f).\end{aligned}$$

Consider now the bimodule $D_{X \to X}$ attached to the map F. The linear map $\varphi : f \otimes T \longmapsto \beta(f)T$, identifies it with D_X. The D_X-module structures given by right multiplication are identical. On the other hand,

$$\varphi(\partial_i(1 \otimes T)) = \varphi\left(\sum_{j=1}^{n}\partial_i F_j \otimes \partial_j T\right) = \sum_{j=1}^{n}\beta(\partial_i F_j)\partial_j T = \beta(\partial_i)\varphi(1 \otimes T)$$

for any $T \in D_X$ and $1 \leq i \leq n$. Therefore, the bimodule $D_{X \to X}$ is isomorphic to D_X with right action by right multiplication and left action of by the composition of β and left multiplication. This in turn implies that $F^+(M)$ is isomorphic to M with the D_X-module structure given by $(T, m) \longmapsto \beta(T)m$ for $T \in D_X$ and $m \in M$.

Therefore, by 3.10, we established the following result.

10.8. LEMMA. *Let $F : X \longrightarrow X$ be an isomorphism of X.*

(i) *Let M be a D_X-module. Then $F^+(M)$ is equal to M as a linear space with the D_X-action given by $(T, m) \longmapsto \beta(T)m$ for $T \in D_X$ and $m \in M$.*
(ii) *The functor $F^+ : \mathcal{M}^L(D_X) \longrightarrow \mathcal{M}^L(D_X)$ is exact.*
(iii) *The functor F^+ maps finitely generated D_X-modules into finitely generated D_X-modules. If M is a finitely generated D_X-module, we have $d(F^+(M)) = d(M)$.*

In particular, F^+ maps holonomic modules into holonomic modules.

We can make the above statement more precise by describing the characteristic variety $Ch(F^+(M))$ for a finitely generated D_X-module M. First, from the above calculations we see that the automorphism β of D_X induces an automorphism $\operatorname{Gr}\beta$ of $\operatorname{Gr} D_X = k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ which is defined by $X_i \longmapsto \beta(X_i) = G_i$ and $\xi_i \longmapsto \sum_{j=1}^n \beta(\partial_i F_j)\xi_j = \sum_{j=1}^n ((\partial_i F_j) \circ G)\,\xi_j$ for $1 \le i \le n$.

Now we want ot describe this construction in more geometric terms. If $x = (x_1, x_2, \dots, x_n)$ is a point in $X = k^n$, we identify the cotangent space $T_x^*(X)$ at x with k^n via the map $df(x) \longmapsto ((\partial_1 f)(x), (\partial_2 f)(x), \dots, (\partial_n f)(x))$. Therefore, the cotangent bundle $T^*(X)$ of X can be identified with k^{2n} via the map $(x, df(x)) \longmapsto (x_1, \dots, x_n, (\partial_1 f)(x), \dots, (\partial_n f)(x))$ for $x \in X$. Let $F : X \longmapsto X$ be an isomorphism of X and G its inverse. Then the map G maps a point x in X into $G(x)$ and F maps $G(X)$ into x. Their differentials $T_x(G)$ and $T_{G(x)}(F)$ are mutually inverse linear isomorphisms between the tangent spaces $T_x(X)$ and $T_{G(x)}(X)$. Therefore, their adjoints $T_x(G)^* : T^*_{G(x)}(X) \longrightarrow T_x^*(X)$ and $T_{G(x)}(F)^* : T_x^*(X) \longrightarrow T^*_{G(x)}(X)$ are mutually inverse linear isomorphisms. This implies that we can define an isomorphism γ of the cotangent bundle $T^*(X)$ of X by $(x, \xi) \longmapsto (G(x), T_{G(x)}(F)^*\xi)$ for $\xi \in T_x^*(X)$ and $x \in X$. If we identify $T^*(X)$ with k^{2n}, by inspecting the above formulas, we see that $(\operatorname{Gr}\beta)(P) = P \circ \gamma$ for any $P \in k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$.

Let M be a finitely generated D_X-module with a good filtration $\mathrm{F}\, M$. Then we can realize $F^+(M)$ as M with the action described above. Clearly, $\mathrm{F}\, M$ is a good filtration of $F^+(M)$ realized that way. Therefore, $\operatorname{Gr} F^+(M)$ can be identified with $\operatorname{Gr} M$ equipped with the action $(Q, m) \longmapsto (\operatorname{Gr}\beta)(Q)m$ for $Q \in k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ and $m \in \operatorname{Gr} M$. Hence, if Q is in the annihilator of $\operatorname{Gr} F^+(M)$ if and only if $(\operatorname{Gr}\beta)(Q)$ is in the annihilator of $\operatorname{Gr} M$. If I is the annihilator of $\operatorname{Gr} F^+(M)$, $(\operatorname{Gr}\beta)I$ is the annihilator of $\operatorname{Gr} M$. Hence, (x, ξ) is in $Ch(F^+(M))$ if and only if $\gamma^{-1}(x, \xi)$ is in $Ch(M)$.

10.9. Lemma. *Let M be a finitely generated D_X-module. Then*

$$Ch(F^+(M)) = \gamma(Ch(M)).$$

Finally, this allows to give an estimate of the left cohomological dimension of the inverse image functor.

10.10. Theorem. *Let $X = k^n$, $Y = k^m$ and $F : X \longrightarrow Y$ a polynomial map. Then the left cohomological dimension of F^+ is $\le \dim Y$.*

Proof. To prove this statement we use the graph construction. Let $i : X \times Y$ be the morphism given by $i(x) = (x, 0)$ for $x \in X$. Let $\Phi : X \times Y \longrightarrow X \times Y$ be the morphism given by $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$. Finally, let $p : X \times Y \longrightarrow Y$ be the projection given by $p(x, y) = y$ for all $x \in X$ and $y \in Y$. Then $F = p \circ \Phi \circ i$. Moreover, Φ is an isomorphism of $X \times Y$ with the inverse $(x, y) \longmapsto (x, y - F(x))$.

By 10.3, $F^+ = i^+ \circ \Phi^+ \circ p^+$. Moreover, by 10.5 and 10.8, the functors p^+ and Φ^+ are exact. Therefore, $L^q F^+ = L^q i^+ \circ \Phi^+ \circ p^+$ for all $q \in \mathbb{Z}$. By 10.7, it follows that $L^q F^+ = 0$ for $q < -\dim Y$. □

11. Direct images

Let $X = k^n$, $Y = k^n$ and $F : X \longrightarrow Y$ a polynomial map, as in the last section. The composition with F defines a natural ring homomorphism $\hat{F} : R(Y) \longrightarrow R(X)$.

This homomorphism in turn defines a functor F_* form the category of $R(X)$-modules into the category of $R(Y)$-modules. For any $R(X)$-module M we define $F_*(M)$ as the module which is equal to M as a linear space over k, and the action of $R(Y)$ is given by $(f, m) \longmapsto \hat{F}(f) \cdot m$, for any $f \in R(Y)$ and $m \in M$. The functor $F_* : \mathcal{M}(R(X)) \longrightarrow \mathcal{M}(R(Y))$ is called the *direct image* functor. Clearly, F_* is an exact functor.

Unfortunately, if M is a D_X-module, the direct image $F_*(M)$ doesn't allow a D_Y-module structure in general. For example, if we consider the inclusion i of $X = \{0\}$ into $Y = k$, $D_X = R(X)$ is equal to k and D_Y is the algebra of all differential operators with polynomial coefficients in one variable. The category of D_X-modules is just the category of linear spaces over k. By 6.2, the inverse image of a nonzero finite-dimensional D_X-module M cannot have a structure of a D_Y-module. Therefore, the direct images for D-modules will not be related to direct images for modules over the rings of regular functions, as in the case of inverse images.

If we apply the transposition to the both actions on $D_{X \to Y}$ we get the (left D_Y, right D_X)-bimodule $D_{Y \leftarrow X}$. This allows the definition of the left D_Y-module

$$F_+(M) = D_{Y \leftarrow X} \otimes_{D_X} M$$

for any left D_X-module M. Clearly, F_+ is a right exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Y)$. We call it the *direct image* functor. The left derived functors $L^i F_+$ of F_+ are given by

$$L^{-j} F_+(M) = \operatorname{Tor}_j^{D_X}(D_{Y \leftarrow X}, M)$$

for a left D_X-module M.

Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. If we transpose the actions 10.4 implies the following statements

$$D_{Z \leftarrow X} = D_{Z \leftarrow Y} \otimes_{D_Y} D_{Y \leftarrow X}$$

and

$$\operatorname{Tor}_j^{D_Y}(D_{Z \leftarrow Y}, D_{Y \leftarrow X}) = 0$$

for $j \in \mathbb{N}$.

If P is a projective left D_X-module, $P \oplus Q = D_X^{(I)}$ for some left D_X-module Q and some I. Therefore, $F_+(P) \oplus F_+(Q) = F_+(D_X^{(I)}) = (D_{Y \leftarrow X})^{(I)}$. This implies the following result.

11.1. Lemma. *Let P be a projective left D_X-module. Then*

$$\operatorname{Tor}_j^{D_Y}(D_{Z \leftarrow Y}, F_+(P)) = 0$$

for $j \in \mathbb{N}$.

11.2. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the direct image functor $(G \circ F)_+$ from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Z)$ is isomorphic to $G_+ \circ F_+$;*
(ii) *for any left D_X-module M there exist a spectral sequence with E_2-term $E_2^{pq} = L^p G_+(L^q F_+(M))$ which converges to $L^{p+q}(G \circ F)_+(M)$.*

PROOF. (i) For any left D_X-module M by 10.4.(i) we have

$$\begin{aligned}(G\circ F)_+(M) &= D_{Z\leftarrow X}\otimes_{D_X} M = (D_{Z\leftarrow Y}\otimes_{D_Y} D_{Y\leftarrow X})\otimes_{D_X} M \\ &= D_{Z\leftarrow Y}\otimes_{D_Y}(D_{Y\leftarrow X}\otimes_{D_X} M) = D_{Z\leftarrow Y}\otimes_{D_Y} F_+(M) = G_+(F_+(M)).\end{aligned}$$

(ii) By 11.1, for any projective D_X-module P, the direct image $F_+(P)$ is G_+-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

Now we consider a simple example. Let i be the canonical injection of X into $X\times Y$ given by $i(x)=(x,0)$ for any $x\in X$. Then

$$D_{X\to X\times Y} = i^+(D_{X\times Y}) = i^+(D_X\boxtimes D_Y) = D_X\boxtimes D_Y/((y_1,y_2,\dots,y_m)D_Y)$$

and

$$D_{X\times Y\leftarrow X} = D_X\boxtimes D_Y/(D_Y(y_1,y_2,\dots,y_m)).$$

This implies that

$$i_+(M) = M\boxtimes D_Y/(D_Y(y_1,y_2,\dots,y_m))$$

for any left D_X-module M. Moreover, the module $D_Y/(D_Y(y_1,y_2,\dots,y_m))$ is isomorphic to Δ_m discussed in 8.3.

11.3. PROPOSITION. *Let $i: X\longrightarrow X\times Y$ be the injection defined by $i(x)=(x,0)$ for $x\in X$. Then,*

(i) *i_+ is an exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_{X\times Y})$;*
(ii) *$i_+(M) = M\boxtimes D_Y/(D_Y(y_1,y_2,\dots,y_m))$ for any left D_X-module M;*
(iii) *$i_+(M)$ is finitely generated $D_{X\times Y}$-module if M is a finitely generated D_X-module;*
(iv) *$d(i_+(M)) = d(M)+m$ for any finitely generated left D_X-module M.*

In particular, a finitely generated D_X-module M is holonomic if and only if $i_+(M)$ is holonomic.

PROOF. We already proved (ii), and it immediately implies (i). As we remarked in 8.3, $D_Y/(D_Y(y_1,y_2,\dots,y_m))$ is an irreducible holonomic D_Y-module, hence (iii) follows from 9.2. To prove (iv) we first remark that by 9.3, we have

$$d(i_+(M)) = d(M) + d(D_Y/(D_Y(y_1,y_2,\dots,y_m))).$$

Since $(D_Y/(D_Y(y_1,y_2,\dots,y_m))$ is holonomic, its dimension is equal to m. □

Now we want to study the direct image of a projection $p: X\times Y\longrightarrow Y$ given by $p(x,y)=y$ for $x\in X$ and $y\in Y$.

Consider first the case of $\dim X = 1$. Then

$$D_{X\times Y\to Y} = p^+(D_Y) = D_X/D_X(\partial_1)\boxtimes D_Y.$$

Hence, $D_{Y\leftarrow X\times Y} = D_X/((\partial_1)D_X)\boxtimes D_Y$. We have an exact sequence

$$0\longrightarrow D_{X\times Y}\xrightarrow{\partial_1} D_{X\times Y}\longrightarrow D_{Y\leftarrow X\times Y}\longrightarrow 0$$

of (left D_Y, right $D_{X\times Y}$)-bimodules, where the second arrow represents left multiplication by ∂_1. Clearly, this is a left resolution of $D_{Y\leftarrow X\times Y}$ by free right $D_{X\times Y}$-modules, hence the cohomology of the complex

$$\dots\longrightarrow 0\longrightarrow M\xrightarrow{\partial_1} M\longrightarrow 0\longrightarrow\dots$$

is $\mathrm{Tor}_{-\cdot}^{D_{X\times Y}}(D_{Y\leftarrow X\times Y}, M) = L^{\cdot}p_{+}(M)$ for any $D_{X\times Y}$-module M. It follows that that $L^q p_+(M) = 0$ for $q \notin \{0, -1\}$.

Therefore, we established the following result.

11.4. Lemma. *Let $\dim X = 1$. Let p be the canonical projection of $X \times Y$ onto Y. Then, for any $D_{X\times Y}$-module M we have*

(i) $p_+(M) = \operatorname{coker} \partial_1$;
(ii) $L^{-1}p_+(N) = \ker \partial_1$;
(iii) $L^q p_+(N) = 0$ *for q different from 0 or -1.*

In particular, the left cohomological dimension of p_+ is ≤ 1.

The last statement has the following generalization for arbitrary X.

11.5. Lemma. *Let p be the canonical injection of $X \times Y$ onto Y. Then, the left cohomological dimension of p_+ is $\leq \dim X$.*

Proof. Let $X' = \{x_n = 0\} \subset X$, and denote by p' the canonical projection of $X' \times Y$ onto Y. Also, denote by p'' the canonical projection of $X \times Y$ onto $X' \times Y$. Then $p = p' \circ p''$. Hence, by 11.2.(ii), 11.4 and the induction assumption we conclude that $L^q p_+(M) =$ for $q < -\dim X$. □

Let $F : X \longrightarrow X$ be an isomorphism of X and G its inverse. As in §10, we define the automorphisms α and β of D_X. We identified there the bimodule $D_{X\to X}$ attached to F with D_X equipped with actions given by right multiplication and left multiplication composed with β. Applying α to it, we see that it is also isomorphic to D_X equipped with actions given by right multiplication composed with α and left multiplication. By applying the principal antiautomorphism we see that the bimodule $D_{X\leftarrow X}$ is isomorphic to D_X with actions given by left multiplication composed with α and right multiplication. This in turn implies that for any D_X-module M, the direct image $F_+(M)$ is isomorphic to M with the action given by $(T, m) \longmapsto \alpha(T)m$. In particular, $F_+(M) = G^+(M)$.

Therefore, from 10.8, we immediately deduce the following result.

11.6. Lemma. *Let $F : X \longrightarrow X$ be an isomorphism of X and $G : X \longrightarrow X$ its inverse.*

(i) *Let M be a D_X-module. Then $F_+(M)$ is equal to M as a linear space with the D_X-action given by $(T, m) \longmapsto \alpha(T)m$ for $T \in D_X$ and $m \in M$.*
(ii) *The functor $F_+ : \mathcal{M}^L(D_X) \longrightarrow \mathcal{M}^L(D_X)$ is exact.*
(iii) *The functor F_+ maps finitely generated D_X-modules into finitely generated D_X-modules. If M is a finitely generated D_X-module, we have $d(F_+(M)) = d(M)$.*
In particular, F_+ maps holonomic modules into holonomic modules.

In addition, $F_+ = G^+$ and $F^+ = G_+$, and these functors are mutiually quasi-inverse equivalences of categories.

As in the last section, this allows to give an estimate of the left cohomological dimension of the direct image functor.

11.7. Theorem. *Let $X = k^n$, $Y = k^m$ and $F : X \longrightarrow Y$ a polynomial map. Then the left cohomological dimension of F_+ is $\leq \dim X$.*

Proof. As in the proof of 10.10, we use the graph construction. Let $i : X \times Y$ be the morphism given by $i(x) = (x, 0)$ for $x \in X$. Let $\Phi : X \times Y \longrightarrow X \times Y$ be the morphism given by $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$. Finally, let $p : X \times Y \longrightarrow Y$ be the projection given by $p(x, y) = y$ for all $x \in X$ and $y \in Y$. Then $F = p \circ \Phi \circ i$. Moreover, Φ is an isomorphism of $X \times Y$ with the inverse $(x, y) \longmapsto (x, y - F(x))$.

By 11.2, $F_+ = p_+ \circ \Phi_+ \circ i_+$. Moreover, by 11.3 and 11.6, the functors i_+ and Φ_+ are exact. Therefore, $L^q F_+ = L^q p_+ \circ \Phi_+ \circ i_+$ for all $q \in \mathbb{Z}$. By 11.5, it follows that $L^q F_+ = 0$ for $p < -\dim X$. ☐

12. Kashiwara's theorem

Let $X = k^n$ and $Y = \{x_n = 0\} \subset X$. We put also $Z = \{x_1 = x_2 = \cdots = x_{n-1} = 0\} \cong k$. Hence $X = Y \times Z$. This also implies that $D_X = D_Y \boxtimes D_Z$. Let M be a D_X-module and put

$$\Gamma_{[Y]}(M) = \{m \in M \mid x_n^p m = 0 \text{ for some } p \in \mathbb{N}\}.$$

12.1. Lemma. *Let M be a D_X-module. Then:*

(i) *$\Gamma_{[Y]}(M)$ is a D_X-submodule of M;*
(ii) *$\operatorname{supp}(\Gamma_{[Y]}(M)) \subset Y$;*
(iii) *if N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$, then $N \subset \Gamma_{[Y]}(M)$.*

Proof. (i) Let $m \in \Gamma_{[Y]}(M)$. Then $x_i m \in \Gamma_{[Y]}(M)$ and $\partial_j m \in \Gamma_{[Y]}(M)$ for $1 \leq i \leq n$ and $1 \leq j < n$. It remains to check that $\partial_n m \in \Gamma_{[Y]}(M)$. We have

$$x_n^{j+1} \partial_n m = [x_n^{j+1}, \partial_n] m + \partial_n x_n^{j+1} m = -(j+1) x_n^j m + \partial_n x_n^{j+1} m$$

for any $j \in \mathbb{N}$. Hence, if $x_n^j m = 0$, we see that $x_n^{j+1} \partial_n m = 0$.

(ii) If $x \notin Y$, $x_n \notin \mathbf{m}_x$ and the localization $\Gamma_{[Y]}(M)_x = 0$.

(iii) Assume that N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$. Let $m \in N$ and denote by N' the $R(X)$-submodule generated by m. Then $\operatorname{supp}(N') \subset Y$. Since N' is finitely generated, by 4.2, its support is equal to the variety determined by its annihilator I in $R(X)$. By Nullstelensatz we see that $r(I) \supset (x_n)$. This implies that x_n^j annihilates N' for some $j \in \mathbb{N}$, i.e., $m \in \Gamma_{[Y]}(M)$. ☐

Therefore $\Gamma_{[Y]}(M)$ is the largest D_X-submodule of M supported in Y.

The multiplication by x_n defines an endomorphism of M as D_Y-module. Let

$$M_0 = \ker x_n \subset \Gamma_{[Y]}(M)$$

and

$$M_1 = \operatorname{coker} x_n = M / x_n M.$$

Denote by i the natural inclusion of Y into X. As we established in 10.6, $i^+(M) = M_1$, $L^{-1} i^+(M) = M_0$ and all other inverse images vanish.

Consider the biadditive map $D_X \times M_0 \longrightarrow M$. Clearly, it factors through $D_X \otimes_{D_Y} M_0 \longrightarrow M$. Moroever, by the definition of M_0, the latter morphism vanishes on the image of $D_X x_n \otimes_{D_Y} M_o$ in $D_X \otimes_{D_Y} M_o$. As we remarked in §11,

$$D_{X \leftarrow Y} = D_Y \boxtimes D_Z / D_Z x_n = \bigoplus_{j=0}^{\infty} \partial_n^j D_Y.$$

Therefore, the above morphism induces a natural D_X-module morphism

$$i_+(M_0) = D_{X \leftarrow Y} \otimes_{D_Y} M_0 \longrightarrow M.$$

Clearly, its image is contained in $\Gamma_{[Y]}(M)$. It is easy to check that this is actually a morphism of the functor $i_+ \circ L^{-1}i^+$ into $\Gamma_{[Y]}$.

The critical result of this section is the next lemma.

12.2. Lemma. *The morphism $i_+(M_0) \longrightarrow \Gamma_{[Y]}(M)$ is an isomorphism of D_X-modules.*

Proof. We first show that the morphism is surjective. We claim that

$$\{m \in M \mid x_n^p m = 0\} \subset D_X \cdot M_0$$

for any $p \in \mathbb{N}$. This is evident for $p = 1$. If $p > 1$ and $x_n^p m = 0$ we see that

$$0 = \partial_n(x_n^p m) = x_n^{p-1}(pm + x_n \partial_n m),$$

and by the induction hypothesis,

$$pm + x_n \partial_n m \in D_X \cdot M_0.$$

Also, by the induction hypothesis, $x_n m \in D_X \cdot M_0$. This implies that

$$(p-1)m = pm + [x_n, \partial_n]m = pm + x_n\partial_n m - \partial_n x_n m \in D_X \cdot M_0$$

and $m \in D_X \cdot M_0$. Hence the map is surjective.

Now we prove injectivity. By the preceding discussion

$$i_+(M_0) = D_{X \leftarrow Y} \otimes_{D_Y} M_0 = \bigoplus_{j=0}^{\infty} \partial_n^j M_0.$$

Let $(m_0, \partial_n m_1, \ldots, \partial_n^q m_q, 0, \ldots)$ be a nonzero element of this direct sum which maps into 0, i.e.,

$$m_0 + \partial_n m_1 + \cdots + \partial_n^q m_q = 0,$$

with minimal possible q. Then

$$0 = x_n(\sum_{j=0}^{q} \partial_n^j m_j) = \sum_{j=1}^{q} [x_n, \partial_n^j] m_j = -\sum_{j=1}^{q} j \partial_n^{j-1} m_j$$

and we have a contradiction. Therefore, the kernel of the map is zero. □

12.3. Corollary. $x_n \Gamma_{[Y]}(M) = \Gamma_{[Y]}(M)$.

Proof. By 12.2 any element of $\Gamma_{[Y]}(M)$ has the form $\sum_{j \in \mathbb{Z}_+} \partial_n^j m_j$ with $m_j \in M_0$. On the other hand,

$$x_n \sum_{j \in \mathbb{Z}_+} \frac{1}{j+1} \partial_n^{j+1} m_j = -\sum_{j \in \mathbb{Z}_+} \partial_n^j m_j.$$

□

12.4. Corollary. *Let M be a D_X-module. Then*

(i) *$\Gamma_{[Y]}(M)$ is a finitely generated D_X-module if and only if M_0 is a finitely generated D_Y-module;*
(ii) *$d(\Gamma_{[Y]}(M)) = d(M_0) + 1$.*

In particular, $\Gamma_{[Y]}(M)$ is holonomic if and only if $L^{-1}i^+(M) = M_0$ is holonomic.

PROOF. (i) From 12.2 and 11.3.(iii) we see that $\Gamma_{[Y]}(M)$ is finitely generated if M_0 is finitely generated. Assume that $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module. Let N_j, $j \in \mathbb{N}$, be an increasing sequence of D_Y-submodules of M_0. Then they generate D_X-submodules $i_+(N_j) = \bigoplus_{p=0}^{\infty} \partial_n^p N_j$ of $\Gamma_{[Y]}(M)$. Since $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module, the increasing sequence $i_+(N_j)$, $j \in \mathbb{N}$, stabilizes. Moreover, N_j is the kernel of x_n in $i_+(N_j)$ and the sequence N_j, $j \in \mathbb{N}$, must also stabilize. Therefore, M_0 is finitely generated.

(ii) Follows from 12.2 and 11.3.(iv). □

12.5. COROLLARY. *Let M be a holonomic D_X-module. Then M_0 is a holonomic D_Y-module.*

PROOF. If M is holonomic, $\Gamma_{[Y]}(M)$ is also holonomic. Therefore, the assertion follows from 12.4. □

Let $\mathcal{M}_Y(D_X)$ be the full subcategory of $\mathcal{M}(D_X)$ consisting of D_X-modules with supports in Y. Denote by $\mathcal{M}_{fg,Y}(D_X)$ and $\mathcal{H}\mathrm{ol}_Y(D_X)$ the corresponding subcategories of finitely generated, resp. holonomic, D_X-modules with supports in Y. Then, by 12.1, we have $M = \Gamma_{[Y]}(M)$ for any M in $\mathcal{M}_Y(D_X)$. By 10.6 and 12.3 we see that $i^+(M) = 0$ for any M in $\mathcal{M}_Y(D_X)$, hence $L^{-1}i^+$ is an exact functor from $\mathcal{M}_Y(D_X)$ into $\mathcal{M}(D_Y)$. On the other hand, i_+ defines an exact functor in the opposite direction, and by 12.2 the composition $i_+ \circ L^{-1}i^+$ is isomorphic to the identity functor on $\mathcal{M}_Y(D_X)$. Also it is evident that $L^{-1}i^+ \circ i_+$ is isomorphic to the identity functor on $\mathcal{M}(D_Y)$.

This leads us to the following basic result.

12.6. THEOREM (Kashiwara). *The direct image functor i_+ defines an equivalence of the category $\mathcal{M}(D_Y)$ (resp. $\mathcal{M}_{fg}(D_Y)$, $\mathcal{H}\mathrm{ol}(D_Y)$) with the category $\mathcal{M}_Y(D_X)$ (resp. $\mathcal{M}_{fg,Y}(D_X)$, $\mathcal{H}\mathrm{ol}_Y(D_X)$). Its inverse is the functor $L^{-1}i^+$.*

PROOF. It remains to show only the statements in parentheses. They follow immediately from 12.4. □

13. Preservation of holonomicity

In this section we prove that direct and inverse images preserve holonomic modules. We start with a simple criterion for holonomicity.

Let $X = k^n$ and $Y = k^m$. Let $F : X \longrightarrow Y$ be a polynomial map. We want to study the behavior of holonomic modules under the action of inverse and direct image functors.

First we use again graph construction to reduce the problem to special maps. As in the proof of 10.10 and 11.7:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle i}\big\downarrow & & \big\uparrow{\scriptstyle p} \\ X \times Y & \xrightarrow{\Phi} & X \times Y \end{array}$$

where $i(x) = (x, 0)$ for all $x \in X$, $p(x, y) = y$ for all $x \in X$ and $y \in Y$; and $\Phi(x, y) = (x, y + F(x))$ for $x \in X$ and $y \in Y$.

By 10.5, we know that p^+ is exact and maps holonomic modules into holonomic modules. By 11.3, we know that i_+ is exact and maps holonomic modules into holonomic modules. Moreover, by 10.8 and 11.6 we know that Φ^+ and Φ_+ are exact and map holonomic modules into holonomic modules.

Therefore, it remains to study the derived functors of i^+ and p_+.

We first discuss the immersion $i : X \longrightarrow X \times Y$.

13.1. Lemma. *Let N is a holonomic $D_{X\times Y}$-module. Then the D_X-modules $L^q i^+(N)$, $q \in \mathbb{Z}$, are holonomic.*

Since the submodules, quotient modules and extensions of holonomic modules are holonomic by 8.1.(ii), as in the proof of 10.7 by the spectral sequence argument we can reduce the proof to the case $\dim Y = 1$. In this situation, if we denote by y the natural coordinate on Y, and consider the D_X-module morphism $N \xrightarrow{y} N$, we have $i^+(N) = \operatorname{coker} y$ and $L^{-1}i^+(N) = \ker y$ and all other derived inverse images vanish, as we established in 10.6. Moreover, if N is holonomic $L^{-1}i^+(N)$ is holonomic by 12.4. Hence, it remains to treat $i^+(N)$.

13.2. Lemma. *Let N be a holonomic $D_{X\times Y}$-module. Then $i^+(N)$ is holonomic.*

Proof. Let $\bar{N} = N/\Gamma_{[X]}(N)$. Then ve can consider the short exact sequence

$$0 \longrightarrow \Gamma_{[X]}(N) \longrightarrow N \longrightarrow \bar{N} \longrightarrow 0.$$

Since i^+ is a right exact functor, this leads to the exact sequence

$$i^+(\Gamma_{[X]}(N)) \longrightarrow i^+(N) \longrightarrow i^+(\bar{N}) \longrightarrow 0.$$

On the other hand, by 12.3, we see that $i^+(\Gamma_{[X]}(N)) = 0$. Therefore, the natural map $i^+(N) \longrightarrow i^+(\bar{N})$ is an isomorphism.

Let $\bar{v} \in \Gamma_{[X]}(\bar{N}) \subset \bar{N}$ and denote by $v \in N$ the representative of $\bar{v}$. Then $y^p \bar{v} = 0$ for sufficiently large $p \in \mathbb{Z}_+$. Therefore, $y^p v \in \Gamma_{[X]}(N)$. This in turn implies that $y^{p+q} v = y^q(y^p n) = 0$ for sufficiently large $q \in \mathbb{Z}_+$. Hence, $v \in \Gamma_{[X]}(N)$ and $\bar{v} = 0$. It follows that $\Gamma_{[X]}(\bar{N}) = 0$.

In addition, if N is a holonomic $D_{X\times Y}$-module, $\bar{N}$ is a holonomic $D_{X\times Y}$-module.

Therefore, we can assume from the beginning that $\Gamma_{[X]}(N) = 0$. This means that the multiplication by y is injective on N, and N imbeds into its localization N_y. Consider the exact sequence

$$0 \longrightarrow N \longrightarrow N_y \longrightarrow L \longrightarrow 0.$$

Since N is a holonomic $D_{X\times Y}$-module, from 8.6 we know that N_y is a holonomic. Hence, L is a holonomic $D_{X\times Y}$-module. By the above discussion, this implies $L^{-1}i^+(L)$ is a holonomic D_X-module.

Applying the long exact sequence of inverse images of i to our short exact sequence, we get

$$\cdots \to L^{-1}i^+(N_y) \to L^{-1}i^+(L) \to i^+(N) \to i^+(N_y) \to i^+(L) \to 0.$$

Since the multiplication by y on N_y is invertible, by 10.6 we see that

$$i^+(N_y) = L^{-1}i^+(N_y) = 0.$$

Hence, it follows that $i^+(N) \cong L^{-1}i^+(L)$. By the preceding discussion we conclude that $i^+(N)$ is a holonomic D_X-module. □

Therefore, by 10.3, we get the following result.

13.3. Theorem. *Let $F : X \longrightarrow Y$ be a polynomial map and M a holonomic D_Y-module. Then $L^q F^+(M)$, $q \in \mathbb{Z}$, are holonomic D_X-modules.*

Now we want to study the direct images of p.

13.4. Lemma. *Let M is a holonomic $D_{X\times Y}$-module. Then the D_Y-modules $L^q p_+(M)$, $q \in \mathbb{Z}$, are holonomic.*

Proof. Since the submodules, quotient modules and extensions of holonomic modules are holonomic by 8.1.(ii), as in the proof of 11.5 by the spectral sequence argument we can reduce the proof to the case $\dim X = 1$. In this situation, if we denote by ∂ the derivative with respect to the coordinate x on X, and consider the D_Y-module morphism $M \xrightarrow{\partial} M$, we have $p_+(M) = \operatorname{coker} \partial$ and $L^{-1}p_+(M) = \ker \partial$ and all other derived inverse images vanish, as we established in 11.4. By applying the Fourier transform we get the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{F}(M) \xrightarrow{x} \mathcal{F}(M) \longrightarrow 0 \longrightarrow \ldots$$

which calculates $\mathcal{F}(L^{\cdot} p_+(M))$. By the arguments from the proof of 13.2, we see that this complex calculates the inverse images of the canonical inclusion $j : Y \longrightarrow X \times Y$ given by $j(y) = (0, y)$ for $y \in Y$. Therefore, its cohomologies are holonomic by 13.2. By 6.4, we see that $L^q p_+(M)$ are holonomic for all $q \in \mathbb{Z}$. □

Therefore, by 11.2, we get the following result.

13.5. Theorem. *Let $F : X \longrightarrow Y$ be a polynomial map and M a holonomic D_X-module. Then $L^q F_+(M)$, $q \in \mathbb{Z}$, are holonomic D_Y-modules.*

13.6. Remark. The statements analogous to 13.4 and 13.5 for finitely generated modules are false. For example, if we put $X = \{0\}$, $Y = k$ and denote by $i : X \longrightarrow Y$ the natural inclusion, the inverse image $i^+(D_Y)$ is an infinite-dimensional vector space over k. Analogously, if p is the projection of Y into a point, $p_+(D_Y)$ is an infinite-dimensional vector space over k.