Lectures on Algebraic Theory of D-Modules

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CHAPTER I

MODULES OVER RINGS OF DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

1. Hilbert polynomials

Let $A = \oplus_{n=0}^{\infty} A_n$ be a graded nötherian ring.

1.1. Lemma.

(i) *A_0 is a nötherian ring.*
(ii) *A is a finitely generated A_0-algebra.*

Proof. (i) Put $A_+ = \oplus_{n=1}^{\infty} A_n$. Then A_+ is an ideal in A and $A_0 = A/A_+$.

(ii) A_+ is finitely generated. Let $x_1, x_2, \dots, x_s$ be a set of homogeneous generators of A_+ and denote $d_i = \deg x_i$, $1 \leq i \leq s$. Let B be the A_0-subalgebra generated by $x_1, \dots, x_s$. We claim that $A_n \subseteq B$, $n \in \mathbb{Z}_+$. Clearly, $A_0 \subseteq B$. Assume that $n > 0$ and $y \in A_n$. Then $y \in A_+$ and therefore $y = \sum_{i=1}^{s} y_i x_i$ where $y_i \in A_{n-d_i}$. It follows that the induction assumption applies to y_i, $1 \leq i \leq s$. This implies that $y \in B$. $\square$

The converse of 1. follows by Hilbert's theorem which states that the polynomial ring $A_0[X_1, \dots, X_n]$ is nötherian if the ring A_0 is nötherian. Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded A-module. Then each M_n, $n \in \mathbb{Z}$, is an A_0-module. Also, $M_n = 0$ for sufficiently negative $n \in \mathbb{Z}$.

1.2. Lemma. *M_n, $n \in \mathbb{Z}$, are finitely generated A_0-modules.*

Proof. Let m_i, $1 \leq i \leq k$, be homogeneous generators of M and deg $m_i = r_i$, $1 \leq i \leq k$. For $j \in \mathbb{Z}_+$ denote by $z_i(j)$, $1 \leq i \leq \ell(j)$, all homogeneous monomials in $x_1, x_2, \dots, x_s$ of degree j. Let $m \in M_n$. Then $m = \sum_{i=1}^{k} y_i m_i$ where $y_i \in A_{n-r_i}$, $i \leq i \leq k$. By 1, $y_i = \sum_j a_{ij} z_j(n - r_i)$, with $a_{ij} \in A_0$. This implies that $m = \sum_{i,j} a_{ij} z_j(n - r_i)\, m_i$; hence M_n is generated by $(z_j(n - r_i)\, m_i;\ 1 \leq j \leq \ell(n - r_i),\ 1 \leq i \leq k)$. $\square$

Let $\mathcal{M}_{fg}(A_0)$ be the category of finitely generated A_0-modules. Let λ be a function on $\mathcal{M}_{fg}(A_0)$ with values in $\mathbb{Z}$. The function λ is called *additive* if for any short exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

1.3. Lemma. *Let*

$$0 \to M_0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$$

be an exact sequence in $\mathcal{M}_{fg}(A_0)$. *Then*

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0.$$

Proof. Evident. □

Let $\mathbb{Z}[[t]]$ be the ring of formal power series in t with coefficients in $\mathbb{Z}$. Denote by $\mathbb{Z}((t))$ the localization of $\mathbb{Z}[[t]]$ with respect to the multiplicative system $\{t^n \mid n \in \mathbb{Z}_+\}$.

Let M be a finitely generated graded A-module. Then the *Poincaré series* $P(M,t)$ of M (with respect to λ) is

$$P(M,t) = \sum_{n \in \mathbb{Z}} \lambda(M_n)\, t^n \in Z((t)).$$

For example, let $A = k[X_1, X_2, \dots, X_s]$ be the algebra of polynomials in s variables with coefficients in a field k graded by the total degree. Then, $A_0 = k$ and for every finitely generated graded A-module M, we have $\dim_k M_n < \infty$. Hence, we can define the Poincaré series for $\lambda = \dim_k$. In particular, for the A-module A itself, we have

$$P(A,t) = \sum_{n \in \mathbb{Z}} \dim_k A_n\, t^n = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} t^n = \frac{1}{(1-t)^s}.$$

The next result shows that general Poincaré series have an analogous form.

1.4. Theorem (Hilbert, Serre). *For any finitely generated graded A-module M we have*

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{d_i})}$$

where $f(t) \in \mathbb{Z}[t, t^{-1}]$.

Proof. We prove the theorem by induction in s. If $s = 0$, $A = A_0$ and M is a finitely generated A_0-module. This implies that $M_n = 0$ for sufficiently large n. Therefore, $\lambda(M_n) = 0$ except for finitely many $n \in \mathbb{Z}$ and $P(M,t)$ is in $\mathbb{Z}[t, t^{-1}]$.

Assume now that $s > 0$. The multiplication by x_s defines an A-module endomorphism f of M. Let $K = \ker f$, $I = \operatorname{im} f$ and $L = M/I$. Then K, I and L are graded A-modules and we have an exact sequence

$$0 \to K \to M \xrightarrow{f} M \to L \to 0.$$

This implies that

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{x_s} M_{n+d_s} \longrightarrow L_{n+d_s} \longrightarrow 0$$

is an exact sequence of A_0-modules for all $n \in \mathbb{Z}$. In particular, by 3,

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+d_s}) - \lambda(L_{n+d_s}) = 0,$$

for all $n \in \mathbb{Z}$. This implies that

$$\begin{aligned}(1 - t^{d_s})\, P(M,t) &= \sum_{n\in\mathbb{Z}} \lambda(M_n)\, t^n - \sum_{n\in\mathbb{Z}} \lambda(M_n)\, t^{n+d_s} \\ &= \sum_{n\in\mathbb{Z}} (\lambda(M_{n+d_s}) - \lambda(M_n))\, t^{n+d_s} \\ &= \sum_{n\in\mathbb{Z}} (\lambda(L_{n+d_s}) - \lambda(K_n))\, t^{n+d_s} \\ &= P(L,t) - P(K,t)\, t^{d_s},\end{aligned}$$

i.e.,

$$(1 - t^{d_s})\, P(M,t) = P(L,t) - t^{d_s} P(K,t).$$

From the construction it follows that x_s act as multiplication by 0 on L and K, i.e., we can view them as $A/(x_s)$-modules. Hence, the induction assumption applies to them. This immediately implies the assertion. □

Let $d_\lambda(M)$ be the order of the pole of $P(M,t)$ at 1.

1.5. Corollary. *If $d_i = 1$ for $1 \leq i \leq s$, $\lambda(M_n)$ is equal to a polynomial in n with rational coefficients of degree $d_\lambda(M) - 1$ for sufficiently large n.*

Proof. Let k be the order of zero of f at 1. Then we can write $f(t) = (t-1)^k g(t)$ with $g(1) \neq 0$. In addition, we put $d = d_\lambda(M) = s - k$, hence

$$P(M,t) = \frac{g(t)}{(1-t)^d}.$$

Now,

$$(1-t)^{-d} = \sum_{k=0}^{\infty} \frac{d(d+1)\dots(d+k-1)}{k!}\, t^k = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^k,$$

and if we put $g(t) = \sum_{k=-N}^{N} a_k t^k$ we get

$$\lambda(M_n) = \sum_{k=-N}^{N} a_k \binom{d+n-k-1}{d-1}$$

for all $n \geq N$. This is equal to

$$\sum_{k=-N}^{N} a_k \, \frac{(d+n-k-1)!}{(d-1)!(n-k)!} = \sum_{k=-N}^{N} a_k \, \frac{(n-k+1)(n-k+2)\dots(n-k+d-1)}{(d-1)!},$$

hence $\lambda(M_n)$ is a polynomial in n with the leading term

$$\left(\sum_{k=-N}^{N} a_k\right) \frac{n^{d-1}}{(d-1)!} = g(1)\frac{n^{d-1}}{(d-1)!} \neq 0. \quad \square$$

We call the polynomial which gives $\lambda(M_n)$ for large $n \in \mathbb{Z}$ the *Hilbert polynomial* of M (with respect to λ). From the proof we see that the leading coefficient of the Hilbert polynomial of M is equal to $\frac{g(1)}{(d-1)!}$.

Returning to our example of $A = k[X_1, X_2, \dots, X_s]$, we see that

$$\dim_k A_n = \binom{s+n-1}{s-1} = \frac{n^{s-1}}{(s-1)!} + \dots .$$

Hence, the degree of the Hilbert polynomial for $A = k[X_1, X_2, \dots, X_s]$ is equal to $s-1$.

Evidently, for any $s \in \mathbb{Z}_+$ and $q \geq s$ we have

$$q^s = s!\binom{q}{s} + Q(q)$$

where Q is a polynomial of degree $s-1$. Therefore any polynomial P of degree d, for large q, can be uniquely written as

$$P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \dots + c_{d-1}\binom{q}{1} + c_d,$$

with suitable coefficients $c_i, 0 \leq i \leq d$. Since binomial coefficients are integers, if c_i, $0 \leq i \leq d$, are integers, the polynomial P has integral values $P(n)$ for integral $n \geq d$. The next result is a converse of this observation.

1.6. Lemma. *If the polynomial*

$$q \longmapsto P(q) = c_0\binom{q}{d} + c_1\binom{q}{d-1} + \dots + c_{d-1}\binom{q}{1} + c_d$$

takes integral values $P(n)$ for large $n \in \mathbb{Z}$, all its coefficients $c_i, 0 \leq i \leq d$, are integers.

Proof. We prove the statement by induction in d. If $d = 0$ the assertion is obvious. Also

$$\begin{aligned} P(q+1) - P(q) &= \sum_{i=0}^{d} c_i \binom{q+1}{d-i} - \sum_{i=0}^{d} c_i \binom{q}{d-i} \\ &= \sum_{i=0}^{d} c_i \left(\binom{q+1}{d-i} - \binom{q}{d-i}\right) = \sum_{i=0}^{d-1} c_i \binom{q}{d-i-1}, \end{aligned}$$

using the identity

$$\binom{q+1}{s} = \binom{q}{s} + \binom{q}{s-1}$$

for $q \geq s$. Therefore, $q \longmapsto P(q+1) - P(q)$ is a polynomial with coefficients $c_0, c_1, \ldots, c_{d-1}$, and $P(n) \in \mathbb{Z}$ for large $n \in \mathbb{Z}$. By the induction assumption all c_i, $0 \leq i \leq d-1$, are integers. This immediately implies that c_d is an integer too. $\square$

We shall need another related remark. If F is a polynomial of degree d with the leading coefficient a_0,

$$\begin{aligned} G(n) &= F(n) - F(n-1) \\ &= (a_0 n^d + a_1 n^{d-1} + \ldots) - (a_0 (n-1)^d + a_1 (n-1)^{d-1} + \ldots) = a_0 d n^{d-1} + \ldots \end{aligned}$$

is polynomial in n of degree $d-1$ with the leading coefficient $d a_0$. The next result is a converse of this fact.

1.7. Lemma. *Let F be a function on $\mathbb{Z}$ such that*

$$G(n) = F(n) - F(n-1),$$

is equal to a polynomial in n of degree $d-1$ for large $n \in \mathbb{Z}$. Then F is equal to a polynomial in n of degree d for large $n \in \mathbb{Z}$.

Proof. Assume that $G(n) = P(n-1)$ for $n \geq N \geq d$, where P is a polynomial in n of degree $d-1$. Then by 6. we have

$$P(n) = \sum_{i=0}^{d-1} c_i \binom{n}{d-i-1}$$

Hence, for $n \geq N+1$,

$$F(n) = \sum_{k=N+1}^{n} (F(k) - F(k-1)) + F(N) = \sum_{k=N+1}^{n} G(k) + F(N) = \sum_{k=d}^{n} P(k-1) + C$$

where C is a constant. Also, by the identity used in the previous proof,

$$\binom{q}{s} = \sum_{j=s+1}^{q} \left(\binom{j}{s} - \binom{j-1}{s} \right) + 1 = \sum_{j=s+1}^{q} \binom{j-1}{s-1} + 1 = \sum_{j=s}^{q} \binom{j-1}{s-1}.$$

This implies that

$$\begin{aligned} \sum_{k=d}^{n} P(k) &= \sum_{k=d}^{n} \sum_{i=0}^{d-1} c_i \binom{k-1}{d-i-1} = \sum_{i=0}^{d-1} c_i \left(\sum_{k=d}^{n} \binom{k-1}{d-i-1} \right) \\ &= \sum_{i=0}^{d-1} c_i \left(\sum_{k=d-i}^{n} \binom{k-1}{d-i-1} \right) - \sum_{i=1}^{d-1} c_i \left(\sum_{k=d-i}^{d-1} \binom{k-1}{d-i-1} \right) = \sum_{i=0}^{d-1} c_i \binom{n}{d-i} + C' \end{aligned}$$

for some constant C'. $\square$

In particular, it follows that the sum $\sum_{n\leq N}\lambda(M_n)$ is equal to a polynomial of degree $d_\lambda(M)$ for large $N\in\mathbb{Z}$. In addition, if we put

$$\sum_{n\leq N}\lambda(M_n)=a_0N^d+a_1N^{d-1}+\ldots+a_{d-1}N+a_d$$

for large $N\in\mathbb{Z}$, then $d!\,a_0$ is an integer.

For example, if $A=k[X_1,X_2,\ldots,X_s]$, the dimension of the space of all polynomials of degree $\leq N$ is equal to

$$\sum_{n=0}^{N}\dim_k(M_n)=\sum_{n=0}^{N}\binom{s+n-1}{s-1}=\binom{s+N}{s}=\frac{N^s}{s!}+\ldots.$$

2. Dimension of modules over local rings

2.1. Lemma (Nakayama). *Let A be a local ring with the maximal ideal $\mathbf{m}$. Let V be a finitely generated A-module such that $\mathbf{m}V=V$. Then $V=0$.*

Proof. Assume that $V\neq 0$. Then we can find a minimal system of generators $v_1,,\ldots,v_s$ of V as an A-module. By the assumption, $v_s=\sum_{i=1}^{s}m_iv_i$ for some $m_i\in\mathbf{m}$. Therefore, $(1-m_s)v_s=\sum_{i=1}^{s-1}m_iv_i$. Since $1-m_s$ is invertible, this implies that $v_1,\ldots,v_{s-1}$ generate V, contrary to our assumption. $\square$

In the following we assume that A is a nötherian local ring, $\mathbf{m}$ its maximal ideal and $k=A/\mathbf{m}$ the residue field of A.

2.2. Lemma. $\dim_k(\mathbf{m}/\mathbf{m}^2)<+\infty$.

Proof. By the nötherian assumption $\mathbf{m}$ is finitely generated. If $a_1,\ldots,a_p$ are generators of $\mathbf{m}$, their images $\bar{a}_1,\ldots,\bar{a}_p$ in $\mathbf{m}/\mathbf{m}^2$ span it as a vector space over k. $\square$

Let $s=\dim_k(\mathbf{m}/\mathbf{m}^2)$. Then we can find $a_1,\ldots,a_s\in\mathbf{m}$ such that $\bar{a}_1,\ldots,\bar{a}_s$ form a basis of $\mathbf{m}/\mathbf{m}^2$. We claim that they generate $\mathbf{m}$. Let I be the ideal generated by $a_1,\ldots,a_s$. Then $I+\mathbf{m}^2=\mathbf{m}$ and $\mathbf{m}(\mathbf{m}/I)=\mathbf{m}/I$, hence, by 1, $\mathbf{m}/I=0$. Therefore, we proved:

2.3. Lemma. $\dim_k(\mathbf{m}/\mathbf{m}^2)$ *is the minimal number of generators of* $\mathbf{m}$.

Any s-tuple $(a_1,\ldots,a_s)$ of elements from $\mathbf{m}$ such that $(\bar{a}_1,\ldots,\bar{a}_s)$ form a basis of $\mathbf{m}/\mathbf{m}^2$ is called a *coordinate system* in A.

Clearly, $(\mathbf{m}^p\,;\,p\in\mathbb{Z}_+)$ is a decreasing filtration of A. Therefore, we can form $\operatorname{Gr}A=\oplus_{p=0}^{\infty}\mathbf{m}^p/\mathbf{m}^{p+1}$. We claim that $\operatorname{Gr}A$ is a finitely generated algebra over k and therefore a nötherian graded ring. Actually, the map $X_i\longmapsto\bar{a}_i\in\mathbf{m}/\mathbf{m}^2\subset\operatorname{Gr}A$ extends to a surjective morphism of $k[X_1,\ldots,X_s]$ onto $\operatorname{Gr}A$.

Let M be a finitely generated A-module. Then we can define a decreasing filtration of M by $(\mathbf{m}^pM\,;\,p\in\mathbb{Z}_+)$ and consider the graded $\operatorname{Gr}A$-module $\operatorname{Gr}M=\oplus_{p=0}^{\infty}\mathbf{m}^pM/\mathbf{m}^{p+1}M$.

2.4. LEMMA. *If M is a finitely generated A-module,* $\operatorname{Gr} M$ *is a finitely generated* $\operatorname{Gr} A$-*module.*

PROOF. From the definition of the graded module $\operatorname{Gr} M$ we see that $\mathbf{m} \cdot \operatorname{Gr}_p M = \operatorname{Gr}_{p+1} M$ for all $p \in \mathbb{Z}_+$. Hence $\operatorname{Gr}_0 M = M/\mathbf{m}M$ generates $\operatorname{Gr} M$. On the other hand, $M/\mathbf{m}M$ is a finite dimensional vector space over k. $\square$

This implies, by 1.2, that $\dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M) < +\infty$, in particular $\mathbf{m}^p M/\mathbf{m}^{p+1} M$ are A-modules of finite length. Since length is clearly an additive function, by 1.5. we see that $p \longmapsto \operatorname{length}_A(\mathbf{m}^p M/\mathbf{m}^{p+1} M) = \dim_k(\mathbf{m}^p M/\mathbf{m}^{p+1} M)$ is equal to a polynomial in p with rational coefficients for large $p \in \mathbb{Z}_+$. Moreover, the function

$$p \longmapsto \operatorname{length}_A(M/\mathbf{m}^p M) = \sum_{q=0}^{p-1} \operatorname{length}_A(\mathbf{m}^q M/\mathbf{m}^{q+1} M)$$

is equal to a polynomial with rational coefficients for large $p \in \mathbb{Z}_+$, and its leading coefficient is of the form $e\frac{p^d}{d!}$, where $e, d \in \mathbb{Z}_+$. We put $d(M) = d$ and $e(M) = e$, and call these numbers the *dimension* and *multiplicity* of M.

Now we want to discuss some properties of the function $M \longmapsto d(M)$. The critical result in controlling the filtrations of A-modules is the Artin-Rees lemma.

2.5. THEOREM (ARTIN, REES). *Let M be a finitely generated A-module and N its submodule. Then there exists $m_0 \in \mathbb{Z}_+$ such that*

$$\mathbf{m}^{p+m_0} M \cap N = \mathbf{m}^p(\mathbf{m}^{m_0} M \cap N)$$

for all $p \in \mathbb{Z}_+$.

PROOF. Put $A^* = \oplus_{n=0}^{\infty} \mathbf{m}^n$. Then A^* has a natural structure of a graded ring. Let $(a_1, \dots, a_s)$ be a coordinate system in A. Then we have a natural surjective morphism $A[a_1, \dots, a_s] \longrightarrow A^*$, and A^* is a graded nötherian ring. Let $M^* = \oplus_{n=0}^{\infty} \mathbf{m}^n M$. Then M^* is a graded A^*-module. It is clearly generated by $M_0^* = M$ as an A^*-module. Since M is a finitely generated A-module, we conclude that M^* is a finitely generated A^*-module.

In addition, put $N^* = \oplus_{n=0}^{\infty}(N \cap \mathbf{m}^n M) \subset M^*$. Then

$$\mathbf{m}^p(N \cap \mathbf{m}^n M) \subset \mathbf{m}^p N \cap \mathbf{m}^{n+p} M \subset N \cap \mathbf{m}^{n+p} M$$

implies that N^* is an A^*-submodule of M^*. Since A^* is a nötherian ring, N^* is finitely generated. There exists $m_0 \in \mathbb{Z}_+$ such that $\oplus_{n=0}^{m_0}(N \cap \mathbf{m}^n M)$ generates N^*. Then for any $p \in \mathbb{Z}_+$,

$$N \cap \mathbf{m}^{p+m_0} M = \sum_{s=0}^{m_0} \mathbf{m}^{p+m_0-s}(N \cap \mathbf{m}^s M) \subset \mathbf{m}^p(N \cap \mathbf{m}^{m_0} M) \subset N \cap \mathbf{m}^{p+m_0} M. \quad \square$$

This result has the following consequence — the Krull intersection theorem.

2.6. Theorem (Krull). *Let M be a finitely generated A-module. Then $\cap_{p=0}^{\infty} \mathbf{m}^p M = 0$.*

Proof. Put $E = \cap_{p=0}^{\infty} \mathbf{m}^p M$. Then, by 5.,

$$E = \mathbf{m}^{p+m_0} M \cap E = \mathbf{m}^p (\mathbf{m}^{m_0} M \cap E) = \mathbf{m}^p E,$$

in particular, $\mathbf{m}E = E$, and $E = 0$ by Nakayama lemma. □

2.7. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated A-modules. Then

(i) $d(M) = \max(d(M'), d(M''))$;
(ii) *if* $d(M) = d(M') = d(M'')$, *we have* $e(M) = e(M') + e(M'')$.

Proof. We can view M' as a submodule of M. If we equip M with the filtration $(\mathbf{m}^p M\,;\, p \in \mathbb{Z}_+)$ and M' and M'' with the induced filtrations $(M' \cap \mathbf{m}^p M\,;\, p \in \mathbb{Z}_+)$ and $(\mathbf{m}^p M''\,;\, p \in \mathbb{Z}_+)$ we get the exact sequence

$$0 \longrightarrow \operatorname{Gr} M' \longrightarrow \operatorname{Gr} M \longrightarrow \operatorname{Gr} M'' \longrightarrow 0.$$

This implies that for any $p \in \mathbb{Z}_+$

$$\begin{aligned}&\operatorname{length}_A(\mathbf{m}^p M / \mathbf{m}^{p+1} M)\\ &\qquad = \operatorname{length}_A((M' \cap \mathbf{m}^p M)/(M' \cap \mathbf{m}^{p+1} M)) + \operatorname{length}_A(\mathbf{m}^p M'' / \mathbf{m}^{p+1} M'')\end{aligned}$$

and, by summation,

$$\operatorname{length}_A(M/\mathbf{m}^p M) = \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M)) + \operatorname{length}_A(M''/\mathbf{m}^p M'').$$

Therefore the function $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ is equal to a polynomial in p for large $p \in \mathbb{Z}_+$. On the other hand, by 5,

$$\mathbf{m}^{p+m_0} M' \subset \mathbf{m}^{p+m_0} M \cap M' \subset \mathbf{m}^p M';$$

hence, for large $p \in \mathbb{Z}_+$, the functions $p \longmapsto \operatorname{length}_A(M'/(M' \cap \mathbf{m}^p M))$ and $p \longmapsto \operatorname{length}_A(M'/\mathbf{m}^p M')$ are given by polynomials in p with equal leading terms. □

2.8. Corollary. *Let A be a nötherian local ring with $s = \dim_k(\mathbf{m}/\mathbf{m}^2)$. Then, for any finitely generated A-module M we have $d(M) \leq s$.*

Proof. By 7. it is enough to show that $d(A) \leq s$. This follows immediately from the existence of a surjective homomorphism of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$, and the fact that the dimension of the space of polynomials of degree $\leq n$ in s variables is a polynomial in n of degree s. □

A nötherian local ring is called *regular* if $d(A) = \dim_k(\mathbf{m}/\mathbf{m}^2)$.

2.9. Theorem. *Let A be a nötherian local ring and $(a_1, a_2, \dots, a_s)$ a coordinate system in A. Then the following conditions are equivalent:*

(i) *A is a regular local ring,*
(ii) *the canonical morphism of $k[X_1, X_2, \dots, X_s]$ into $\operatorname{Gr} A$ defined by $X_i \longmapsto \bar{a}_i$, $1 \leq i \leq s$, is an isomorphism.*

Proof. By definition, the canonical morphism of $k[X_1, \dots, X_s]$ into $\operatorname{Gr} A$ is surjective. Let I be the graded ideal which is the kernel of the natural surjection of $k[X_1, \dots, X_s]$ onto $\operatorname{Gr} A$. If $I \neq 0$, it contains a homogeneous polynomial P of degree $d > 0$. Let J be the ideal in $k[X_1, X_2, \dots, X_s]$ generated by P. Then its Poincaré series is $P(J,t) = \frac{t^d}{(1-t)^s}$. Clearly,

$$\begin{aligned} P(k[X_1, X_2, \dots, X_s]/J, t) &= P(k[X_1, X_2, \dots, X_s], t) - P(J,t) \\ &= \frac{1-t^d}{(1-t)^s} = \frac{1 + t + \ldots + t^{d-1}}{(1-t)^{s-1}}. \end{aligned}$$

The order of the pole of the Poincaré series $P(k[X_1, X_2, \dots, X_s]/J, t)$ at 1 is $s-1$, and by 1.5 the function $\dim_k (k[X_1, X_2, \dots, X_s]/J)_n$ is given by a polynomial in n of degree $s-2$ for large $n \in \mathbb{Z}_+$. It follows that the function $\dim_k (k[X_1, X_2, \dots, X_s]/I)_n = \dim_k \operatorname{Gr}_n A$ is given by a polynomial in n of degree $\leq s-2$ for large $n \in \mathbb{Z}_+$. This implies that $d(A) \leq s-1$. Therefore, $I = 0$ if and only if $d(A) = s$. $\square$

2.10. Theorem. *Let A be a regular local ring. Then A is integral.*

Proof. Let $a, b \in A$ and $a \neq 0$, $b \neq 0$. Then, by 6., we can find $p, q \in \mathbb{Z}_+$ such that $a \in \mathbf{m}^p$, $a \notin \mathbf{m}^{p+1}$ and $b \in \mathbf{m}^q$, $b \notin \mathbf{m}^{q+1}$. Then their images $\bar{a} \in \operatorname{Gr}_p A$ and $\bar{b} \in \operatorname{Gr}_q A$ are different form zero, and since $\operatorname{Gr} A$ is integral by 9., we see that $\bar{a}\bar{b} \neq 0$. Therefore, $ab \neq 0$. $\square$

Finally we want to discuss an example which will play an important role later. Let k be a field, $A = k[X_1, X_2, \dots, X_n]$ be the ring of polynomials in n-variables with coefficients in k and $\hat{A} = k[[X_1, X_2, \dots, X_n]]$ the ring of formal power series in n-variables with coefficients in k. It is easy to check that $\hat{A}$ is a local ring with maximal ideal $\hat{\mathbf{m}}$ generated by $X_1, X_2, \dots X_n$. Also, the canonical morphism from $k[X_1, X_2, \dots, X_n]$ into $\operatorname{Gr} \hat{A}$ is clearly an isomorphism.

For any $x \in k^n$ we denote by $\mathbf{m}_x$ the maximal ideal in A generated by $X_i - x_i$, $1 \leq i \leq n$. Then its complement in A is a multiplicative system in A, and we denote by A_x the corresponding localization of A. It is isomorphic to the ring of all rational functions on k^n regular at x. This is clearly a nötherian local ring. The localization of $\mathbf{m}_x$ is the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ of all rational functions vanishing at x. The automorphism of A defined by $X_i \longmapsto X_i - x_i$, $1 \leq i \leq n$, gives an isomorphism of A_0 with A_x for any $x \in k^n$. On the other hand, the natural homomorphism of A into $\hat{A}$ extends to an injective homomorphism of A_0 into $\hat{A}$. This homomorphism preserves the filtrations on these local rings and induces a canonical isomorphism of $\operatorname{Gr} A_0$ onto $\operatorname{Gr} \hat{A}$. Therefore we have the following result.

2.11. Proposition. *The rings A_x, $x \in k^n$, are n-dimensional regular local rings.*

3. Dimension of modules over filtered rings

Let D be a ring with identity and $(D_n\,;\,n\in\mathbb{Z})$ an increasing filtration of D by additive subgroups such that

(i) $D_n=\{0\}$ for $n<0$;
(ii) $\bigcup_{n\in\mathbb{Z}} D_n = D$;
(iii) $1\in D_0$;
(iv) $D_n\cdot D_m\subset D_{n+m}$, for any $n,m\in\mathbb{Z}$;
(v) $[D_n,D_m]\subset D_{n+m-1}$, for any $n,m\in\mathbb{Z}$.

Then $\operatorname{Gr} D=\oplus_{n\in\mathbb{Z}}\operatorname{Gr}_n D=\oplus_{n\in\mathbb{Z}}D_n/D_{n-1}$ is a graded ring with identity. The property (v) implies that it is commutative. In particular, D_0 is a commutative ring with identity. Therefore, we can view $\operatorname{Gr} D$ as an algebra over D_0. Let's assume in addition that D satisfies

(vi) $\operatorname{Gr} D$ is a nötherian ring;
(vii) $\operatorname{Gr}_1 D$ generates $\operatorname{Gr} D$ as a D_0-algebra.

Then, by 1.1, D_0 is a nötherian ring. Moreover, by (vi), (vii) and 1.2 we know that we can choose finitely many elements $x_1,x_2,\dots,x_s\in\operatorname{Gr}_1 D$ such that $\operatorname{Gr} D$ is generated by them as a D_0-algebra. Clearly, by (vii), we also have

$$\operatorname{Gr}_{n+1} D=\operatorname{Gr}_1 D\cdot\operatorname{Gr}_n D \text{ for } n\in\mathbb{Z}_+$$

and therefore

$$D_{n+1}=D_n\cdot D_1 \text{ for } n\in\mathbb{Z}_+.$$

Let M be a D-module. An increasing filtration $\mathrm{F}\,M=(\mathrm{F}_n\,M\,;\,n\in\mathbb{Z})$ of M by additive subgroups is a *D-module filtration* if $D_n\cdot\mathrm{F}_m\,M\subset\mathrm{F}_{m+n}\,M$, for $n,m\in\mathbb{Z}$. In particular, $\mathrm{F}_n\,M$ are D_0-modules. A D-module filtration is called *stable* if there exists $m_0\in\mathbb{Z}$ such that $D_n\cdot\mathrm{F}_m\,M=\mathrm{F}_{m+n}\,M$ for all $n\in\mathbb{Z}_+$ and $m\geq m_0$. A D-module filtration is called *good* if

(i) $\mathrm{F}_n\,M=\{0\}$ for sufficiently negative $n\in\mathbb{Z}$;
(ii) the filtration $\mathrm{F}\,M$ is exhaustive (i.e. $\cup_{n\in\mathbb{Z}}\mathrm{F}_n\,M=M$);
(iii) $\mathrm{F}_n\,M$, $n\in\mathbb{Z}$, are finitely generated D_0-modules;
(iv) the filtration $\mathrm{F}\,M$ is stable.

3.1. Lemma. *Let $\mathrm{F}\,M$ be an exhaustive hausdorff D-module filtration of M. Then the following statements are equivalent:*

(i) *$\mathrm{F}\,M$ is a good filtration;*
(ii) *$\operatorname{Gr} D$-module $\operatorname{Gr} M$ is finitely generated.*

Proof. First we prove (i)⇒(ii). There exists $m_0\in\mathbb{Z}$ such that $D_n\cdot\mathrm{F}_{m_0}\,M=\mathrm{F}_{n+m_0}\,M$ for all $n\in\mathbb{Z}_+$. Therefore $\operatorname{Gr}_n D\cdot\operatorname{Gr}_{m_0} M=\operatorname{Gr}_{n+m_0} M$ for all $n\in\mathbb{Z}_+$. It follows that $\oplus_{n\leq m_0}\operatorname{Gr}_n M$ generates $\operatorname{Gr} M$ as a $\operatorname{Gr} D$-module. Since $\mathrm{F}_n\,M$ are finitely generated D_0-modules, $\operatorname{Gr}_n M$ are finitely generated D_0-modules too. This implies, since $\mathrm{F}_n\,M=\{0\}$ for sufficiently negative $n\in\mathbb{Z}$, that $\oplus_{n\leq m_0}\operatorname{Gr}_n M$ is a finitely generated D_0-module.

(ii)⇒(i). Clearly, $\operatorname{Gr}_n M = \{0\}$ for sufficiently negative $n \in \mathbb{Z}$. Also, by 1.2, all $\operatorname{Gr}_n M$ are finitely generated D_0-modules. The exact sequence

$$0 \longrightarrow \mathrm{F}_{n-1} M \longrightarrow \mathrm{F}_n M \longrightarrow \operatorname{Gr}_n M \longrightarrow 0$$

implies that $\mathrm{F}_n M = \mathrm{F}_{n-1} M$ for sufficiently negative n, hence there exists $n_0 \in \mathbb{Z}$ such that $\cap_{n\in\mathbb{Z}} \mathrm{F}_n M = \mathrm{F}_{n_0} M$. Since the filtration $\mathrm{F} M$ is hausdorff, $\mathrm{F}_{n_0} M = \{0\}$. This implies, by induction in n, that all $\mathrm{F}_n M$ are finitely generated D_0-modules. Let $m_0 \in \mathbb{Z}$ be such that $\oplus_{n\le m_0} \operatorname{Gr}_n M$ generates $\operatorname{Gr} M$ as $\operatorname{Gr} D$-module. Let $m \ge m_0$. Then

$$\begin{aligned}\operatorname{Gr}_{m+1} M &= \bigoplus_{k\le m_0} \operatorname{Gr}_{m+1-k} D \cdot \operatorname{Gr}_k M \\ &= \bigoplus_{k\le m_0} \operatorname{Gr}_1 D \cdot \operatorname{Gr}_{m-k} D \cdot \operatorname{Gr}_k M \subset \operatorname{Gr}_1 D \cdot \operatorname{Gr}_m M \subset \operatorname{Gr}_{m+1} M,\end{aligned}$$

i.e., $\operatorname{Gr}_1 D \cdot \operatorname{Gr}_m M = \operatorname{Gr}_{m+1} M$. This implies that

$$\mathrm{F}_{m+1} M = D_1 \cdot \mathrm{F}_m M + \mathrm{F}_m M = D_1 \cdot \mathrm{F}_m M$$

and by induction in n,

$$\mathrm{F}_{m+n} M = D_1 \cdot D_1 \cdot \ldots \cdot D_1 \cdot \mathrm{F}_m M = D_n \cdot \mathrm{F}_m M \subset \mathrm{F}_{m+n} M.$$

Therefore $\mathrm{F} M$ is a good filtration. □

In particular, $(D_n\,;\, n \in \mathbb{Z})$ is a good filtration of D considered as a D-module for left multiplication.

Remark. From the proof it follows that the stability condition in the definition of a good filtration can be replaced by an apparently weaker condition:

(iv)′ There exists $m_0 \in \mathbb{Z}$ such that $D_n \cdot F_{m_0} M = F_{m_0+n} M$ for all $n \in \mathbb{Z}_+$.

3.2. Lemma. *Let M be a D-module with a good filtration $\mathrm{F} M$. Then M is finitely generated.*

Proof. By definition, $\cup_{n\in\mathbb{Z}} \mathrm{F}_n M = M$ and $\mathrm{F}_{n+m_0} M = D_n \cdot F_{m_0} M$ for $n \in \mathbb{Z}_+$ and some sufficiently large $m_0 \in \mathbb{Z}$. Therefore, $\mathrm{F}_{m_0} M$ generates M as a D-module. Since $\mathrm{F}_{m_0} M$ is a finitely generated D_0-module, the assertion follows. □

3.3. Lemma. *Let M be a finitely generated D-module. Then M admits a good filtration.*

Proof. Let U be a finitely generated D_0-module which generates M as a D-module. Put $\mathrm{F}_n M = 0$ for $n < 0$ and $\mathrm{F}_n M = D_n \cdot U$ for $n \ge 0$. Then $U = \operatorname{Gr}_0 M$, and

$$\operatorname{Gr}_n M = \mathrm{F}_n M / \mathrm{F}_{n-1} M = (D_n \cdot U)/(D_{n-1} \cdot U) \subset \operatorname{Gr}_n D \cdot \operatorname{Gr}_0 M,$$

hence $\operatorname{Gr} M$ is finitely generated as a $\operatorname{Gr} D$-module. The statement follows from 1. □

The lemmas 2. and 3. imply that the D-modules admitting good filtrations are precisely the finitely generated D-modules.

3.4. Proposition. *D is a left and right nötherian ring.*

Proof. Let L be a left ideal in D. The natural filtration of D induces a filtration $(L_n = L \cap D_n\,;\, n \in \mathbb{Z})$, on L. This is evidently a D-module filtration. The graded module $\operatorname{Gr} L$ is naturally an ideal in $\operatorname{Gr} D$, and since $\operatorname{Gr} D$ is a nötherian ring, it is finitely generated as $\operatorname{Gr} D$-module. Therefore, the filtration $(L_n\,;\, n \in \mathbb{Z})$ is good by 1, and L is finitely generated by 2. This proves that D is left nötherian.

To get the right nötherian property one has to replace D with its opposite ring D°. $\square$

If we have two filtrations $\mathrm{F}\,M$ and $\mathrm{F}'\,M$ of a D-module M, we say that $\mathrm{F}\,M$ is *finer* than $\mathrm{F}'\,M$ if there exists a number $k \in \mathbb{Z}_+$ such that $\mathrm{F}_n\,M \subset \mathrm{F}'_{n+k}\,M$ for all $n \in \mathbb{Z}$. If $\mathrm{F}\,M$ is finer than $\mathrm{F}'\,M$ and $\mathrm{F}'\,M$ finer than $\mathrm{F}\,M$, we say that they are *equivalent*.

3.5. Lemma. *Let $\mathrm{F}\,M$ be a good filtration on a finitely generated D-module M. Then $\mathrm{F}\,M$ is finer than any other exhaustive D-module filtration on M.*

Proof. Fix $m_0 \in \mathbb{Z}_+$ such that $D_n \cdot \mathrm{F}_{m_0}\,M = \mathrm{F}_{n+m_0}\,M$ for all $n \in \mathbb{Z}_+$. Let $\mathrm{F}'\,M$ be another exhaustive D-module filtration on M. Then $\mathrm{F}_{m_0}\,M$ is finitely generated as a D_0-module. Since $\mathrm{F}'\,M$ is exhaustive, it follows that there exists $p \in \mathbb{Z}$ such that $\mathrm{F}_{m_0}\,M \subset \mathrm{F}'_p\,M$. Since $\mathrm{F}\,M$ is a good filtration, there exists n_0 such that $\mathrm{F}_{n_0}\,M = \{0\}$. Let $k = p + |n_0|$. Then, for $n_0 \leq m \leq m_0$ we have

$$\mathrm{F}_m\,M \subset \mathrm{F}_{m_0}\,M \subset \mathrm{F}'_p\,M \subset \mathrm{F}'_{m+k}\,M,$$

and for $m > m_0$,

$$\mathrm{F}_m\,M = D_{m-m_0} \cdot \mathrm{F}_{m_0}\,M \subset D_{m-m_0} \cdot \mathrm{F}'_k\,M \subset \mathrm{F}'_{m-m_0+k}\,M \subset \mathrm{F}'_{m+k}\,M. \quad \square$$

3.6. Corollary. *Any two good filtrations on a finitely generated D-module are equivalent.*

Let M be a finitely generated D-module and $\mathrm{F}\,M$ a good filtration on M. Then $\operatorname{Gr} M$ is a finitely generated $\operatorname{Gr} D$-module, hence we can apply the results on Hilbert polynomials from 1.. Let λ be an additive function on finitely generated D_0-modules. Assume also that λ takes only nonnegative values on objects of $\mathcal{M}_{fg}(D_0)$. Then, by 1.5,

$$\lambda(\mathrm{F}_n\,M) - \lambda(\mathrm{F}_{n-1}\,M) = \lambda(\operatorname{Gr}_n M)$$

is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. By 1.7 this implies that $\lambda(\mathrm{F}_n\,M)$ is equal to a polynomial in n for large $n \in \mathbb{Z}_+$. If $\mathrm{F}'\,M$ is another good filtration on M, by 6. we know that $\mathrm{F}\,M$ an $\mathrm{F}'\,M$ are equivalent, i.e., there is a number $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}_n\,M \subset \mathrm{F}'_{n+k}\,M \subset \mathrm{F}_{n+2k}\,M$$

for all $n \in \mathbb{Z}$. Since λ is additive and takes nonnegative values only, we conclude that

$$\lambda(\mathrm{F}_n\,M) \leq \lambda(\mathrm{F}'_{n+k}\,M) \leq \lambda(\mathrm{F}_{n+2k}\,M)$$

for all $n \in \mathbb{Z}$. This implies that the polynomials representing $\lambda(\mathrm{F}_n M)$ and $\lambda(\mathrm{F}'_n M)$ for large n have equal leading terms. Let's denote the common degree of these polynomials by $d_\lambda(M)$ and call it the *dimension* of the D-module M (with respect to λ). By 1.6 the leading coefficient of these polynomials has the form $e_\lambda(M)/d_\lambda(M)!$ where $e_\lambda(M) \in \mathbb{N}$. We call $e_\lambda(M)$ the *multiplicity* of the D-module M (with respect to λ).

Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of D-modules. If M is equipped by a D-module filtration $\mathrm{F}\, M$, it induces filtrations $\mathrm{F}\, M' = (f^{-1}(f(M') \cap \mathrm{F}_n M)\,;\, n \in \mathbb{Z})$ on M' and $\mathrm{F}\, M'' = (\,g(F_n M)\,;\, n \in \mathbb{Z})$ on M''. Clearly, these filtrations are D-module filtrations. Moreover, the sequence

$$0 \longrightarrow \mathrm{Gr}\, M' \xrightarrow{\mathrm{Gr}\, f} \mathrm{Gr}\, M \xrightarrow{\mathrm{Gr}\, g} \mathrm{Gr}\, M'' \longrightarrow 0$$

is exact. If the filtration $\mathrm{F}\, M$ is good, $\mathrm{Gr}\, M$ is a finitely generated $\mathrm{Gr}\, D$-module, hence both $\mathrm{Gr}\, M'$ and $\mathrm{Gr}\, M''$ are finitely generated $\mathrm{Gr}\, D$-modules. By 1, $\mathrm{F}\, M'$ and $\mathrm{F}\, M''$ are good filtrations. Therefore, we proved the following result.

3.7. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of D-modules. If $\mathrm{F}\, M$ is a good filtration on M, the induced filtrations $\mathrm{F}\, M'$ and $\mathrm{F}\, M''$ are good.

By the preceding discussion

$$\lambda(\mathrm{Gr}_n M) = \lambda(\mathrm{Gr}_n M') + \lambda(\mathrm{Gr}_n M'')$$

for all $n \in \mathbb{Z}$. This implies, by induction in n, that

$$\lambda(\mathrm{F}_n M) = \lambda(\mathrm{F}_n M') + \lambda(\mathrm{F}_n M'')$$

for all $n \in \mathbb{Z}$. This leads to the following result.

3.8. Proposition. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finitely generated D-modules. Then

(i) $d_\lambda(M) = \max(d_\lambda(M'), d_\lambda(M''))$;
(ii) *if* $d_\lambda(M) = d_\lambda(M') = d_\lambda(M'')$, *then* $e_\lambda(M) = e_\lambda(M') + e_\lambda(M'')$.

Finally, let ϕ be an automorphism of the ring D such that $\phi(D_0) = D_0$. We can define a functor $\tilde{\phi}$ from $\mathcal{M}(D)$ into itself which attaches to a D-module M a D-module $\tilde{\phi}(M)$ with the same underlying additive group structure and with the action of D given by $(T, m) \longmapsto \phi(T)m$ for $T \in D$ and $m \in M$. Clearly, $\tilde{\phi}$ is an automorphism of the category $\mathcal{M}(D)$, and it preserves finitely generated D-modules.

3.9. Proposition. *Let M be a finitely generated D-module. Then*

$$d_\lambda(\tilde{\phi}(M)) = d_\lambda(M).$$

Proof. Let $T_1, T_2, \ldots, T_s$ be the representatives in D_1 of classes in $\operatorname{Gr}_1 D$ generating $\operatorname{Gr} D$ as a D_0-algebra. Then there exists $d \in \mathbb{N}$ such that $\phi(T_i) \in D_d$ for $1 \le i \le s$. Since $T_1, T_2, \ldots, T_s$ and 1 generate D_1 as a D_0-module, we conclude that $\phi(D_1) \subset D_d$.

Let $\mathrm{F}\, M$ be a good filtration of M. Define a filtration $\mathrm{F}\, \tilde{\phi}(M)$ by

$$\mathrm{F}_p\, \tilde{\phi}(M) = \mathrm{F}_{dp}\, M \text{ for } p \in \mathbb{Z}.$$

Clearly, $\mathrm{F}\, \tilde{\phi}(M)$ is an increasing filtration of $\tilde{\phi}(M)$ by finitely generated D_0-submodules. Also,

$$D_1 \cdot \mathrm{F}_m\, \tilde{\phi}(M) = \phi(D_1)\, \mathrm{F}_{dm}\, M \subset D_d\, \mathrm{F}_{dm}\, M \subset \mathrm{F}_{d(m+1)}\, M = \mathrm{F}_{m+1}\, \tilde{\phi}(M)$$

for $m \in \mathbb{Z}$. Hence, by induction, we have

$$D_n \cdot \mathrm{F}_m\, \tilde{\phi}(M) = D_1 \cdot D_{n-1} \cdot \mathrm{F}_m\, \tilde{\phi}(M) \subset D_1\, \mathrm{F}_{m+n-1}\, \tilde{\phi}(M) \subset \mathrm{F}_{m+n}\, \tilde{\phi}(M)$$

for all $n, m \in \mathbb{Z}$, i.e., $\mathrm{F}\, \tilde{\phi}(M)$ is a D-module filtration. By 5, there exists a good filtration $\mathrm{F}'\, \tilde{\phi}(M)$ which is finer than this filtration, i.e, there exists $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}'_n\, \tilde{\phi}(M) \subset \mathrm{F}_{n+k}\, \tilde{\phi}(M) = \mathrm{F}_{d(n+k)}\, M$$

for all $n \in \mathbb{Z}$. Therefore,

$$\lambda(\mathrm{F}'_n\, \tilde{\phi}(M)) \le \lambda(\mathrm{F}_{d(n+k)}\, M)$$

for $n \in \mathbb{Z}$. For large $n \in \mathbb{Z}$, $\lambda(\mathrm{F}_{d(n+k)}\, M)$ is equal to a polynomial in n with the leading term equal to

$$\frac{e_\lambda(M) d^{d_\lambda(M)}}{d_\lambda(M)!} n^{d_\lambda(M)}.$$

Since $\lambda(\mathrm{F}'_n\, \tilde{\phi}(M))$ is also given by a polynomial of degree $d_\lambda(\tilde{\phi}(M))$ for large $n \in \mathbb{Z}$, we conclude that $d_\lambda(\tilde{\phi}(M)) \le d_\lambda(M)$. By applying the same reasoning to ϕ^{-1} we also conclude that

$$d_\lambda(M) = d_\lambda(\tilde{\phi}^{-1}(\tilde{\phi}(M))) \le d_\lambda(\tilde{\phi}(M)). \quad \square$$

4. Dimension of modules over polynomial rings

Let $A = k[X_1, \dots, X_n]$ where k is an algebraically closed field. We can filter A by degree of polynomials, i.e., we can put $A_m = \{\sum a_I x^I \mid |I| \leq m\}$. Then $\operatorname{Gr} A = k[X_1, \dots, X_n]$, hence A satisfies properties (i)-(vii) from the preceding section.

Since $A_0 = k$ we can take for the additive function λ the function $\dim_k$. This leads to notions of dimension $d(M)$ and multiplicity $e(M)$ of a finitely generated A-module M. We know that for any $p \in \mathbb{Z}_+$, we have

$$\dim_k A_p = \binom{n+p}{n} = \frac{p^n}{n!} + \text{ lower order terms in } p,$$

i.e., $d(A) = n$ and $e(A) = 1$. In addition, for any finitely generated A-module M we have an exact sequence

$$0 \longrightarrow K \longrightarrow A^p \longrightarrow M \longrightarrow 0,$$

hence, by 3.8, $d(M) \leq n$. We shall give later a geometric interpretation of $d(M)$.

Let $x \in k^n$ and denote by $\mathbf{m}_x$ be the maximal ideal in $k[X_1, \dots, X_n]$ of all polynomials vanishing at x. We denote by A_x the localization of A at x, i.e., the ring of all rational k-valued functions on k^n regular at x. As we have seen in 2.11, A_x is an n-dimensional regular local ring with the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ consisting of all rational k-valued functions on k^n vanishing at x. Let M be an A-module. Its localization M_x at x is an A_x-module. We define the *support* of M by $\operatorname{supp}(M) = \{x \in k^n \mid M_x \neq 0\}$.

4.1. Lemma. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then

$$\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'').$$

Proof. By exactness of localization we see that

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0$$

is an exact sequence of A_x-modules. This immediately implies our statement. □

For an ideal $I \subset k[X_1, \dots, X_n]$ we denote $V(I) = \{x \in k^n \mid f(x) = 0 \text{ for } f \in I\}$.

4.2. Proposition. *Let M be a finitely generated A-module and I its annihilator in A. Then* $\operatorname{supp}(M) = V(I)$.

Proof. We prove the statement by induction in the number of generators of M.

Assume first that M has one generator, i.e., $M = A/I$. Then $M_x = (A/I)_x = A_x/I_x$. Let $x \in V(I)$. Then $I \subset \mathbf{m}_x$ and $I_x \subset \mathbf{n}_x$. Hence $I_x \neq A_x$. It follows that $(A/I)_x \neq 0$ and $x \in \operatorname{supp}(M)$. Conversely, if $x \notin V(I)$, there exists $f \in I$ such that $f(x) \neq 0$, i.e., $f \notin \mathbf{m}_x$.

It follows that $f(g+I)=0$ in A/I for any $g \in A$, hence $(A/I)_x = 0$ and $x \notin \operatorname{supp}(A/I)$. Therefore, $\operatorname{supp}(A/I) = V(I)$.

Now we consider the general situation. Let $m_1, \dots, m_p$ be a set of generators of M. Denote by M' the submodule generated by $m_1, \dots, m_{p-1}$. Then we have the exact sequence

$$0 \to M' \to M \to M'' \to 0$$

and M'' is cyclic. Moreover, by 1, $\operatorname{supp}(M) = \operatorname{supp}(M') \cup \operatorname{supp}(M'')$. Hence by the induction assumption $\operatorname{supp}(M) = V(I') \cup V(I'')$ where I' and I'' are the annihilators of M' and M'' respectively. Clearly, $I' \cdot I''$ is in the annihilator I of M and $I' \cdot I'' \subset I \subset r(I' \cap I'')$. This implies that $V(I' \cap I'') \subset V(I) \subset V(I' \cdot I'') \subset V(I') \cup V(I'') = V(I' \cap I'')$; i.e., $\operatorname{supp}(M) = V(I)$. □

The next lemma is useful in some reduction arguments.

4.3. Lemma. *Let B be a nötherian commutative ring and $M \neq 0$ be a finitely generated B-module. Then there exist a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ of M by B-submodules, and prime ideals J_i of B such that $M_i/M_{i-1} \cong B/J_i$, for $1 \leq i \leq n$.*

Proof. For any $x \in M$ we put $\operatorname{Ann}(x) = \{a \in B \mid ax = 0\}$. Let $\mathcal{A}$ be the family of all such ideals $\operatorname{Ann}(x)$, $x \in M$, $x \neq 0$. Because B is a nötherian ring, $\mathcal{A}$ has maximal elements. Let I be a maximal element in $\mathcal{A}$. We claim that I is prime. Let $x \in M$ be such that $I = \operatorname{Ann}(x)$. Then $ab \in I$ implies $abx = 0$. Assume that $b \notin I$, i.e., $bx \neq 0$. Then $I \subset \operatorname{Ann}(bx)$ and $a \in \operatorname{Ann}(bx)$. By the maximality of I, $a \in \operatorname{Ann}(bx) = I$, and I is prime. Therefore, there exists $x \in M$ such that $J_1 = \operatorname{Ann}(x)$ is prime. If we put $M_1 = Bx$, $M_1 \cong B/J_1$. Now, denote by $\mathcal{F}$ the family of all B-submodules of M having filtrations $0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$ such that $N_i/N_{i-1} \cong B/J_i$ for some prime ideals J_i. Since M is a nötherian module, $\mathcal{F}$ contains a maximal element L. Assume that $L \neq M$. Then we would have the exact sequence:

$$0 \to L \to M \to L' \to 0,$$

and by the first part of the proof, L' would have a submodule N' of the form B/J' for some prime ideal J', contradicting the maximality of L. Hence, $L = M$. This proves the existence of the filtration with required properties. □

4.4. Theorem. *Let M be a finitely generated A-module. Then $d(M) = \dim \operatorname{supp}(M)$.*

This result has the following companion local version. The localization A_x of A at $x \in k^n$ is a nötherian local ring. Moreover, its maximal ideal $\mathbf{n}_x$ is the ideal generated by the polynomials $X_i - x_i$, $1 \leq i \leq n$, and their images in $\mathbf{n}_x/\mathbf{n}_x^2$ span it as a vector space over k. Therefore, $X_i - x_i$, $1 \leq i \leq n$, form a coordinate system in A_x. For any finitely generated A-module M, its localization M_x at x is a finitely generated A_x-module, hence we can consider its dimension $d(M_x)$.

For any algebraic variety V over k and $x \in V$ we denote by $\dim_x V$ the local dimension of V at x.

4.5. THEOREM. *Let M be a finitely generated A-module and $x \in \operatorname{supp}(M)$. Then $d(M_x) = \dim_x(\operatorname{supp}(M))$.*

We shall simultaneously prove 4.4 and 4.5. First we observe that if we have an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and 4.4 and 4.5 hold for M' and M'', we have, by 3.8 and 1, that

$$\begin{aligned} d(M) &= \max(d(M'), d(M'')) = \max(\dim \operatorname{supp}(M'), \dim \operatorname{supp}(M'')) \\ &= \dim(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim \operatorname{supp}(M). \end{aligned}$$

Also, for any $x \in \operatorname{supp}(M)$, by the exactness of localization we have the exact sequence:

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0;$$

hence, by 2.7 and 1,

$$\begin{aligned} d(M_x) &= \max(d(M'_x), d(M''_x)) = \max(\dim_x \operatorname{supp}(M'), \dim_x \operatorname{supp}(M'')) \\ &= \dim_x(\operatorname{supp}(M') \cup \operatorname{supp}(M'')) = \dim_x \operatorname{supp}(M). \end{aligned}$$

Assume that 4.4 and 4.5 hold for all $M = A/J$ where J is a prime ideal. Then the preceding remark, 3. and an induction in the length of the filtration would prove the statements in general.

Hence we can assume that $M = A/J$ with J prime. Assume first that J is such that A/J is a finite-dimensional vector space over k. Then A/J is an integral ring and it is integral over k. Hence it is a field which is an algebraic extension of k. Since k is algebraically closed, $A/J = k$ and J is a maximal ideal. In this case, by Hilbert Nulstellenatz, $\operatorname{supp}(M) = V(J)$ is a point x in k^n, i.e., $\dim \operatorname{supp}(M) = 0$. On the other hand, since M_x is one-dimensional linear space, $d(M_x) = 0$, and the assertion is evident. It follows that we can assume that J is not of finite codimension in A, in particular it is not a maximal ideal. Let $J_1 \supset J$ be a prime ideal different form J. Then there exists $f \in J_1$ such that $f \notin J$. It follows that $J \subset (f) + J \subset J_1$ and $J \neq (f) + J$. Therefore, A/J_1 is a quotient of $A/((f) + J)$, and $A/((f) + J)$ is a quotient of A/J. In addition, $A/((f) + J) = M/fM$. Consider the endomorphism of M given by multiplication by f. Then, if $g + J$ is in the kernel of this map, $0 = f(g + J) = fg + J$ and $fg \in J$. Since J is prime and $f \notin J$ it follows that $g \in J$, $g + J = 0$ and the map is injective. Therefore, we have an exact sequence of A-modules:

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

This implies, by 3.8, that $d(M/fM) \leq d(M)$. If $d(M/fM) = d(M)$, we would have in addition that $e(M) = e(M) + e(M/fM)$, hence $e(M/fM) = 0$. This is possible only if $d(M/fM) = 0$, and in this case it would also imply that $d(M) = 0$ and M is finite-dimensional, which is impossible by our assumption. Therefore, $d(M/fM) < d(M)$. Since A/J_1 is a quotient of M/fM, this implies that $d(A/J_1) < d(A/J)$.

Let $x \in V(J_1)$. Then, by localization, we get the exact sequence:

$$0 \longrightarrow M_x \xrightarrow{f} M_x \longrightarrow M_x/fM_x \longrightarrow 0$$

of A_x-modules. This implies, by 2.7, that $d(M_x/fM_x) \leq d(M_x)$. If $d(M_x/fM_x) = d(M_x)$, we would have in addition that $e(M_x) = e(M_x) + e(M_x/fM_x)$, hence $e(M_x/fM_x) = 0$. This is possible only if $d(M_x/fM_x) = 0$, and in this case it would imply that $\mathbf{m}_x(M_x/fM_x) = M_x/fM_x$ and, by Nakayama lemma, $M_x/fM_x = 0$. It would follow that the multiplication by f is surjective on M_x, and, since $f \in \mathbf{m}_x$, by Nakayama lemma this would imply that $M_x = 0$ contrary to our assumptions. Therefore, $d(M_x/fM_x) < d(M_x)$. Since A/J_1 is a quotient of M/fM this implies that $d((A/J_1)_x) < d((A/J)_x)$.

Let

$$Z_0 = \{x\} \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = k^n$$

be a maximal chain of nonempty irreducible closed subsets of k^n. Then

$$I(Z_0) = \mathbf{m}_x \supset I(Z_1) \supset \cdots \supset I(Z_{n-1}) \supset I(Z_n) = \{0\}$$

is a maximal chain of prime ideals in A. By the preceding arguments and 2.8, we have the following sequences of strict inequalities

$$0 \leq d(A/I(Z_0)) < d(A/I(Z_1)) < \cdots < d(A/I(Z_n)) = d(A) = n,$$

and

$$0 \leq d((A/I(Z_0))_x) < d((A/I(Z_1))_x) < \cdots < d((A/I(Z_n))_x) = d(A_x) = n,$$

by 2.11. It follows that

$$d((A/I(Z_j))_x) = d(A/I(Z_j)) = j = \dim Z_j$$

for $0 \leq j \leq n$. Since every closed irreducible subset Z can be put in a maximal chain, it follows that $d((A/I(Z))_x) = d(A/I(Z)) = \dim Z$ for any closed irreducible subset $Z \subset k^n$ and any $x \in Z$. On the other hand, this implies that $d((A/J)_x) = d(A/J) = \dim V(J)$ for any prime ideal J in A and $x \in V(J)$. By 2, this ends the proof of 4.4 and 4.5.

Next result follows immediately from 4.4 and 4.5.

4.6. Corollary. *Let M be a finitely generated A-module. Then*

$$d(M) = \sup_{x \in \operatorname{supp}(M)} d(M_x).$$

5. Homological dimension

Let A be a ring. For any left A-module M we define the *projective dimension* $\mathrm{pd}_A(M)$ as the infimum in $\bar{\mathbb{Z}}$ of the lengths of left projective resolutions of M.

5.1. Proposition. *Let M be a left A-module. Then the following statements are equivalent:*

(i) $\mathrm{pd}_A(M) \le n$;
(ii) $\mathrm{Ext}_A^r(M, N) = 0$ *for any left A-module N and any $r > n$;*
(iii) $\mathrm{Ext}_A^{n+1}(M, N) = 0$ *for any left A-module N;*
(iv) *for any exact sequence of left A-modules*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with projective P_i, $0 \le i \le n-1$, the A-module K is projective too.

Proof. Implications (i)⇒(ii)⇒(iii) and (iv)⇒(i) are evident.

(iii)⇒(iv). First, we claim that for any exact sequence of left A-modules

$$0 \longrightarrow K \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with projective P_i, $0 \le i \le k-1$, we have $\mathrm{Ext}_A^j(K, N) \cong \mathrm{Ext}_A^{k+j}(M, N)$ for $j \ge 1$. We prove this statement by induction in k. If $k = 1$ we have the short exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

and by the long exact sequence of $\mathrm{Ext}_A^{\cdot}(-, N)$ we see that $\mathrm{Ext}_A^j(K, N) \cong \mathrm{Ext}_A^{j+1}(M, N)$ for $j \ge 1$. Assume that the statement holds for $k \ge 1$. Let

$$0 \longrightarrow K \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence of left A-modules with projective P_i, $0 \le i \le k$. Then we can split it into the exact sequences

$$0 \longrightarrow K \longrightarrow P_k \longrightarrow K' \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

and by the induction assumption

$$\mathrm{Ext}_A^j(K, N) \cong \mathrm{Ext}_A^{j+1}(K', N) \cong \mathrm{Ext}_A^{j+k+1}(M, N).$$

This proves our claim.

Assume that (iii) holds. Then, from the exact sequence in (iv) we conclude that $\mathrm{Ext}_A^1(K, N) \cong \mathrm{Ext}_A^{n+1}(M, N) = 0$ for any left A-module N. Therefore, K is projective. □

Let M and N be left A-modules. Then $\mathrm{Hom}_A(M, A)$ is a right A-module with the A-action defined by right multiplication on A. We define a biadditive map of $\mathrm{Hom}_A(M, A) \times N$ into $\mathrm{Hom}_A(M, N)$ which attaches to $(T, n) \in \mathrm{Hom}_A(M, A) \times N$ the morphism $m \longmapsto T(m)n$ of M into N. This map induces a natural additive map of $\mathrm{Hom}_A(M, A) \otimes_A N$ into $\mathrm{Hom}_A(M, N)$.

5.2. Lemma. *Let M and N be left A-modules. If M is a finitely generated projective A-module, the natural map of* $\mathrm{Hom}_A(M, A) \otimes_A N$ *into* $\mathrm{Hom}_A(M, N)$ *is surjective.*

Proof. If $M = A$ the statement is obvious. Moreover, if $M = M' \oplus M''$,

$$\mathrm{Hom}_A(M, A) \otimes_A N = \mathrm{Hom}_A(M', A) \otimes_A N \oplus \mathrm{Hom}_A(M'', A) \otimes_A N$$

and

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(M', N) \oplus \mathrm{Hom}_A(M'', N).$$

Hence, the statement of lemma is true for M if and only if it is true for M' and M''. This implies first, by induction in p, that the statement is true for $M = A^p$. Moreover, if M is a finitely generated projective A-module, we can identify it with a direct summand of a free A-module A^p. Hence, the assertion holds for M. $\square$

The next result is a converse of 2.

5.3. Lemma. *Let M be a left A-module such that the natural map of* $\mathrm{Hom}_A(M, A) \otimes_A M$ *into* $\mathrm{Hom}_A(M, M)$ *is surjective. Then M is a finitely generated projective left A-module.*

Proof. If the map is surjective, one can find $T_i \in \mathrm{Hom}_A(M, A)$ and $m_i \in M$, $1 \leq i \leq p$, such that $\sum T_i \otimes m_i \in \mathrm{Hom}_A(M, A) \otimes_A M$ maps into the identity map in $\mathrm{Hom}_A(M, M)$. Therefore, we have $m = \sum T_i(m) m_i$ for any $m \in M$. Define a morphism i of M into A^p by $m \longmapsto (T_i(m)\,;\, 1 \leq i \leq p)$ and a morphism j of A^p into M by $(a_i\,;\, 1 \leq i \leq p) \longmapsto \sum a_i m_i$. Then, $j \circ i = 1_M$. Therefore i is injective and M is isomorphic to the image of i. Moreover, $A^p = \mathrm{im}\, i \oplus \ker j$, hence M is projective and finitely generated. $\square$

5.4. Lemma. *Let A be a left nötherian ring and M a finitely generated left A-module. Then the following conditions are equivalent:*

(i) *M is a projective;*
(ii) *M is flat.*

Proof. (i)⇒(ii) is evident.

(ii)⇒(i). Since A is a left nötherian ring and M finitely generated, we can find free left A-modules F_0 and F_1 of finite rank such that

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence of left A-modules. If we apply the functor $\mathrm{Hom}_A(-, A)$ to this exact sequence, we get the exact sequence of right A-modules

$$0 \longrightarrow \mathrm{Hom}_A(M, A) \longrightarrow \mathrm{Hom}_A(F_0, A) \longrightarrow \mathrm{Hom}_A(F_1, A).$$

Since M is flat, by tensoring with it we get the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_A(M, A) \otimes_A M \longrightarrow \mathrm{Hom}_A(F_0, A) \otimes_A M \longrightarrow \mathrm{Hom}_A(F_1, A) \otimes_A M.$$

This finally leads to the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}_A(M,A)\otimes_A M & \longrightarrow & \operatorname{Hom}_A(F_0,A)\otimes_A M & \longrightarrow & \operatorname{Hom}_A(F_1,A)\otimes_A M \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Hom}_A(M,M) & \longrightarrow & \operatorname{Hom}_A(F_0,M) & \longrightarrow & \operatorname{Hom}_A(F_1,M)
\end{array} .$$

Both rows in this diagram are exact and the last two vertical arrows are surjective by 2. This implies that the first vertical arrow is surjective. Hence, by 3, M is a projective A-module. □

5.5. Proposition. *Let A be a left nötherian ring and M be a finitely generated left A-module. Then the following statements are equivalent:*

(i) $\operatorname{pd}_A(M) \le n$;
(ii) *M has a projective resolution of length $\le n$ consisting of finitely generated A-modules;*
(iii) $\operatorname{Ext}^r_A(M,N) = 0$ *for any finitely generated left A-module N and any $r > n$;*
(iv) $\operatorname{Ext}^{n+1}_A(M,N) = 0$ *for any finitely generated left A-module N;*
(v) $\operatorname{Tor}^A_r(P,M) = 0$ *for any finitely generated right A-module P and any $r > n$;*
(vi) $\operatorname{Tor}^A_{n+1}(P,M) = 0$ *for any finitely generated right A-module P.*

Proof. (ii)⇒(i), (i)⇒(iii)⇒(iv) and (i)⇒(v)⇒(vi) are evident.

(i)⇒(ii). Since M is a finitely generated left A-module and A a left nötherian ring, we can construct a left resolution

$$0 \to K \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

where F_i, $0 \le i \le n-1$, are free left A-modules of finite rank and K is finitely generated. By 1, we conclude that K is also projective.

(iv)⇒(i). Since M is finitely generated we can construct a left resolution $F^\cdot$ of M consisting of free A-modules of finite rank. Therefore, $\operatorname{Ext}^j_A(M,Q) = H^j(\operatorname{Hom}_A(F^\cdot,Q))$ for any left A-module Q. Now, any left A-module Q is a direct limit of a directed system $\{Q_i; i \in I\}$ of its finitely generated submodules, which leads to

$$\begin{aligned}\operatorname{Ext}^j_A(M,Q) &= H^j(\operatorname{Hom}_A(F^\cdot,Q)) = H^j(\operatorname{Hom}_A(F^\cdot,\varinjlim Q_i)) \\ &= \varinjlim H^j(\operatorname{Hom}_A(F^\cdot,Q_i)) = \varinjlim \operatorname{Ext}^j_A(M,Q_i).\end{aligned}$$

Hence, $\operatorname{Ext}^{n+1}_A(M,Q) = 0$ for any left A-module Q and, by 1, $\operatorname{pd}_A M \le n$.

(vi)⇒(i). Let

$$0 \to K \to F_{k-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

be a left resolution of M such that F_i, $0 \le i \le k-1$, are free A-modules of finite rank and K is a finitely generated left A-module. We claim that $\operatorname{Tor}^A_j(P,K) \cong \operatorname{Tor}^A_{j+k}(P,M)$ for any right A-module P and $j \geq 1$. Assume first that $k = 1$. In this case, we have a short exact sequence

$$0 \longrightarrow K \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

and by the long exact sequence of $\operatorname{Tor}^A_\cdot(P, -)$ we see that $\operatorname{Tor}^A_j(P, K) \cong \operatorname{Tor}^A_{j+1}(P, M)$ for $j \geq 1$. Assume now that $k \geq 1$ and consider the exact sequence

$$0 \longrightarrow K \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Then we can split it into two exact sequences

$$0 \longrightarrow K \longrightarrow F_k \longrightarrow K' \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow F_{k-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

By the induction assumption we now have

$$\operatorname{Tor}^A_j(P, K) \cong \operatorname{Tor}^A_{j+1}(P, K') \cong \operatorname{Tor}^A_{j+k+1}(P, M)$$

for any $j \geq 1$. This proves our claim.

Applying this to $k = n$ we get a resolution

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i, $0 \leq i \leq n-1$, are free A-modules of finite rank and K is a finitely generated left A-module. By (vi) we conclude that $\operatorname{Tor}^A_1(P, K) \cong \operatorname{Tor}^A_{n+1}(P, M) = 0$ for any finitely generated right A-module P.

Since K is finitely generated we can construct a left resolution $F^\cdot$ of K consisting of free A-modules of finite rank. Therefore, $\operatorname{Tor}^A_j(Q, K) = H_j(Q \otimes_A F^\cdot)$ for any right A-module Q. Now, any right A-module Q is a direct limit of a directed system $\{Q_i\,;\, i \in I\}$ of its finitely generated submodules, which leads to

$$\begin{aligned}\operatorname{Tor}^A_1(Q, K) &= H_1(Q \otimes_A F^\cdot) = H_1((\varinjlim Q_i) \otimes_A F^\cdot) \\ &= \varinjlim H_1(Q_i \otimes_A F^\cdot) = \varinjlim \operatorname{Tor}^A_1(Q_i, K) = 0.\end{aligned}$$

Hence, K is a flat A-module, and by 4, it is a projective A-module. □

Homological dimension $\operatorname{hd}(A)$ of a ring A is the supremum in $\bar{\mathbb{Z}}$ of $n \in \mathbb{Z}_+$ such that there exist left A-modules M, N with $\operatorname{Ext}^n_A(M, N) \neq 0$.

5.6. Lemma. *Let J be a left A-module such that $\operatorname{Ext}^1_A(C, J) = 0$ for every cyclic left A-module C. Then J is injective.*

Proof. We can assume that $C = A/L$ for some left ideal L of A. Let M be a left A-module, N its submodule and f a morphism $f : N \longrightarrow J$. We have to show that f is a restriction of a morphism of M into J. Denote by $\mathcal{F}$ the family of all pairs (N', f'), where N' is a submodule of M such that $N \subset N' \subset M$ and f' is a morphism of N' into J which extends f. We define a partial order on $\mathcal{F}$ by: $(N', f') \leq (N'', f'')$ if $N' \subset N''$ and f'' extends f'. By Zorn lemma, there exists a maximal element (P, g) in $\mathcal{F}$. Assume that $P \neq M$. Then we can find $m \in M$ such that $m \notin P$. Let $L = \{a \in A \mid am \in P\}$. Then $\phi(a) = g(am)$ defines a morphism $\phi : L \longrightarrow J$. Since $\operatorname{Ext}^1_A(A/L, J) = 0$, the long exact sequence of $\operatorname{Ext}^{\cdot}_A(-, J)$ applied to the short exact sequence

$$0 \longrightarrow L \longrightarrow A \longrightarrow A/L \longrightarrow 0$$

implies that every morphism of L into J extends to the morphism of A into J. Therefore, there exists $j \in J$ such that $\phi(a) = aj$ for $a \in L$. Let $P' = P + Am$. Since $p \in P \cap Am$ implies that $p = bm$ for some $b \in L$, we conclude that $g(p) = g(bm) = \phi(b) = bj$. Therefore, we can define a morphism $g' : P' \longrightarrow J$ by $g'(p + am) = g(p) + aj$. Clearly, $(P', g') \in \mathcal{F}$ and $(P, g) \leq (P', g')$ contradicting the maximality of (P, g). Therefore, $P = M$ and J is injective. $\square$

5.7. Proposition. *Let $n \in \mathbb{Z}_+$. Then the following conditions are equivalent*

(i) $\operatorname{hd}(A) \leq n$;
(ii) *for every left A-module M, $\operatorname{pd}_A(M) \leq n$;*
(iii) *for every finitely generated left A-module M, $\operatorname{pd}_A(M) \leq n$;*
(iv) *for every exact sequence of left A-modules*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0$$

with projective P_i, K is projective too;
(v) *for every exact sequence of left A-modules*

$$I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{n-1} \longrightarrow J \longrightarrow 0$$

with injective I^i, J is injective too;
(vi) *every left A-module has an injective resolution of length $\leq n$.*

Proof. The conditions (i), (ii) and (iv) are equivalent by 1. Evidently, (i)⇒(iii) and (v)⇒(vi)⇒(i). Hence, it remains to show that (iii) implies (v).

Consider the exact sequence

$$0 \longrightarrow K \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{k-1} \longrightarrow J \longrightarrow 0$$

with injective I^i, $0 \leq i \leq k-1$. We claim that $\operatorname{Ext}^j_A(M, J) \cong \operatorname{Ext}^{j+k}_A(M, K)$ for any left A-module M and $j \in \mathbb{N}$. If $k = 1$, we have the short exact sequence

$$0 \longrightarrow K \longrightarrow I^0 \longrightarrow J \longrightarrow 0$$

and from the long exact sequence of $\operatorname{Ext}^{\cdot}_A(M,-)$ we conclude that for any $j \in \mathbb{N}$ we have $\operatorname{Ext}^j_A(M,J) \cong \operatorname{Ext}^{j+1}_A(M,K)$. Assume that the statement holds for $k \geq 1$, and consider the exact sequence

$$0 \longrightarrow K \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{k-1} \longrightarrow I^k \longrightarrow J \longrightarrow 0,$$

with injective I^i, $0 \leq i \leq k$. We can split it into two exact sequences

$$0 \longrightarrow K \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{k-1} \longrightarrow J' \longrightarrow 0,$$

and

$$0 \longrightarrow J' \longrightarrow I^k \longrightarrow J \longrightarrow 0.$$

By the induction assumption we conclude that

$$\operatorname{Ext}^j_A(M,J) \cong \operatorname{Ext}^{j+1}_A(M,J') \cong \operatorname{Ext}^{j+k+1}_A(M,K),$$

and this proves our claim.

Now, (iii) implies that for any finitely generated left A-module M and any left A-module K we have $\operatorname{Ext}^{n+1}_A(M,K) = 0$. Hence, if we apply this and the preceding discussion to the exact sequence

$$0 \longrightarrow K \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{n-1} \longrightarrow J \longrightarrow 0,$$

with injective I^i, $0 \leq i \leq n-1$, we conclude that $\operatorname{Ext}^1_A(M,J) \cong \operatorname{Ext}^{n+1}_A(M,K) = 0$. From 6. it follows that J is injective. □

5.8. Corollary. *Let A be a left nötherian ring and $n \in \mathbb{Z}_+$. Then the following conditions are equivalent:*

(i) $\operatorname{hd}(A) \leq n$;
(ii) *for any two finitely generated left A-modules M and N we have* $\operatorname{Ext}^{n+1}_A(M,N) = 0$;
(iii) *for any finitely generated right A-module P and any finitely generated left A-module M we have* $\operatorname{Tor}^A_{n+1}(P,M) = 0$.

Proof. Clearly, (i) implies (ii) and (iii). By 5. and 7, (ii) and (iii) imply (i). □

Since the condition (iii) is symmetric with respect to left and right modules, 8. has the following consequence.

5.9. Proposition. *Let A be a left and right nötherian ring and A° its opposite ring. Then* $\operatorname{hd}(A) = \operatorname{hd}(A^\circ)$.

5.10. Proposition. *Let A be a commutative ring. Then* $\operatorname{hd}(A[X]) = \operatorname{hd}(A) + 1$.

The proof is based on the following results. Let M be an $A[X]$-module. We can view it also as an A-module.

5.11. Lemma. *Let A be a commutative ring and M an $A[X]$-module. Then*

$$\operatorname{pd}_A(M) \le \operatorname{pd}_{A[X]}(M) \le \operatorname{pd}_A(M) + 1.$$

Proof. To prove the first inequality we can assume that $\operatorname{pd}_{A[X]}(M)$ is finite and put $\operatorname{pd}_{A[X]}(M) = q$. Then we can find a projective resolution

$$0 \longrightarrow Q_q \longrightarrow Q_{q-1} \longrightarrow \dots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

of the $A[X]$-module M. Since any projective $A[X]$-module Q is a direct summand of a free $A[X]$-module, and any free $A[X]$-module is free as an A-module, we conclude that Q is a projective A-module. It follows that $\operatorname{pd}_A(M) \le q$, i.e., $\operatorname{pd}_A(M) \le \operatorname{pd}_{A[X]}(M)$.

To prove the other inequality we can assume that $\operatorname{pd}_A(M)$ is finite and put $\operatorname{pd}_A(M) = m$. Then we have a projective resolution

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of the A-module M and, since $A[X]$ is a free A-module, the sequence

$$0 \longrightarrow A[X]\otimes_A P_m \longrightarrow A[X]\otimes_A P_{m-1} \longrightarrow \dots \longrightarrow A[X]\otimes_A P_1 \longrightarrow A[X]\otimes_A P_0 \longrightarrow A[X]\otimes_A M \longrightarrow 0$$

is exact. We also claim that, for any projective A-module P, $A[X] \otimes_A P$ is a projective $A[X]$-module. This follows immediately from the natural isomorphism

$$\operatorname{Hom}_{A[X]}(A[X] \otimes_A M, N) = \operatorname{Hom}_A(M, N)$$

for any A-module M and $A[X]$-module N. Therefore we see that $\operatorname{pd}_{A[X]}(A[X] \otimes_A M) \le m$ and $\operatorname{Ext}^j_{A[X]}(A[X] \otimes_A M, N) = 0$ for any $A[X]$-module N and $j > m$.

Define the maps $\phi : A[X] \otimes_A M \longrightarrow A[X] \otimes_A M$ by $\phi(P \otimes m) = XP \otimes m - P \otimes Xm$ and $\psi : A[X] \otimes_A M \longrightarrow M$ by $\psi(P \otimes m) = Pm$ for $P \in A[X]$ and $m \in M$. Then we have the exact sequence

$$0 \longrightarrow A[X] \otimes_A M \xrightarrow{\phi} A[X] \otimes_A M \xrightarrow{\psi} M \longrightarrow 0.$$

Clearly, ψ is surjective and $\psi \circ \phi = 0$. Moreover, an arbitrary element s of $A[X] \otimes_A M$ can be written as $s = \sum X^n \otimes m_n$. Therefore,

$$\phi(s) = \phi\left(\sum X^n \otimes m_n\right) = \sum (X^{n+1} \otimes m_n - X^n \otimes X m_n) = \sum X^n \otimes (m_{n-1} - X m_n),$$

and s is in the kernel of ϕ if and only if $m_{n-1} = X m_n$ for all $n \in \mathbb{Z}_+$. Since $m_n = 0$ for large n, by downward induction in n, this implies that $m_n = 0$ for all $n \in \mathbb{Z}_+$, i.e., $s = 0$. Hence, ϕ is injective.

Assume that s is in the kernel of ψ. Then $\sum X^n m_n = 0$. Therefore,

$$s = \sum X^n \otimes m_n = \sum (X^n \otimes m_n - 1 \otimes X^n m_n).$$

Denote by T and S the A-module endomorphisms of $A[X] \otimes_A M$ defined by $T(P \otimes m) = XP \otimes m$ and $S(P \otimes m) = P \otimes Xm$ for $P \in A[X]$ and $m \in M$. Then T and S commute and $\phi = T - S$. This immediately implies that

$$s = \sum (T^n - S^n)(1 \otimes m_n) = (T - S) \sum_n \left(\sum_{j=0}^{n} T^{n-j} S^j \right) (1 \otimes m_n) \in \operatorname{im} \phi$$

This completes the proof of the exactness.

From the long exact sequence of $\operatorname{Ext}^{\cdot}_{A[X]}(-, N)$ corresponding to this short exact sequence and the preceding result we see that $\operatorname{Ext}^j_{A[X]}(M, N) = 0$ for any $A[X]$-module N and $j > m + 1$. Hence, by 1, $\operatorname{pd}_{A[X]}(M) \leq \operatorname{pd}_A(M) + 1$. □

As a consequence, we see that $\operatorname{hd}(A[X]) \leq \operatorname{hd}(A) + 1$. To finish the proof of 10. it remains to show that $\operatorname{hd}(A[X]) \geq \operatorname{hd}(A) + 1$.

Let M be an $A[X]$-module. Denote by X_M the morphism of M induced by the multiplication by X.

5.12. Lemma. *Let M be an $A[X]$-module. Then:*

(i) *if X_M is injective, we have* $\operatorname{pd}_A(M/XM) \leq \operatorname{pd}_{A[X]}(M)$*;*
(ii) *if $M \neq 0$ and $X_M = 0$, we have* $\operatorname{pd}_A(M) = \operatorname{pd}_{A[X]}(M) - 1$.

Proof. (i) We can assume that $\operatorname{pd}_{A[X]}(M)$ is finite. We prove, by induction in k, that $\operatorname{pd}_A(M/XM) \leq \operatorname{pd}_{A[X]}(M)$ if $\operatorname{pd}_{A[X]}(M) \leq k$. If M is a projective $A[X]$-module, M is a direct summand of a free $A[X]$-module. On the other hand, if F is a free $A[X]$-module, F/XF is a free A-module. Therefore, M/XM is a direct summand of a free A-module and hence projective. It follows that $\operatorname{pd}_A(M/XM) \leq \operatorname{pd}_{A[X]}(M)$ if $\operatorname{pd}_{A[X]}(M) = 0$. Assume now that $\operatorname{pd}_{A[X]}(M) = k$ and consider a short exact sequence of $A[X]$-modules

$$0 \to N \to F \to M \to 0$$

with F free. This leads to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle X_N} & & \Big\downarrow{\scriptstyle X_F} & & \Big\downarrow{\scriptstyle X_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array} .$$

We can consider it as a short exact sequence of complexes, and from the associated long exact sequence conclude that X_N is injective and that

$$0 \to N/XN \to F/XF \to M/XM \to 0$$

is an exact sequence of A-modules. From the long exact sequence of $\operatorname{Ext}^{\cdot}_{A[X]}(-, L)$ we see that the connecting morphism of $\operatorname{Ext}^{j-1}_{A[X]}(N, L)$ into $\operatorname{Ext}^j_{A[X]}(M, L)$ is surjective for any

$A[X]$-module L and $j \in \mathbb{N}$. Hence, by 1, we conclude that $\mathrm{pd}_{A[X]}(N) = k-1$, and by the induction assumption $\mathrm{pd}_A(N/XN) \leq \mathrm{pd}_{A[X]}(N) = k-1$. Analogously, from the long exact sequence of $\mathrm{Ext}^{\cdot}_A(-,Q)$ we deduce that the connecting morphism of $\mathrm{Ext}^{j-1}_A(N/XN,Q)$ into $\mathrm{Ext}^j_A(M/XM,Q)$ is surjective for any A-module Q and $j \in \mathbb{N}$, which implies that $\mathrm{pd}_A(M/XM) \leq k$.

(ii) By 11. the assertion is trivial if $\mathrm{pd}_{A[X]}(M)$ is infinite. Also, if $\mathrm{pd}_{A[X]}(M) = 0$, M is projective and a direct summand of a free $A[X]$-module. Since $Xu = 0$ implies $u = 0$ in a free $A[X]$-module, $XM = 0$ would imply that $M = 0$, contrary to our assumptions. Therefore, we can assume that $\mathrm{pd}_{A[X]}(M) = k > 0$. Considering again the short exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

with a free $A[X]$-module F, we see that $\mathrm{pd}_{A[X]}(N) = k-1$, X_N is injective and the sequence of A-modules

$$0 \longrightarrow M \longrightarrow N/XN \longrightarrow F/XF \longrightarrow M \longrightarrow 0$$

is exact. Hence, by (i), we conclude that $\mathrm{pd}_A(N/XN) \leq k-1$. We can split the above exact sequence into two short exact sequences of A-modules:

$$0 \longrightarrow M \longrightarrow N/XN \longrightarrow K \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow F/XF \longrightarrow M \longrightarrow 0.$$

By using the long exact sequence of $\mathrm{Ext}^{\cdot}_A(-,L)$ and 11. we conclude from the last one that $\mathrm{pd}_A(K) \leq k-1$. Now, the same argument applied to the first one implies that $\mathrm{pd}_A(M) \leq k-1$. By 11. we conclude that $\mathrm{pd}_A(M) = k-1$. □

Let M be an A-module. We can consider it as an $A[X]$-module by defining $X \cdot M = 0$. Denote this $A[X]$-module by $\bar{M}$. By 12. we see that $\mathrm{pd}_A(M) = \mathrm{pd}_{A[X]}(\bar{M}) - 1$, hence $\mathrm{hd}(A) \leq \mathrm{hd}(A[X]) - 1$. This ends the proof of 10.

5.13. THEOREM. *Let k be a field. Then*

$$\mathrm{hd}(k[X_1, X_2, \ldots, X_n]) = n.$$

Now we want to study the homological dimension of local rings. We start with the following characterization of free A-modules of finite rank.

5.14. PROPOSITION. *Let A be a nötherian local ring and $\mathbf{m}$ its maximal ideal. Let M be a finitely generated A-module. Then the following conditions are equivalent:*

(i) *M is a free A-module;*
(ii) *M is a projective A-module;*
(iii) $\mathrm{Tor}^A_1(M, A/\mathbf{m}) = 0$.

Proof. Clearly, we have (i)⇒(ii)⇒(iii). Therefore, it remains to prove that (iii)⇒(i). Let $x_1, x_2, \dots, x_n$ be elements of M such that their images in $M/\mathbf{m}M$ form a basis of the k-vector space $M/\mathbf{m}M$. Let F be the free A-module with the basis $e_1, e_2, \dots, e_n$. Denote by ϕ the A-module morphism from F into M defined by $\phi(e_i) = x_i$ for $1 \le i \le n$. If we put $C = \ker \phi$ and $K = \operatorname{coker} \phi$ we get the exact sequence

$$0 \to C \to F \xrightarrow{\phi} M \to K \to 0$$

of finitely generated A-modules. By tensoring with $A/\mathbf{m} \otimes_A -$ it gives the exact sequence

$$F/\mathbf{m}F \xrightarrow{1\otimes\phi} M/\mathbf{m}M \to K/\mathbf{m}K \to 0$$

of vector spaces over k. By the construction, $1 \otimes \phi$ is surjective and $K/\mathbf{m}K = 0$. By Nakayama lemma this implies that $K = 0$. Therefore, we have the exact sequence

$$0 \to C \to F \xrightarrow{\phi} M \to 0.$$

By applying to it the long exact sequence of $\operatorname{Tor}_j^A(A/\mathbf{m}, -)$ and (iii), we conclude that

$$0 \to C/\mathbf{m}C \to F/\mathbf{m}F \xrightarrow{1\otimes\phi} M/\mathbf{m}M \to 0$$

is an exact sequence of vector spaces over k. Since $1 \otimes \phi$ is injective, $C/\mathbf{m}C = 0$ and by Nakayama lemma, $C = 0$. Therefore, M is a free A-module. □

5.15. Proposition. *Let A be a nötherian local ring, $\mathbf{m}$ its maximal ideal and $n \in \mathbb{Z}_+$. Then the following conditions are equivalent:*

(i) $\operatorname{hd}(A) \le n$;
(ii) $\operatorname{Tor}_{n+1}^A(A/\mathbf{m}, A/\mathbf{m}) = 0$.

Proof. Clearly (i)⇒(ii). Assume that (ii) holds. Let M be a finitely generated A-module and

$$0 \longrightarrow K \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

a left resolution of M such that the modules F_i, $0 \le i \le n-1$, are free A-modules of finite rank. Then, as in the proof of 5, we conclude that $\operatorname{Tor}_1^A(K, A/\mathbf{m}) \cong \operatorname{Tor}_{n+1}^A(M, A/\mathbf{m})$. Applying this to $M = A/\mathbf{m}$ we conclude that in this case $\operatorname{Tor}_1^A(K, A/\mathbf{m}) = 0$, and by 14, K is a free A-module of finite rank. It follows that $A/\mathbf{m}$ has a free left resolution of length n, i. e. $\operatorname{pd}_A(A/\mathbf{m}) \le n$. This implies that $\operatorname{Tor}_{n+1}^A(M, A/\mathbf{m}) = 0$ for any finitely generated A-module M, and by repeating the argument, $\operatorname{pd}_A(M) \le n$. By 7. we conclude that $\operatorname{hd}(A) \le n$. □

Finally, we make two technical remarks we shall use in the following.

5.16. Lemma. *Let A be a left nötherian ring with* $\mathrm{hd}(A) < \infty$*. Let M be a finitely generated A-module and $n \in \mathbb{Z}_+$. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}_A^j(M, A) = 0$ *for $j > n$;*
(ii) $\mathrm{pd}_A(M) \leq n$.

Proof. Clearly (ii)⇒(i). It remains to show the opposite implication. Let N be a finitely generated A-module. Since A is left nötherian, we have an exact sequence $0 \to K \to A^p \to N \to 0$ with finitely generated A-module K. By the long exact sequence of $\mathrm{Ext}_A^{\cdot}(M, -)$ we conclude that the connecting morphism $\mathrm{Ext}_A^j(M, N) \to \mathrm{Ext}_A^{j+1}(M, K)$ is an isomorphism for $j > n$. Since the homological dimension of A is finite, by downward induction in j we see that $\mathrm{Ext}_A^j(M, N) = 0$ for $j > n$. By 5. this implies that $\mathrm{pd}_A(M) \leq n$. □

5.17. Corollary. *Let A be a left nötherian ring with* $\mathrm{hd}(A) < \infty$ *and $n \in \mathbb{Z}_+$. Then the following statements are equivalent:*

(i) $\mathrm{hd}(A) \leq n$*;*
(ii) $\mathrm{Ext}_A^j(M, A) = 0$ *for any finitely generated left A-module M and any $j > n$.*

Proof. This follows immediately from 7. and 16. □

5.18. Corollary. *Let A be a left nötherian ring with* $\mathrm{hd}(A) < \infty$ *and M a finitely generated left A-module. Then the following statements are equivalent:*

(i) $M = 0$*;*
(ii) $\mathrm{Ext}_A^j(M, A) = 0$ *for all $j \in \mathbb{Z}_+$.*

Proof. Clearly, (i)⇒(ii). If (ii) holds, M is projective by 16, and therefore a direct summand of some A^p. This implies that $\mathrm{Hom}_A(M, M) = 0$ which is possible only if $M = 0$. □

6. Some homological algebra of polynomial rings

Let k be an algebraically closed field. Denote by A the ring of polynomials in n variables $k[X_1, X_2, \ldots, X_n]$ with coefficients in k.

Since A is commutative, for any two A-modules M and N, $\mathrm{Hom}_A(M, N)$ has a natural structure of an A-module given by $(p \cdot T)(m) = pT(m)$ for any $p \in A$, $T \in \mathrm{Hom}_A(M, N)$ and $m \in M$. In addition, we can view $\mathrm{Ext}_A^j(-, -)$, $1 \leq j \leq \infty$, as rightderived functors of this functor $\mathrm{Hom}_A(-, -)$, i.e., all $\mathrm{Ext}_A^j(M, N)$ have natural structures of A-modules.

Also, $M \otimes_A N$ hasa a natural structure of an A-module given by $p(m \otimes n) = pm \otimes n = m \otimes pn$, for any $p \in A$, $m \in M$ and $n \in N$. Therefore, we can view $\mathrm{Tor}_j^A(-, -)$ as left derived functors of $- \otimes_A -$, i.e., all $\mathrm{Tor}_j^A(M, N)$ have natural structures of A-modules.

We want now to analyze these modules more carefully. First, by 5.13 we know that $\mathrm{hd}(A) = n$. Therefore, for any two A-modules M and N, we have $\mathrm{Ext}_A^j(M, N) = 0$ and $\mathrm{Tor}_j^A(M, N) = 0$ for $j > n$.

Also, for any $x \in k^n$ we can consider the localization A_x of A with respect to the maximal ideal $\mathbf{m}_x$.

6.1. LEMMA. *Let $x \in k^n$ and M and N be A-modules. Then*

$$\operatorname{Tor}_j^{A_x}(M_x, N_x) = \operatorname{Tor}_j^A(M, N)_x$$

for all $j \in \mathbb{Z}_+$.

PROOF. Let

$$\ldots \longrightarrow F_k \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a resolution of M by free, finitely generated A-modules. Then $\operatorname{Tor}_j^A(M, N)$ is the j^{th}-homology module of the complex

$$\ldots \longrightarrow F_k \otimes_A N \longrightarrow \ldots \longrightarrow F_1 \otimes_A N \longrightarrow F_0 \otimes_A N \longrightarrow 0,$$

and since localization is exact, $\operatorname{Tor}_j^A(M, N)_x$ is the j^{th}-homology module of the complex

$$\ldots \longrightarrow (F_k \otimes_A N)_x \longrightarrow \ldots \longrightarrow (F_1 \otimes_A N)_x \longrightarrow (F_0 \otimes_A N)_x \longrightarrow 0.$$

On the other hand, if $F = A^{(I)}$,

$$(F \otimes_A N)_x = (A^{(I)} \otimes_A N)_x = N_x^{(I)} = (N_x)^{(I)} = (A_x)^{(I)} \otimes_{A_x} N_x = F_x \otimes_{A_x} N_x,$$

which implies that $\operatorname{Tor}_j^A(M, N)_x$ is the j^{th}-homology module of the complex

$$\ldots \longrightarrow (F_k)_x \otimes_{A_x} N_x \longrightarrow \ldots \longrightarrow (F_1)_x \otimes_{A_x} N_x \longrightarrow (F_0)_x \otimes_{A_x} N_x \longrightarrow 0.$$

Since localization is exact,

$$\ldots \longrightarrow (F_k)_x \longrightarrow \ldots \longrightarrow (F_1)_x \longrightarrow (F_0)_x \longrightarrow M_x \longrightarrow 0$$

is a free resolution of the A_x-module M_x, and $\operatorname{Tor}_j^A(M, N)_x = \operatorname{Tor}_j^{A_x}(M_x, N_x)$ for all $j \in \mathbb{Z}$. $\square$

By 5.8, it follows that $\operatorname{Tor}_{n+1}^{A_x}(M_x, N_x) = 0$ for any two A-modules M and N. The local ring A_x of rational functions on k^n regular at x has the maximal ideal $\mathbf{n}_x = (\mathbf{m}_x)_x$ consisting of all rational functions vanishing at x. Clearly,

$$(A/\mathbf{m}_x)_x = A_x/(\mathbf{m}_x)_x = A_x/\mathbf{n}_x.$$

This implies that

$$\operatorname{Tor}_j^{A_x}(A_x/\mathbf{n}_x, A_x/\mathbf{n}_x) = \operatorname{Tor}_j^{A_x}((A/\mathbf{m}_x)_x, (A/\mathbf{m}_x)_x) = \operatorname{Tor}_j^A(A/\mathbf{m}_x, A/\mathbf{m}_x)_x$$

and by the preceeding discussion

$$\operatorname{Tor}_{n+1}^{A_x}(A_x/\mathbf{n}_x, A_x/\mathbf{n}_x) = 0.$$

Hence, by 5.15, we have the following result.

6.2. LEMMA. *$\operatorname{hd}(A_x) \leq n$ for any $x \in k^n$.*

We shall see later in 8. that this inequality is actually an equality.

6.3. LEMMA. *Let $x \in k^n$ and M be a finitely generated A-module and N any A-module. Then*

$$\operatorname{Ext}^j_{A_x}(M_x, N_x) = \operatorname{Ext}^j_A(M, N)_x$$

for all $j \in \mathbb{Z}_+$.

PROOF. Let

$$\ldots \longrightarrow F_k \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a resolution of M by free, finitely generated A-modules. Then $\operatorname{Ext}^j_A(M, N)$ is the j^{th}-cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_A(F_0, N) \longrightarrow \operatorname{Hom}_A(F_1, N) \longrightarrow \ldots \longrightarrow \operatorname{Hom}_A(F_k, N) \longrightarrow \ldots ,$$

and since localization is exact, $\operatorname{Ext}^j_A(M, N)_x$ is the j^{th}-cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_A(F_0, N)_x \longrightarrow \operatorname{Hom}_A(F_1, N)_x \longrightarrow \ldots \longrightarrow \operatorname{Hom}_A(F_k, N)_x \longrightarrow \ldots .$$

On the other hand, if $F = A^p$,

$$\operatorname{Hom}_A(F, N)_x = (N^p)_x = N^p_x = \operatorname{Hom}_{A_x}(F_x, N_x),$$

which implies that $\operatorname{Ext}^j_A(M, N)_x$ is the j^{th}-cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_{A_x}((F_0)_x, N_x) \longrightarrow \operatorname{Hom}_{A_x}((F_1)_x, N_x) \longrightarrow \ldots \longrightarrow \operatorname{Hom}_{A_x}((F_k)_x, N_x) \longrightarrow \ldots .$$

Since localization is exact,

$$\ldots \longrightarrow (F_k)_x \longrightarrow \ldots \longrightarrow (F_1)_x \longrightarrow (F_0)_x \longrightarrow M_x \longrightarrow 0$$

is a free resolution of the A_x-module M_x, and $\operatorname{Ext}^j_A(M, N)_x = \operatorname{Ext}^j_{A_x}(M_x, N_x)$ for all $j \in \mathbb{Z}$. □

6.4. PROPOSITION. *Let M be a finitely generated A-module. Then*

(i) *$\operatorname{Ext}^j_A(M, A)$, $0 \leq j \leq n$, are finitely generated A-modules;*

(ii)

$$\operatorname{supp}(M) = \bigcup_{j=0}^{n} \operatorname{supp}(\operatorname{Ext}^j_A(M, A)).$$

Proof. (i) Let $0 \to K \to A^p \to M \to 0$ be an exact sequence of A-modules. By the long exact sequence of $\mathrm{Ext}^{\cdot}_A(-,A)$, $\mathrm{Hom}_A(M,A)$ is a submodule of A^p. This implies that $\mathrm{Hom}_A(M,A)$ is finitely generated. Moreover, $\mathrm{Ext}^j_A(M,A)$ is a quotient of $\mathrm{Ext}^{j-1}_A(K,A)$ for any $1 \le j \le n$. By the induction in j, it follows that all $\mathrm{Ext}^j_A(M,A)$ are finitely generated A-modules.

(ii) Let I be the annihilator of M and $f \in I$. Then the multiplication by f induces a morphism of a projective resolution $P^{\cdot}$ of M which is homotopic to zero. Therefore, f annihilates $\mathrm{Ext}^j_A(M,A)$ for any $0 \le j \le n$. It follows that the annihilators of $\mathrm{Ext}^j_A(M,A)$, $0 \le j \le n$, contain I. By 4.2, $\mathrm{supp}(\mathrm{Ext}^j_A(M,A)) \subset \mathrm{supp}(M)$ for $0 \le j \le n$. Therefore, $\cup_{j=0}^n \mathrm{supp}(\mathrm{Ext}^j_A(M,A)) \subset \mathrm{supp}(M)$. Take now $x \in k^n$ such that $x \notin \mathrm{supp}(\mathrm{Ext}^j_A(M,A))$ for $0 \le j \le n$. Then we have $\mathrm{Ext}^j_A(M,A)_x = 0$ for $0 \le j \le n$ and, by 3, this implies $\mathrm{Ext}^j_{A_x}(M_x,A_x) = 0$ for $0 \le j \le n$. Since the homological dimension of A_x is finite by 2, we conclude that $M_x = 0$ by 5.18. Hence, $x \notin \mathrm{supp}(M)$. $\square$

By 5.18, for any finitely generated A-module M different from zero we can define

$$j(M) = \min\{j \in \mathbb{Z}_+ \mid \mathrm{Ext}^j_A(M,A) \neq 0\}$$

and for any $x \in \mathrm{supp}(M)$,

$$j(M_x) = \min\{j \in \mathbb{Z}_+ \mid \mathrm{Ext}^j_{A_x}(M_x,A_x) \neq 0\} = \min\{j \in \mathbb{Z}_+ \mid \mathrm{Ext}^j_A(M,A)_x \neq 0\}.$$

Clearly, $0 \le j(M_x) \le n$ and

$$j(M) = \min_{x \in \mathrm{supp}(M)} j(M_x).$$

The main result of this section is the next theorem.

6.5. Theorem. *Let M be a finitely generated A-module. Then*

(i) $\mathrm{Ext}^j_A(M,A) = 0$ *for* $j < n - d(M)$;

(ii) $d(\mathrm{Ext}^j_A(M,A)) \le n - j$ *for all* $0 \le j \le n$;

(iii) $d(\mathrm{Ext}^{n-d(M)}_A(M,A)) = d(M)$.

In addition, if $M \neq 0$, $\mathrm{Ext}^{n-d(M)}_A(M,A) \neq 0$ *, i.e.,*

$$d(M) + j(M) = n.$$

In fact we shall show the following localized version of 5.

6.6. Theorem. *Let M be a finitely generated A-module and $x \in \mathrm{supp}(M)$. Then*

(i) $\mathrm{Ext}^j_A(M,A)_x = 0$ *for* $j < n - d(M_x)$;

(ii) $d(\mathrm{Ext}^j_A(M,A)_x) \le n - j$ *for all* $0 \le j \le n$;

(iii) $\mathrm{Ext}^{n-d(M_x)}_A(M,A)_x \neq 0$ *and* $d(\mathrm{Ext}^{n-d(M_x)}_A(M,A)_x) = d(M_x)$.

In particular,

$$d(M_x) + j(M_x) = n.$$

Clearly, 6. combined with 4.6 implies 5. Therefore, it remains to prove 6.

First we show that (iii) follows from (i) and (ii). By (ii), $d(\operatorname{Ext}_A^{n-d(M_x)}(M,A)_x) \leq d(M_x)$. Assume that $d(\operatorname{Ext}_A^{n-d(M_x)}(M,A)_x) < d(M_x)$. Then, by (i) and (ii), we would have $d(\operatorname{Ext}_A^j(M,A)_x) < d(M_x)$ for $0 \leq j \leq n$, which contradicts 4. and 4.5. Therefore, $d(\operatorname{Ext}_A^{n-d(M_x)}(M,A)_x) = d(M_x)$.

If $d(M_x) \neq 0$, $d(\operatorname{Ext}_A^{n-d(M_x)}(M,A)_x) = d(M_x)$ implies that $\operatorname{Ext}_A^{n-d(M_x)}(M,A)_x \neq 0$. If $d(M_x) = 0$, $\operatorname{Ext}_A^j(M,A)_x = 0$ for $j < n$ by (i), and $\operatorname{Ext}_A^n(M,A)_x \neq 0$ by 4.

We shall prove (i) and (ii) by induction in $d(M)$. We begin with a lemma.

6.7. Lemma. *Let $x \in k^n$. Then*

$$\operatorname{Ext}_A^j(A/\mathbf{m}_x, A) = 0 \textit{ for } 0 \leq j < n \textit{ and } \operatorname{Ext}_A^n(A/\mathbf{m}_x, A) \cong A/\mathbf{m}_x.$$

Proof. By induction in p we shall prove the following statement:

(∗) Let $M_p = A/(X_1 - x_1, \dots, X_p - x_p)$. Then

$$\operatorname{Ext}_A^j(M_p, A) = 0 \text{ for } j \neq p,$$

and

$$\operatorname{Ext}_A^p(M_p, A) \cong M_p.$$

For $p = n$ this proves our assertion.

Because of $M_0 = A$, the statement (∗) holds for $p = 0$. Assume that it holds for $p - 1 \geq 0$. Then we have the exact sequence

$$0 \longrightarrow M_{p-1} \xrightarrow{X_p - x_p} M_{p-1} \longrightarrow M_p \longrightarrow 0.$$

By the long exact sequence of $\operatorname{Ext}_A^{\cdot}(-, A)$ we have

$$0 \to \operatorname{Ext}_A^{p-1}(M_p, A) \to \operatorname{Ext}_A^{p-1}(M_{p-1}, A) \xrightarrow{X_p - x_p} \operatorname{Ext}_A^{p-1}(M_{p-1}, A) \to \operatorname{Ext}_A^p(M_p, A) \to 0$$

and all other $\operatorname{Ext}_A^j(M_p, A)$ vanish. By the induction assumption, $\operatorname{Ext}_A^{p-1}(M_p, A)$ is isomorphic to the kernel of the map induced on M_{p-1} by multiplication by $X_p - x_p$ and $\operatorname{Ext}_A^p(M_p, A)$ is isomorphic to the cokernel of this map. It follows that $\operatorname{Ext}_A^{p-1}(M_p, A) = 0$ and $\operatorname{Ext}_A^p(M_p, A) \cong M_p$. □

By 3, this implies, that

$$\begin{aligned}\operatorname{Ext}_{A_x}^n(A_x/\mathbf{n}_x, A_x) &= \operatorname{Ext}_{A_x}^n(A_x/(\mathbf{m}_x)_x, A_x) = \operatorname{Ext}_{A_x}^n((A/\mathbf{m}_x)_x, A_x) \\ &= \operatorname{Ext}_A^n(A/\mathbf{m}_x, A)_x \cong (A/\mathbf{m}_x)_x = A_x/(\mathbf{m}_x)_x = A_x/\mathbf{n}_x \neq 0.\end{aligned}$$

Combined with 2, this implies the following result.

6.8. THEOREM. *For any $x \in k^n$, we have*

$$\operatorname{hd}(k[X_1, X_2, \dots, X_n]_x) = n.$$

Now we can start the proof of 6. First we remark that if we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and (i) and (ii) hold for M' and M'', then they hold for M. In fact, since localization is exact, we have the exact sequence

$$0 \longrightarrow M'_x \longrightarrow M_x \longrightarrow M''_x \longrightarrow 0,$$

and if $j < n - d(M_x)$, we have $j < n - d(M'_x)$, $j < n - d(M''_x)$ and $\operatorname{Ext}^j_A(M', A)_x = \operatorname{Ext}^j_A(M'', A)_x = 0$. Hence, the localization of the long exact sequence of $\operatorname{Ext}^{\cdot}_A(-, A)$ implies that $\operatorname{Ext}^j_A(M, A)_x = 0$ and (i) holds for M. On the other hand, for any $0 \leq j \leq n$,

$$\dots \longrightarrow \operatorname{Ext}^j_A(M'', A)_x \longrightarrow \operatorname{Ext}^j_A(M, A)_x \longrightarrow \operatorname{Ext}^j_A(M', A)_x \longrightarrow \dots$$

also implies that

$$d(\operatorname{Ext}^j_A(M, A)_x) \leq \max(d(\operatorname{Ext}^j_A(M', A)_x), d(\operatorname{Ext}^j_A(M'', A)_x)) \leq n - j,$$

and (ii) holds for M.

Assume that $d(M) = 0$. Then M is a finite-dimensional vector space over k. By 7. we know that (i) and (ii) hold if $\dim_k M = 1$. The general case follows by an induction in $\dim_k$. In fact, M has a one-dimensional A-submodule M', hence we can form an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ such that (i) and (ii) hold for M'' by the induction assumption. Then the preceding discussion applies, and (i) and (ii) hold for M. This ends the proof for $d(M) = 0$.

Now we prove the induction step. Assume now that (i) and (ii) hold for all finitely generated A-modules N with $d(N) < p$ for some $p > 0$. Let M be a finitely generated A-module with $d(M) = p$. Then, by 4.3, we can find a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_{s-1} \subset M_s = M$ such that $M_i/M_{i-1} \cong A/J_i$ for some prime ideals J_i, $1 \leq i \leq s$, and $V(J_i) \subset \operatorname{supp}(M)$ for $1 \leq i \leq s$. It follows that $d(A/J_i) \leq p$ and we have an exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow A/J_i \longrightarrow 0$$

for $1 \leq i \leq s$. Hence, if we assume that (i) and (ii) hold for A/J with J prime and $\dim V(J) = p$, by the induction assumption and an induction in i we see that (i) and (ii) hold for M.

This reduces the proof of the induction step to the case $M = A/J$, J a prime ideal and $\dim V(J) = p$. Let $x \in V(J)$. Since $V(J)$ is irreducible, $\dim_x V(J) = p$ and $d(M_x) = p$ by 4.5. Moreover, since $\dim V(J) > 0$, $\mathbf{m}_x \neq J$ and we can find $f \in A$, $f \notin J$ such that

$f(x) = 0$. Consider the endomorphism of M induced by the multiplication by f. As in the proof of 4.4 and 4.5 we conclude that it is injective, hence we have the exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0.$$

Moreover, $d(M/fM) \leq d(M)$. If we would have $d(M/fM) = d(M)$ it would follow that $e(M/fM) = 0$ and $M/fM = 0$. This would imply that $d(M) = d(M/fM) = 0$ contrary to our assumption. Hence, $d(M/fM) < d(M) = p$ and M/fM satisfies (i) and (ii) by the induction assumption.

Now we can prove (i) for M. By the induction assumption $\operatorname{Ext}^j_A(M/fM, A)_x = 0$ for $j < n - p + 1$. Hence, for any $j < n - p$, from the localization of the long exact sequence of $\operatorname{Ext}^{\cdot}_A(-, A)$ we see that multiplication by f induces an automorphism of $\operatorname{Ext}^j_A(M, A)_x$; and, since $f \in \mathbf{m}_x$, we have $\mathbf{n}_x \operatorname{Ext}^j_A(M, A)_x = \operatorname{Ext}^j_A(M, A)_x$. By Nakayama lemma we conclude that $\operatorname{Ext}^j_A(M, A)_x = 0$.

It remains to prove (ii). We consider the following part of the long exact sequence of $\operatorname{Ext}^{\cdot}_A(-, A)$

$$\ldots \longrightarrow \operatorname{Ext}^j_A(M/fM, A) \longrightarrow \operatorname{Ext}^j_A(M, A) \xrightarrow{f} \operatorname{Ext}^j_A(M, A) \longrightarrow \operatorname{Ext}^{j+1}_A(M/fM, A) \longrightarrow \ldots.$$

If $j = n$, by 5.13, we have $\operatorname{Ext}^{n+1}_A(M/fM, A) = 0$. Hence the morphism $\operatorname{Ext}^n_A(M, A) \xrightarrow{f} \operatorname{Ext}^n_A(M, A)$ is surjective. By localization at x, we see that $\operatorname{Ext}^n_A(M, A)_x \xrightarrow{f} \operatorname{Ext}^n_A(M, A)_x$ is also surjective. As in the proof of (i), it follows that $\mathbf{n}_x \operatorname{Ext}^n_A(M, A)_x = \operatorname{Ext}^n_A(M, A)_x$. Again, by Nakayama lemma, we conlude that $\operatorname{Ext}^n_A(M, A)_x = 0$. Hence, $d(\operatorname{Ext}^n_A(M, A)_x) = 0$, and (ii) holds in this case.

Consider now the case $j < n$. By the induction assumption we know that

$$d(\operatorname{Ext}^j_A(M/fM, A)_x) \leq n - j \text{ and } d(\operatorname{Ext}^{j+1}_A(M/fM, A)_x) \leq n - j - 1.$$

If we put

$$\phi : \operatorname{Ext}^j_A(M, A) \xrightarrow{f} \operatorname{Ext}^j_A(M, A)$$

and denote $N = \operatorname{Ext}^j_A(M, A)$, $K = \ker \phi$, $C = \operatorname{coker} \phi$ and $I = \operatorname{im} \phi$ we get the exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{\phi} N \longrightarrow C \longrightarrow 0.$$

By localizing we get

$$0 \longrightarrow K_x \longrightarrow N_x \xrightarrow{\phi_x} N_x \longrightarrow C_x \longrightarrow 0$$

with $d(K_x) \leq n - j$ and $d(C_x) \leq n - j - 1$.

Let $I_x = \operatorname{im} \phi_x$. Then

$$I_x \cap \mathbf{n}_x^q N_x \supset f(\mathbf{n}_x^{q-1} N_x) = \mathbf{n}_x^{q-1} I_x$$

for any $q \in \mathbb{Z}_+$, and this implies that

$$\dim_k(I_x/(I_x \cap \mathbf{n}_x^q N_x)) \leq \dim_k(I_x/\mathbf{n}_x^{q-1} I_x)$$

for any $q \in \mathbb{Z}_+$. On the other hand, we have the short exact sequence

$$0 \longrightarrow I_x \longrightarrow N_x \longrightarrow C_x \longrightarrow 0$$

which leads to the short exact sequence

$$0 \longrightarrow I_x/(I_x \cap \mathbf{n}_x^q N_x) \longrightarrow N_x/\mathbf{n}_x^q N_x \longrightarrow C_x/\mathbf{n}_x^q C_x \longrightarrow 0$$

for any $q \in \mathbb{Z}_+$. Hence,

$$\begin{aligned}\dim_k(N_x/\mathbf{n}_x^q N_x) = \dim_k(C_x/\mathbf{n}_x^q C_x) + \dim_k(I_x/(I_x \cap \mathbf{n}_x^q N_x))& \\ \leq \dim_k(C_x/\mathbf{n}_x^q C_x) + \dim_k(I_x/\mathbf{n}_x^{q-1} I_x)&\end{aligned}$$

for any $q \in \mathbb{Z}_+$. Also, the short exact sequence

$$0 \longrightarrow K_x \longrightarrow N_x \longrightarrow I_x \longrightarrow 0$$

leads to the short exact sequence

$$0 \longrightarrow K_x/(K_x \cap \mathbf{n}_x^q N_x) \longrightarrow N_x/\mathbf{n}_x^q N_x \longrightarrow I_x/\mathbf{n}_x^q I_x \longrightarrow 0$$

and

$$\dim_k(N_x/\mathbf{n}_x^q N_x) = \dim_k(I_x/\mathbf{n}_x^q I_x) + \dim_k(K_x/(K_x \cap \mathbf{n}_x^q N_x)),$$

for any $q \in \mathbb{Z}_+$. Putting these formulas together we get

$$\dim_k(N_x/\mathbf{n}_x^q N_x) - \dim_k(N_x/\mathbf{n}_x^{q-1} N_x) \leq \dim_k(C_x/\mathbf{n}_x^q C_x) - \dim_k(K_x/(K_x \cap \mathbf{n}_x^{q-1} N_x))$$

for any $q \in \mathbb{N}$. Clearly, the left side of this inequality is greater or equal to zero for all $q \in \mathbb{N}$. From the discussion in the proof of 2.7 we know that for large $q \in \mathbb{N}$ the right side of this inequality is given by a polynomial of degree $\leq n - j$.

Assume first that its degree is equal to $n - j$, i.e., that $d(K_x) = n - j$ and its leading coefficient is $-e(K_x)\frac{q^{n-j}}{(n-j)!}$. In particular, it takes negative values for large $q \in \mathbb{Z}_+$. This is clearly a contradiction.

Therefore, we must have $d(K_x) \leq n - j - 1$. Then, for large q, the function $q \longmapsto \dim_k(N_x/\mathbf{n}_x^q N_x) - \dim_k(N_x/\mathbf{n}_x^{q-1} N_x)$ is given by a polynomial of degree $\leq n - j - 1$. By 1.7, $d(N_x) \leq n - j$. This ends the proof of 6.

7. Some homological algebra of filtered rings

Let D be a filtered ring with an increasing filtration $(D_n\,;\, n \in \mathbb{Z})$ by abelian subgroups satisfying the properties (i)-(vii) from the beginning of 3. Let M and N be two filtered left D-modules with filtrations $\mathrm{F}\,M$ and $\mathrm{F}\,N$ respectively. We define an increasing filtration of $\mathrm{Hom}_D(M,N)$ by abelian subgroups

$$\mathrm{F}_p \mathrm{Hom}_D(M,N) = \{\phi \in \mathrm{Hom}_D(M,N) \mid \phi(\mathrm{F}_n M) \subset \mathrm{F}_{n+p} N,\ n \in \mathbb{Z}\}$$

for all $p \in \mathbb{Z}$. We first want to study some basic properties of this filtration.

7.1. Lemma. *Assume that:*

(i) *M is a finitely generated D-module and its filtration is exhaustive,*
(ii) *the D-module filtration on N has the property that $\mathrm{F}_q N = 0$ for some $q \in \mathbb{Z}$.*

Then there exists $p \in \mathbb{Z}$ such that $\mathrm{F}_p \mathrm{Hom}_D(M,N) = 0$.

Proof. By the assumption, there exists $m \in \mathbb{Z}$ such that $\mathrm{F}_m M$ generates M as a D-module. On the other hand, for any $\phi \in \mathrm{F}_{q-m} \mathrm{Hom}_D(M,N)$ we have $\phi(\mathrm{F}_m M) \subset \mathrm{F}_q N = 0$. Therefore $\phi = 0$. $\square$

7.2. Lemma. *Assume that:*

(i) *M is a finitely generated D-module with a good filtration,*
(ii) *the D-module filtration on N is exhaustive.*

Then the filtration on $\mathrm{Hom}_D(M,N)$ is exhaustive.

Proof. Take $\phi \in \mathrm{Hom}_D(M,N)$. Let $m \in \mathbb{Z}$ be such that $\mathrm{F}_{n+m} M = D_n \cdot \mathrm{F}_m M$ for all $n \in \mathbb{Z}_+$. Then $\phi(\mathrm{F}_m M)$ is a finitely generated D_0-submodule of N, hence we can find $k \in \mathbb{Z}$ such that $\phi(\mathrm{F}_m M) \subset \mathrm{F}_k N$. This implies that for $n \geq m$, we have

$$\phi(\mathrm{F}_n M) = \phi(D_{n-m} \cdot \mathrm{F}_m M) = D_{n-m}\phi(\mathrm{F}_m M) \subset D_{n-m} \mathrm{F}_k N \subset \mathrm{F}_{n-m+k} N$$

On the other hand, since the filtration of M is good, $\mathrm{F}_q M = 0$ for a sufficiently negative q. For $q \leq n \leq m$, we have

$$\phi(\mathrm{F}_n M) \subset \phi(\mathrm{F}_m M) \subset \mathrm{F}_k N \subset \mathrm{F}_{n+k-q} N.$$

Hence, $\phi \in \mathrm{F}_{k-q} \mathrm{Hom}_D(M,N)$. $\square$

In particular, if M and N are finitely generated D-modules with good filtrations, the filtration on $\mathrm{Hom}_D(M,N)$ is exhaustive and $\mathrm{F}_q \mathrm{Hom}_D(M,N) = 0$ for sufficiently negative $q \in \mathbb{Z}$.

In the following, we want to describe the structure of $\mathrm{Gr}\,\mathrm{Hom}_D(M,N)$. Let $\phi \in \mathrm{F}_p \mathrm{Hom}_D(M,N)$. Then $\phi(\mathrm{F}_n M) \subset \mathrm{F}_{n+p} N$ for any $n \in \mathbb{Z}$. Therefore, ϕ defines a graded morphism $\phi_p : \mathrm{Gr}\, M \longrightarrow \mathrm{Gr}\, N$ of degree p, i.e., $\phi_p(\mathrm{Gr}_n M) \subset \mathrm{Gr}_{n+p} N$ for $n \in \mathbb{Z}$. Also, $\phi_p \in \mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N)$. Clearly, $\phi_p = 0$ if and only if $\phi \in \mathrm{F}_{p-1} \mathrm{Hom}_D(M,N)$, hence $\phi \longmapsto \phi_p$ induces an injective additive map from $\mathrm{Gr}_p \mathrm{Hom}_D(M,N)$ into the subgroup of

$\operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} N)$ consisting of graded homomorphisms of degree p. This defines an imbedding of $\operatorname{Gr} \operatorname{Hom}_D(M, N)$ into $\operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} N)$.

Now we study some functorial properties of our filtration. Let N be a filtered D-module. Assume that M and M' are two filtered D-modules equipped with filtrations $\mathrm{F}\, M$ and $\mathrm{F}\, M'$ and $f : M \to M'$ a morphism compatible with the filtrations, i.e., satisfying $f(\mathrm{F}_n M) \subset \mathrm{F}_n M'$, $n \in \mathbb{Z}$. Then f induces the morphism $\phi \longmapsto \phi \circ f$ of $\operatorname{Hom}_D(M', N)$ into $\operatorname{Hom}_D(M, N)$. If $\phi \in \mathrm{F}_p \operatorname{Hom}_D(M', N)$, we have

$$(\phi \circ f)(\mathrm{F}_n M) \subset \phi(\mathrm{F}_n M') \subset \mathrm{F}_{n+p} N, \ n \in \mathbb{Z};$$

hence $\phi \circ f \in \mathrm{F}_p \operatorname{Hom}_D(M', N)$. Therefore, $\phi \longmapsto \phi \circ f$ is a map compatible with filtrations on $\operatorname{Hom}_D(M, N)$ and $\operatorname{Hom}_D(M', N)$. Moreover, the diagram

$$\begin{array}{ccc} \operatorname{Gr} \operatorname{Hom}_D(M', N) & \longrightarrow & \operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} M', \operatorname{Gr} N) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Gr} \operatorname{Hom}_D(M, N) & \longrightarrow & \operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} N) \end{array}$$

is commutative.

The next step is to analyze the conditions which imply that the canonical map from $\operatorname{Gr} \operatorname{Hom}_D(M, N)$ into $\operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} N)$ is surjective. We start with the following observation.

7.3. Lemma. *Let M be a finitely generated D-module with a good filtration $\mathrm{F}\, M$, such that $\operatorname{Gr} M$ is a free $\operatorname{Gr} D$-module with a basis $e_1, e_2, \ldots, e_s$ consisting of homogeneous elements of degree $r_1, r_2, \ldots, r_s$ respectively. Let $m_1, m_2, \ldots, m_s$ be elements of M such that $\bar{m}_i = e_i$ for $1 \le i \le s$. Then M is a free D-module with basis $m_1, m_2, \ldots, m_s$ and*

$$\mathrm{F}_p M = \bigoplus_{i=1}^{s} D_{p-r_i} m_i$$

for any $p \in \mathbb{Z}$.

Proof. First we claim that $m_1, m_2, \ldots, m_s$ generate M. We shall prove, by induction in p, that $\mathrm{F}_p M \subset \sum_{i=1}^s D_{p-r_i} m_i$. First, $\mathrm{F}_q M = 0$ for sufficiently negative $q \in \mathbb{Z}$. Assume that the statement holds for $p - 1$. Take $v \in \mathrm{F}_p M$. Then $\bar{v} \in \operatorname{Gr}_p M$ and $\bar{v} = \sum_{i=1}^s \bar{d}_i e_i$ with $d_i \in D_{p-r_i}$. It follows that $v - \sum_{i=1}^s d_i m_i \in F_{p-1} M \subset \sum_{i=1}^s D_{p-1-r_i} m_i$, hence $v \in \sum_{i=1}^s D_{p-r_i} m_i$. This proves our statement and, in particular, $m_1, m_2, \ldots, m_s$ generate M.

We show next that $m_1, m_2, \ldots, m_s$ are free generators. Let $\sum d_i m_i = 0$, $d_i \in D_{p_i}$, $d_i \notin D_{p_i - 1}$, be a nontrivial relation. Take $p = \max_{1 \le i \le s}(p_i + r_i)$. Then we get $\sum \bar{d}_i e_i = 0$ where the sum is taken over indices i such that $p_i + r_i = p$. This implies that for such i, $\bar{d}_i = 0$, contrary to our assumption. Therefore, $(m_i, 1 \le i \le s)$ are free generators of M.

This implies, by the first part of the proof, that $\mathrm{F}_p M \subset \bigoplus_{i=1}^s D_{p-r_i} m_i$. The opposite inclusion is evident. □

We call a free $\operatorname{Gr} D$-module with a basis consisting of homogeneous elements, a *free graded* $\operatorname{Gr} D$-module.

7.4. LEMMA. *Let M be a finitely generated D-module with a good filtration $\mathrm{F}\, M$, such that $\mathrm{Gr}\, M$ is a free graded $\mathrm{Gr}\, D$-module. Then the canonical map $\mathrm{Gr}\,\mathrm{Hom}_D(M,N) \longrightarrow \mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N)$ is an isomorphism.*

PROOF. Since $\mathrm{Gr}\, M$ is a free graded $\mathrm{Gr}\, D$-module it has a basis $e_1, \dots, e_s$ as a free $\mathrm{Gr}\, D$-module consisting of homogeneous elements of degrees $r_1, \dots, r_s$. Therefore, we have

$$\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N) = \mathrm{Hom}_{\mathrm{Gr}\, D}\left(\bigoplus_{i=1}^{s} \mathrm{Gr}\, D \cdot e_i, \mathrm{Gr}\, N\right) = \bigoplus_{i=1}^{s} \mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, D \cdot e_i, \mathrm{Gr}\, N).$$

Moreover, the linear map $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, D \cdot e_i, \mathrm{Gr}\, N) \longrightarrow \mathrm{Gr}\, N$ given by $T \longmapsto T(e_i)$ is a linear isomorphism, and it maps the graded homomorphisms of degree $p - r_i$ into homogeneous elements of degree p for all $p \in \mathbb{Z}$. Therefore, $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, D \cdot e_i, \mathrm{Gr}\, N)$ is a direct sum of spaces of graded homomorphisms of degree p, $p \in \mathbb{Z}$, and the same holds for $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N)$.

Hence, it is enough to show that all graded homomorphisms in $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N)$ are in the image. In the following we use the notation from the preceding argument. Let $\Phi \in \mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N)$ be a morphism of order q. This means that $\Phi(e_i) \in \mathrm{Gr}_{r_i+q} N$. Let $w_i \in \mathrm{F}_{r_i+q} N$ such that $\Phi(e_i) = \bar{w}_i$. Define $\phi \in \mathrm{Hom}_D(M,N)$ by $\phi(m_i) = w_i$ for $1 \le i \le s$. We see from 3. that

$$\phi(\mathrm{F}_p\, M) \subset \sum_{i=1}^{s} D_{p-r_i} w_i \subset \mathrm{F}_{p+q}\, N$$

and $\phi \in \mathrm{F}_q \mathrm{Hom}_D(M,N)$. It is evident that $\phi_q = \Phi$. $\square$

7.5. LEMMA. *Let M be a filtered D-module, L a free graded $\mathrm{Gr}\, D$-module and $f : L \longrightarrow \mathrm{Gr}\, M$ a morphism of graded $\mathrm{Gr}\, D$-modules. Then there exists a filtered free D-module P, a morphism $g : P \longrightarrow M$ of filtered D-modules and an isomorphism $j : \mathrm{Gr}\, P \longrightarrow L$ of graded $\mathrm{Gr}\, D$-modules such that the diagram*

$$\begin{tikzcd} \mathrm{Gr}\, P \arrow[r, "\mathrm{Gr}\, g"] \arrow[d, "j"'] & \mathrm{Gr}\, M \arrow[d, equal] \\ L \arrow[r, "f"] & \mathrm{Gr}\, M \end{tikzcd}$$

commutes.

PROOF. Let $(\ell_i\,; i \in I)$ be a basis of L such that ℓ_i are homogeneous and $\deg(\ell_i) = r_i$, $i \in I$. Let P be a free D-module with the basis $(p_i\,; i \in I)$ and define its filtration by $\mathrm{F}_p\, P = \sum_{i\in I} D_{p-r_i} p_i$, for $p \in \mathbb{Z}$. Let $(m_i\,; i \in I)$ be a family of elements of M such that $m_i \in \mathrm{F}_{r_i}\, M$ and $\bar{m}_i = f(\ell_i)$ for $i \in I$. Then there exists a unique morphism $g : P \longrightarrow M$ such that $g(p_i) = m_i$, for $i \in I$. Moreover,

$$g(\mathrm{F}_p\, P) = \sum_{i\in I} D_{p-r_i} g(p_i) \subset \mathrm{F}_p\, M,$$

for any $p \in \mathbb{Z}$, and g is a morphism of filtered D-modules. Clearly, $\operatorname{Gr} P = \sum_{i \in I} \operatorname{Gr} D \cdot \bar{p}_i$ and $\operatorname{Gr}_n P = \sum_{i \in I} \operatorname{Gr}_{n-r_i} D \cdot \bar{p}_i$. This implies that the morphism $j : \operatorname{Gr} P \longrightarrow L$ defined by $j(\bar{p}_i) = \ell_i$, $i \in I$, is an isomorphism of $\operatorname{Gr} D$-modules. Also,

$$(\operatorname{Gr} g)(\bar{p}_i) = \bar{m}_i = f(\ell_i) = (f \circ j)(\bar{p}_i),$$

hence $\operatorname{Gr} g = f \circ j$. $\square$

7.6. Lemma. *Let*

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

be a sequence in the category of filtered D-modules such that $g \circ f = 0$. Assume that the filtration of M is exhaustive and $\mathrm{F}_p M = 0$ for sufficiently negative $p \in \mathbb{Z}$. If the sequence

$$\operatorname{Gr} M' \xrightarrow{\operatorname{Gr} f} \operatorname{Gr} M \xrightarrow{\operatorname{Gr} g} \operatorname{Gr} M''$$

of graded $\operatorname{Gr} D$-modules is exact, the original sequence is exact too.

Proof. By our assumption $\operatorname{im} f \subset \ker g$; i.e.,

$$\operatorname{im} f \cap \mathrm{F}_p M \subset \ker g \cap \mathrm{F}_p M$$

for $p \in \mathbb{Z}$. We shall prove by induction that

$$\operatorname{im} f \cap \mathrm{F}_p M = \ker g \cap \mathrm{F}_p M$$

for $p \in \mathbb{Z}$. Since the filtration of M is exhaustive, this shall imply that $\operatorname{im} f = \ker g$. The assertion is clear for sufficiently negative p. Assume that the assertion holds for $p-1$ and $x \in \ker g \cap \mathrm{F}_p M$. By the assumption there exists $y \in \mathrm{F}_p M$ such that $(\operatorname{Gr} f)(\bar{y}) = \bar{x}$. Hence, $f(y) - x \in \mathrm{F}_{p-1} M$. Also $g(f(y) - x) = g(f(y)) - g(x) = 0$, hence $f(y) - x \in \ker g$. By the induction assumption $x - f(y) \in \operatorname{im} f$, hence there exists $z \in M$ such that $x - f(y) = f(z)$ and $x = f(y + z)$, i.e., $x \in \operatorname{im} f$. $\square$

7.7. Lemma. *Let M be a finitely generated D-module with good filtration $\mathrm{F} M$. Let $L_.$ be a left resolution of $\operatorname{Gr} M$ in the category of graded $\operatorname{Gr} D$-modules by free graded $\operatorname{Gr} D$-modules of finite rank. Then there exists a left resolution $P_.$ of M in the category of filtered D-modules such that:*

(i) *the D-modules P_n, $n \in \mathbb{Z}_+$, are free;*
(ii) *the complex $\operatorname{Gr} P_.$ is isomorphic to $L_.$.*

Proof. By 5. we can construct a filtered free D-module P_0 and a morphism of filtered modules $P_0 \xrightarrow{\epsilon} M$ such that

$$\begin{array}{ccc} \operatorname{Gr} P_0 & \xrightarrow{\operatorname{Gr} \epsilon} & \operatorname{Gr} M \\ {\scriptstyle\cong}\downarrow & & \| \\ L_0 & \longrightarrow & \operatorname{Gr} M \end{array}$$

commutes. By 3.1 the filtration $\mathrm{F}\,P_0$ is a good filtration, and by 3. P_0 is a free D-module of finite rank. In addition, $\operatorname{Gr}\epsilon$ is surjective. By 6, ϵ is also surjective. Assume that we constructed filtered D-modules $P_0, P_1, \dots, P_{k-1}$ and morphisms $d_0, d_1, \dots, d_{k-1}$ such that

$$P_{k-1} \xrightarrow{d_{k-1}} P_{k-2} \xrightarrow{d_{k-2}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is an exact sequence of filtered D-modules, P_i, $0 \le i \le k-1$, are free D-modules of finite rank and

$$\begin{tikzcd}
 & \operatorname{Gr} P_{k-1} \arrow[r, "\operatorname{Gr} d_{k-1}"] \arrow[d, "\cong"'] & \operatorname{Gr} P_{k-2} \arrow[r, "\operatorname{Gr} d_{k-2}"] \arrow[d, "\cong"'] & \dots \arrow[r, "\operatorname{Gr} d_1"] & \operatorname{Gr} P_0 \arrow[r, "\operatorname{Gr}\epsilon"] \arrow[d, "\cong"'] & \operatorname{Gr} M \arrow[d, equal] \\
L_k \arrow[r] & L_{k-1} \arrow[r] & L_{k-2} \arrow[r] & \dots \arrow[r] & L_0 \arrow[r] & \operatorname{Gr} M
\end{tikzcd}$$

is commutative. Let K be the kernel of d_{k-1} equipped with the induced filtration. Then $\operatorname{Gr} K$ is identified with a submodule of $\operatorname{Gr} P_{k-1} \cong L_{k-1}$. Moreover, with this identification, $\operatorname{Gr} K \subset \ker \operatorname{Gr} d_{k-1}$.

We claim that $\operatorname{Gr} K = \ker \operatorname{Gr} d_{k-1}$. Let $\xi \in \ker \operatorname{Gr}_p d_{k-1}$. Let $x \in \mathrm{F}_p\, P_{k-1}$ be such that $\bar{x} = \xi$. Then $(\operatorname{Gr} d_{k-1})(\bar{x}) = 0$ and $d_{k-1}(x) \in \mathrm{F}_{p-1}\, P_{k-2}$. Let $\mathcal{S}$ be the set of all $s \in \mathbb{Z}$, $s \le p-1$, such that there exists $x \in \mathrm{F}_p\, P_{k-1}$ with the property that $\bar{x} = \xi$ and $d_{k-1}x \in \mathrm{F}_s\, P_{k-2}$. Let $q \in \mathcal{S}$. Assume that x' is such that $\bar{x}' = \xi$ and $y = d_{k-1}x' \in \mathrm{F}_q\, P_{k-2}$. Then $d_{k-2}(y) = (d_{k-2} \circ d_{k-1})(x') = 0$ hence $\bar{y} \in \ker \operatorname{Gr} d_{k-2} = \operatorname{im} \operatorname{Gr} d_{k-1}$. It follows that one can find $z \in \mathrm{F}_q\, P_{k-1}$ such that $\bar{y} = (\operatorname{Gr} d_{k-1})(\bar{z})$, i.e., $y - d_{k-1}z \in \mathrm{F}_{q-1}\, P_{k-2}$. Now, if we put $x'' = x' - z$ we get $\bar{x}'' = \bar{x}' = \xi$ and

$$d_{k-1}x'' = d_{k-1}x' - d_{k-1}z = y - d_{k-1}z \in \mathrm{F}_{q-1}\, P_{k-2}$$

and $q-1$ is in $\mathcal{S}$. Therefore, $\mathcal{S} = \{q \in \mathbb{Z} \mid q \le p-1\}$. Since $\mathrm{F}_q\, P_{k-2} = 0$ for sufficiently negative q, we conclude that there exists $x \in \mathrm{F}_p\, P_{k-1}$ with the property that $\bar{x} = \xi$ and $d_{k-1}x = 0$, i. e. $\xi \in \operatorname{Gr} K$.

We proved that $\operatorname{Gr} K = \ker \operatorname{Gr} d_{k-1}$, hence under the isomorphism of $\operatorname{Gr} P_{k-1}$ with L_{k-1}, it corresponds to the image of L_k in L_{k-1}. This implies that, by 5, we can construct P_k and $d_k : P_k \longrightarrow P_{k-1}$ such that

(i) P_k is a filtered free D-module,
(ii) d_k is a morphism of filtered D-modules,
(iii) $\operatorname{im} d_k \subset K$,
(iv) the diagram

$$\begin{tikzcd}
\operatorname{Gr} P_k \arrow[r, "\operatorname{Gr} d_k"] \arrow[d, "\cong"'] & \operatorname{Gr} P_{k-1} \arrow[d, "\cong"'] \\
L_k \arrow[r] & L_{k-1}
\end{tikzcd}$$

is commutative.

Now 6. implies that $\operatorname{im} d_k = \ker d_{k-1}$ and

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is exact. By induction we get the required resolution. □

Now, let M be a finitely generated D-module with a good filtration $\mathrm{F}\, M$. Then $\operatorname{Gr} M$ is a finitely generated $\operatorname{Gr} D$-module. We can find homogeneous generators $e_1, e_2, \dots, e_s$ of $\operatorname{Gr} M$ as $\operatorname{Gr} D$-module. Let $r_1, r_2, \dots, r_s$ be the degrees of these generators respectively. Let L_0 be the free graded $\operatorname{Gr} D$-module with basis $\ell_1, \ell_2, \dots, \ell_s$ consisting of homogeneous elements of degrees $r_1, r_2, \dots, r_s$. Then we can define a $\operatorname{Gr} D$-module morphism $f : L_0 \longrightarrow \operatorname{Gr} M$ by $f(\ell_i) = e_i$, $1 \le i \le s$, and it is a surjective morphism of graded $\operatorname{Gr} D$-modules. Since $\operatorname{Gr} D$ is a nötherian ring, $\ker f$ is a finitely generated $\operatorname{Gr} D$-module and by induction we can construct a left resolution

$$\dots \longrightarrow L_n \longrightarrow \dots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow \operatorname{Gr} M \longrightarrow 0$$

by free graded $\operatorname{Gr} D$-modules of finite rank. By 7. this implies the existence of a left resolution in the category of filtered D-modules

$$\dots \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

by filtered free D-modules, such that the complex $\operatorname{Gr} P.$ is a left resolution of $\operatorname{Gr} M$ and

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \operatorname{Gr} P_n & \longrightarrow & \dots & \longrightarrow & \operatorname{Gr} P_1 & \longrightarrow & \operatorname{Gr} P_0 & \longrightarrow & \operatorname{Gr} M \\
 & & \cong\big\downarrow & & & & \cong\big\downarrow & & \cong\big\downarrow & & \big\| \\
\dots & \longrightarrow & L_n & \longrightarrow & \dots & \longrightarrow & L_1 & \longrightarrow & L_0 & \longrightarrow & \operatorname{Gr} M
\end{array}$$

is a commutative diagram. By 3.1 this implies that $\mathrm{F}\, P_i$ are good filtrations of P_i for any $i \in \mathbb{Z}_+$.

Let N be a finitely generated D-module with good filtration $\mathrm{F}\, N$. Then we can consider the complex $K^\cdot = \operatorname{Hom}_D(P., N)$. Clearly,

$$H^j(K^\cdot) = \operatorname{Ext}^j_D(M, N)$$

for any $j \in \mathbb{Z}_+$. Moreover, by 1. and 2, K^j, $j \in \mathbb{Z}$, have natural exhaustive filtrations compatible with the differentials in the complex and such that $\mathrm{F}_p\, K^j = 0$ for sufficiently negative $p \in \mathbb{Z}$. The corresponding graded complex $\operatorname{Gr} K^\cdot = \operatorname{Gr} \operatorname{Hom}_D(P., N)$ is isomorphic to the complex $\operatorname{Hom}_{\operatorname{Gr} D}(\operatorname{Gr} P., \operatorname{Gr} N) \cong \operatorname{Hom}_{\operatorname{Gr} D}(L., \operatorname{Gr} N)$ by 4. Therefore,

$$H^j(\operatorname{Gr} K^\cdot) = \operatorname{Ext}^j_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} N)$$

for any $j \in \mathbb{Z}_+$.

7.8. Theorem. $\mathrm{hd}(D) \leq \mathrm{hd}(\mathrm{Gr}\, D)$.

Proof. We can assume that the homological dimension of $\mathrm{Gr}\, D$ is finite. Let M and N be two finitely generated D-modules equipped with good filtrations $\mathrm{F}\, M$ and $\mathrm{F}\, N$. By the preceding discussion, $\mathrm{Ext}^j_D(M, N) = H^j(K^\cdot)$ and $\mathrm{Ext}^j_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, N) = H^j(\mathrm{Gr}\, K^\cdot)$. Therefore, $H^j(\mathrm{Gr}\, K^\cdot) = 0$ for $j > \mathrm{hd}(\mathrm{Gr}\, D)$. Hence, if we put $n = \mathrm{hd}(\mathrm{Gr}\, D)$, we have the exact sequence

$$\mathrm{Gr}\, K^n \longrightarrow \mathrm{Gr}\, K^{n+1} \longrightarrow \mathrm{Gr}\, K^{n+2} \longrightarrow \dots.$$

Using 6. (for $D = k$), we conclude that

$$K^n \longrightarrow K^{n+1} \longrightarrow K^{n+2} \longrightarrow \dots$$

is exact. Therefore, $\mathrm{Ext}^j_D(M, N) = 0$ for $j > n$, and the assertion follows from 3.4 and 5.8. □

Assume now that $N = D$ equipped with the natural filtration. Then we can define a structure of a right D-module on $\mathrm{Hom}_D(M, D)$ by $(\phi T)(m) = \phi(m)T$, for $\phi \in \mathrm{Hom}_D(M, D)$, $T \in D$ and $m \in M$. Clearly $\mathrm{Hom}_D(-, D)$ is a contravariant functor from the category of left D-modules into the category of right D-modules. If M is a finitely generated left D-module, we have an exact sequence $D^p \longrightarrow M \longrightarrow 0$ of left D-modules. By applying the contravariant functor $\mathrm{Hom}_D(-, D)$ we get the exact sequence $0 \longrightarrow \mathrm{Hom}_D(M, D) \longrightarrow \mathrm{Hom}_D(D^p, D)$. Since

$$\mathrm{Hom}_D(D^p, D) = \mathrm{Hom}_D(D, D)^p \cong D^p$$

as a right D-module, we conclude that $\mathrm{Hom}_D(M, D)$ is a finitely generated right D-module. Let $\mathrm{F}\, M$ be a D-module filtration on M. Then for any $\phi \in \mathrm{F}_p \mathrm{Hom}_D(M, D)$ and $T \in D_q$ we have

$$(\phi T)(\mathrm{F}_m\, M) = \phi(\mathrm{F}_m\, M)T \subset D_{m+p}T \subset D_{m+p+q}$$

and $\phi T \in \mathrm{F}_{p+q} \mathrm{Hom}_D(M, D)$. Therefore, $\mathrm{F}\,\mathrm{Hom}_D(M, D)$ is a D-module filtration.

7.9. Lemma. *Let M be a finitely generated left D-module with a good filtration $\mathrm{F}\, M$. Then the filtration $\mathrm{F}\,\mathrm{Hom}_D(M, D)$ on the right D-module $\mathrm{Hom}_D(M, D)$ is also good.*

Proof. By 1. and 2. we know that the D-module filtration on $\mathrm{Hom}_D(M, D)$ is hausdorff and exhaustive. From the preceding discussion, we have the natural inclusion of $\mathrm{Gr}\,\mathrm{Hom}_D(M, D)$ into $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, D)$ which is a morphism of $\mathrm{Gr}\, D$-modules. Since $\mathrm{Gr}\, M$ is a finitely generated $\mathrm{Gr}\, D$-module, $\mathrm{Hom}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, D)$ is a finitely generated $\mathrm{Gr}\, D$-module. Moreover, since $\mathrm{Gr}\, D$ is a nötherian ring, this implies that the $\mathrm{Gr}\, D$-module $\mathrm{Gr}\,\mathrm{Hom}_D(M, D)$ is finitely generated. Finally, 3.1 implies now that $\mathrm{F}\,\mathrm{Hom}_D(M, D)$ is a good filtration. □

Therefore, if we apply this to our previous discussion, and denote now by $K^\cdot$ the filtered complex $\mathrm{Hom}_D(P., D)$, we see that $K^\cdot$ is a complex consisting of filtered right D-modules equipped with good filtrations. Its cohomology modules $H^j(K^\cdot) = \mathrm{Ext}^j_D(M, D)$,

$j \in \mathbb{Z}_+$, are filtered right D-modules and the induced filtrations are good by 3.7. The corresponding graded complex is $\operatorname{Gr} K^{\cdot} = \operatorname{Hom}_{\operatorname{Gr} D}(L_{\cdot}, \operatorname{Gr} D)$ and its cohomology modules are $H^j(\operatorname{Gr} K^{\cdot}) = \operatorname{Ext}^j_{\operatorname{Gr} D}(\operatorname{Gr} M, \operatorname{Gr} D)$ for $j \in \mathbb{Z}_+$. To get a more precise relationship between these modules we shall study the corresponding spectral sequence.

We start with the following result.

7.10. Lemma. *Let $f : M \to N$ be a morphism of filtered D-modules with good filtrations $\mathrm{F}\, M$ and $\mathrm{F}\, N$. Let*

$$Z_r^p = \{x \in \mathrm{F}_p\, M \mid f(x) \in \mathrm{F}_{p-r}\, N\}$$

and

$$B_r^p = \{x \in \mathrm{F}_p\, N \mid x = f(y),\ y \in \mathrm{F}_{p+r}\, M\}.$$

Then there exists r_0 such that for $r \geq r_0$ we have

$$Z_r^p + \mathrm{F}_{p-1}\, M = (\ker f \cap \mathrm{F}_p\, M) + \mathrm{F}_{p-1}\, M$$

for all $p \in \mathbb{Z}$, and

$$B_r^p = \operatorname{im} f \cap \mathrm{F}_p\, N$$

for all $p \in \mathbb{Z}$.

Proof. The submodule $\operatorname{im} f$ of N has two good filtrations $\mathrm{F}\, N \cap \operatorname{im} f$ and $f(\mathrm{F}\, M)$. By 3.6 we know that there exists $k \in \mathbb{Z}_+$ such that

$$\mathrm{F}_p\, N \cap \operatorname{im} f \subset f(\mathrm{F}_{p+k}\, M)$$

for any $p \in \mathbb{Z}$. Therefore, if $r \geq k$,

$$B_r^p = \mathrm{F}_p\, N \cap f(\mathrm{F}_{p+r}\, M) \supset \mathrm{F}_p\, N \cap \operatorname{im} f \supset \mathrm{F}_p\, N \cap f(\mathrm{F}_{p+r}\, M);$$

i. e., $B_r^p = \mathrm{F}_p\, N \cap \operatorname{im} f$. Also, if $r \geq k+1$, for any $p \in \mathbb{Z}$ we have

$$\mathrm{F}_{p-r}\, N \cap \operatorname{im} f \subset f(\mathrm{F}_{p+k-r}\, M) \subset f(\mathrm{F}_{p-1}\, M);$$

hence for $x \in Z_r^p$ we have

$$f(x) \in \mathrm{F}_{p-r}\, N \cap \operatorname{im} f \subset f(\mathrm{F}_{p-1}\, M)$$

and we can find $y \in \mathrm{F}_{p-1}\, M$ such that $f(x) = f(y)$. Therefore, $x + \mathrm{F}_{p-1}\, M = x - y + \mathrm{F}_{p-1}\, M$ and $x - y \in \ker f \cap \mathrm{F}_p\, M$. This implies that

$$Z_r^p + \mathrm{F}_{p-1}\, M \subset (\ker f \cap \mathrm{F}_p\, M) + \mathrm{F}_{p-1}\, M.$$

The inclusion $\ker f \cap \mathrm{F}_p\, M \subset Z_r^p$ is clear from the definition. $\square$

Let $K^\cdot$ be a filtered complex. Then we define

$$Z_r^{pq} = \{x \in \mathrm{F}_p K^{p+q} \mid dx \in \mathrm{F}_{p-r} K^{p+q+1}\}.$$

Clearly they satisfy

$$\mathrm{F}_p K^{p+q} = Z_0^{pq} \supset Z_1^{pq} \supset \ldots \supset Z_r^{pq} \supset \ldots \supset Z^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q}.$$

Moreover, if we put

$$B_r^{pq} = \{y \in \mathrm{F}_p K^{p+q} \mid y = dx,\ x \in \mathrm{F}_{p+r} K^{p+q-1}\} = dZ_r^{p+r\ q-r-1}$$

we see that

$$B^{p+q}(\mathrm{F}_p K^\cdot) = B_0^{pq} \subset B_1^{pq} \subset \ldots \subset B_r^{pq} \subset \ldots \subset B^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q}.$$

The E_r-term in the spectral sequence for the filtered complex $K^\cdot$ is defined as

$$E_r^{pq} = (Z_r^{pq} + \mathrm{F}_{p-1} K^{p+q})/(B_{r-1}^{pq} + \mathrm{F}_{p-1} K^{p+q}),$$

and the differential $d_r : E_r^{pq} \longrightarrow E_r^{p-r\ q+r+1}$ is induced by the differential d of $K^\cdot$. In particular the E_1-term is equal to

$$E_1^{pq} = H^{p+q}(\mathrm{Gr}_p K^\cdot)$$

for any $p, q \in \mathbb{Z}$. (To establish the connection of this formula for E_r^{pq} with the one in [Godement] we remark that from the definition of Z_r^{pq} it follows that

$$Z_r^{pq} \cap \mathrm{F}_{p-1} K^{p+q} = Z_{r-1}^{p-1\ q+1},$$

Hence,

$$Z_r^{pq} \cap (B_{r-1}^{pq} + \mathrm{F}_{p-1} K^{p+q}) = B_{r-1}^{pq} + Z_{r-1}^{p-1\ q+1},$$

and we have

$$E_r^{pq} = Z_r^{pq}/(B_{r-1}^{pq} + Z_{r-1}^{p-1\ q+1}),$$

which establishes the connection.)

Assume that $K^\cdot$ is a filtered complex of D-modules, such that the filtrations $\mathrm{F} K^j$ on K^j, $j \in \mathbb{Z}$, are good. Then $\oplus_{p+q=j} E_r^{pq}$, $j \in \mathbb{Z}$, are $\mathrm{Gr}\, D$-modules and $d_r : \oplus_{p+q=j} E_r^{pq} \longrightarrow \oplus_{p+q=j+1} E_r^{pq}$ are morphisms of $\mathrm{Gr}\, D$-modules. Since the filtrations on K^j, $j \in \mathbb{Z}$, are good, $\mathrm{Gr}\, K^\cdot$ is a complex of finitely generated $\mathrm{Gr}\, D$-modules by 3.1. This implies, by induction in r, that $\oplus_{p+q=j} E_r^{pq}$, $j \in \mathbb{Z}$, are finitely generated $\mathrm{Gr}\, D$-modules. Moreover,

$$\begin{aligned}
&\bigoplus_{p+q=j} E_\infty^{pq} \\
&\quad = \bigoplus_{p+q=j} (Z^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q} + \mathrm{F}_{p-1} K^{p+q})/(B^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q} + \mathrm{F}_{p-1} K^{p+q}) \\
&\qquad = \bigoplus_{p+q=j} (Z^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q})/(B^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q} + Z^{p+q}(K^\cdot) \cap \mathrm{F}_{p-1} K^{p+q})
\end{aligned}$$

is the graded $\mathrm{Gr}\, D$-module corresponding to the filtered module $H^j(K^\cdot)$ with the induced filtration. This filtration is good by 3.7.

7.11. Lemma. *Let $K^\cdot$ be a filtered complex of D-modules, such that the filtrations $\mathrm{F}\, K^j$ on K^j, $j \in \mathbb{Z}$, are good. Then the spectral sequence of filtered complex with E_1-term $E_1^{pq} = H^{p+q}(\mathrm{Gr}_p K^\cdot)$ converges to $H^{p+q}(K^\cdot)$.*

Proof. By 10, we know that for $p, q \in \mathbb{Z}$, we can find $r_0(p+q)$ such that for $r > r_0(p+q)$ we have

$$Z_r^{pq} + \mathrm{F}_{p-1} K^{p+q} = (\ker d \cap \mathrm{F}_p K^{p+q}) + \mathrm{F}_{p-1} K^{p+q} = Z^{p+q}(K^\cdot) + \mathrm{F}_{p-1} K^{p+q}$$

and

$$B_{r-1}^{pq} = \mathrm{im}\, d \cap \mathrm{F}_p K^{p+q} = B^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q}.$$

This implies that

$$\begin{aligned} E_r^{pq} &= (Z^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q} + \mathrm{F}_{p-1} K^{p+q})/(B^{p+q}(K^\cdot) \cap \mathrm{F}_p K^{p+q} + \mathrm{F}_{p-1} K^{p+q}) \\ &= \mathrm{Gr}_p H^{p+q}(K^\cdot), \end{aligned}$$

i.e., our spectral sequence converges. □

If we apply this to the complex $K^\cdot = \mathrm{Hom}_D(P., D)$ we see that the spectral sequence of this filtered complex has the E_1-term equal to $\mathrm{Gr}_p \mathrm{Ext}^{p+q}_{\mathrm{Gr}\, D}(\mathrm{Gr}\, M, \mathrm{Gr}\, D)$ and converges to $\mathrm{Ext}^{p+q}_D(M, D)$.

8. Rings of differential operators with polynomial coefficients

Let k be a field of characteristic zero. Let A be a commutative algebra over k. A *k-derivation* of A is a k-endomorphism T of A such that

$$T(ab) = T(a)b + aT(b)$$

for any $a, b \in A$. In particular, $[T, a]b = T(ab) - aT(b) = T(a)b$, i.e., $[T, a] = T(a) \in A$ for any $a \in A$. This implies that $[[T, a_0], a_1] = 0$ for any $a_0, a_1 \in A$.

This leads to the following definition. We say that a k-endomorphism T of A is a (*k-linear*) *differential operator* on A of *order* $\leq n$ if

$$[\dots[[T, a_0], a_1], \dots, a_n] = 0$$

for any $a_0, a_1, \dots, a_n \in A$. We denote by $\mathrm{Diff}_k(A)$ the space of all differential operators on A.

8.1. Lemma. *Let T, S be two differential operators of order $\leq n$, $\leq m$ respectively. Then $T \circ S$ is a differential operator of order $\leq n + m$.*

Proof. We prove the statement by induction in $n + m$. If $n = m = 0$, $T, S \in \mathrm{Hom}_A(A, A)$, hence $T \circ S \in \mathrm{Hom}_A(A, A)$ and it is a differential operator of order 0.

Assume now that $n + m \leq p$. Then

$$[T \circ S, a] = TSa - aTS = T[S, a] + [T, a]S,$$

and $[T, a], [S, a]$ are differential operators of order $\leq n - 1$ and $\leq m - 1$ respectively. By the induction assumption, this differential operator is of order $\leq n + m - 1$. Therefore $T \circ S$ is of order $\leq n + m$. $\square$

Therefore $\mathrm{Diff}_k(A)$ is an algebra over k. Also,

$$\mathrm{F}_n \mathrm{Diff}_k(A) = \{T \in \mathrm{Diff}_k(A) \mid \mathrm{order}(T) \leq n\}$$

is an increasing exhaustive filtration of $\mathrm{Diff}_k(A)$ by vector subspaces over k. This filtration is compatible with the ring structure of $\mathrm{Diff}_k(A)$, i.e., it satisfies

$$\mathrm{F}_n \mathrm{Diff}_k(A) \circ \mathrm{F}_m \mathrm{Diff}_k(A) \subset \mathrm{F}_{n+m} \mathrm{Diff}_k(A)$$

for any $n, m \in \mathbb{Z}_+$.

8.2. Lemma.

(i) $\mathrm{F}_p \mathrm{Diff}_k(A) = 0$ *for* $p < 0$.
(ii) $\mathrm{F}_0 \mathrm{Diff}_k(A) = A$.
(iii) $\mathrm{F}_1 \mathrm{Diff}_k(A) = \mathrm{Der}_k(A) \oplus A$.
(iv) $[\mathrm{F}_n \mathrm{Diff}_k(A), \mathrm{F}_m \mathrm{Diff}_k(A)] \subset \mathrm{F}_{n+m-1} \mathrm{Diff}_k(A)$ *for any* $n, m \in \mathbb{Z}_+$.

Proof. (i) is evident.

(ii) Clearly, $\mathrm{F}_0 \mathrm{Diff}_k(A) = \mathrm{Hom}_A(A, A)$. Hence the evaluation map $T \longmapsto T(1)$ identifies it with A.

(iii) As we remarked before, $\mathrm{Der}_k(A) \subset \mathrm{F}_1 \mathrm{Diff}_k(A)$. Also, for any $T \in \mathrm{Der}_k(A)$, we have $T(1) = T(1 \cdot 1) = 2T(1)$, hence $T(1) = 0$. This implies that $\mathrm{Der}_k(A) \cap A = 0$. Let $S \in \mathrm{F}_1 \mathrm{Diff}_k(A)$ and $T = S - S(1)$. Then $T(1) = 0$, hence $T(a) = [T, a](1)$, and

$$T(ab) = [T, ab](1) = ([T, a]b)(1) + (a[T, b])(1) = (b[T, a])(1) + (a[T, b])(1) = T(a)b + aT(b),$$

i.e., $T \in \mathrm{Der}_k(A)$.

(iv) Let T, S be of order $\leq n$, $\leq m$ respectively. We claim that $[T, S]$ is of order $\leq n + m - 1$. We prove it by induction on $n + m$. If $n = m = 0$, there is nothing to prove. In general

$$[[T, S], a] = [[T, a], S] + [T, [S, a]]$$

where $[T, a]$ and $[S, a]$ are of order $\leq n - 1$ and $\leq m - 1$ respectively. Hence, by the induction assumption, $[[T, S], a]$ is of order $\leq n + m - 2$ and $[T, S]$ is of order $\leq n + m - 1$. $\square$

This implies that the graded ring $\mathrm{Gr}\,\mathrm{Diff}_k(A)$ is a commutative A-algebra. In addition, $\mathrm{Diff}_k(A)$ satisfies properties (i)-(v) from 3.

Let T be a differential operator on A of order $\leq n$. Then we can define a map from A^n into $\mathrm{Diff}_k(A)$ by

$$\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n) = [[\ldots[[T, a_1], a_2], \ldots, a_{n-1}], a_n].$$

Since $\sigma_n(T)(a_1, a_2, \ldots, a_{n-1}, a_n)$ is of order ≤ 0, we can consider this map as a map from A^n into A.

8.3. Lemma. *Let T be a differential operator on A of order $\leq n$. Then:*

(i) *the map $\sigma_n(T) : A^n \longrightarrow A$ is a symmetric k-multilinear map;*
(ii) *the operator T is of order $\leq n-1$ if and only if $\sigma_n(T) = 0$.*

Proof. (i) We have to check the symmetry property only. To show this, we observe that

$$[[S,a],b] = [S,a]b - b[S,a] = Sab - aSb - bSa + baS = [S,b]a - a[S,b] = [[S,b],a]$$

for any $S \in \mathrm{Diff}_k(A)$ and $a, b \in A$. This implies that

$$\begin{aligned}\sigma_n(T)&(a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n-1}, a_n)\\ &= [[\ldots[[\ldots[[T,a_1],a_2],\ldots,a_i],a_{i+1}]\ldots,a_{n-1}],a_n]\\ &= [[\ldots[[\ldots[[T,a_1],a_2],\ldots,a_{i+1}],a_i]\ldots,a_{n-1}],a_n]\\ &= \sigma_n(T)(a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_{n-1}, a_n),\end{aligned}$$

hence $\sigma(T)$ is symmetric.

(ii) is obvious. $\square$

Now we want to discuss a special case. Let $A = k[X_1, X_2, \ldots, X_n]$. Then we put $D(n) = \mathrm{Diff}_k(A)$. Let $\partial_1, \partial_2, \ldots, \partial_n$ be the standard derivations of $k[X_1, X_2, \ldots, X_n]$. For $I, J \in \mathbb{Z}_+^n$ we put

$$X^I = X_1^{i_1} X_2^{i_2} \ldots X_n^{i_n}$$

and

$$\partial^J = \partial_1^{j_1} \partial_2^{j_2} \ldots \partial_n^{j_n}.$$

Then $X^I \partial^J \in D(n)$ and it is a differential operator of order $\leq |J| = j_1 + j_2 + \cdots + j_n$.

8.4. Lemma. *The derivations $(\partial_i\,;\, 1 \leq i \leq n)$ form a basis of the free $k[X_1, \ldots, X_n]$-module* $\mathrm{Der}_k(k[X_1, X_2, \ldots, X_n])$.

Proof. Let $T \in \mathrm{Der}_k(k[X_1, X_2, \ldots, X_n])$. Put $P_i = T(X_i)$ for $1 \leq i \leq n$, and define $S = \sum_{i=1}^n P_i \partial_i$. Clearly,

$$S(X_i) = \sum_{j=1}^n P_j\, \partial_j(X_i) = P_i = T(X_i)$$

for all $1 \leq i \leq n$. Since $X_1, X_2, \ldots, X_n$ generate $k[X_1, X_2, \ldots, X_n]$ as a k-algebra it follows that $T = S$. Therefore, $(\partial_i\,;\, 1 \leq i \leq n)$ generate the $k[X_1, X_2, \ldots, X_n]$-module $\mathrm{Der}_k(k[X_1, X_2, \ldots, X_n])$. Assume that $\sum_{i=1}^n Q_i \partial_i = 0$ for some $Q_i \in k[X_1, X_2, \ldots, X_n]$. Then $0 = (\sum_{j=1}^n Q_j \partial_j)(X_i) = Q_i$ for all $1 \leq i \leq n$. This implies that ∂_i, $1 \leq i \leq n$, are free generators of $\mathrm{Der}_k(k[X_1, X_2, \ldots, X_n])$. $\square$

Let T be a differential operator of order $\leq p$ on $k[X_1, X_2, \dots, X_n]$. Let $(\xi_1, \xi_2, \dots, \xi_n) \in k^n$. Then we can define a linear polynomial $\ell_\xi = \sum_{i=1}^n \xi_i X_i \in k[X_1, X_2, \dots, X_n]$ and the function $\mathrm{Symb}_p(T)$:

$$(\xi_1, \xi_2, \dots, \xi_n) \longmapsto \frac{1}{p!} \sigma_p(T)(\ell_\xi, \ell_\xi, \dots, \ell_\xi)$$

on k^n with values in $k[X_1, X_2, \dots, X_n]$. Clearly, one can view $\mathrm{Symb}_p(T)$ as a polynomial in $X_1, X_2, \dots, X_n$ and $\xi_1, \xi_2, \dots, \xi_n$ homogeneous of degree p in $\xi_1, \xi_2, \dots, \xi_n$. It is called the *$p$-symbol* of the differential operator T. By its definition, $\mathrm{Symb}_p(T)$ vanishes for T of order $< p$. Therefore, it induces a k-linear map of $\mathrm{Gr}_p D(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$. We denote by Symb the corresponding k-linear map of $\mathrm{Gr}\, D(n)$ into $k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$.

8.5. THEOREM. *The map* $\mathrm{Symb} : \mathrm{Gr}\, D(n) \longrightarrow k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ *is a k-algebra isomorphism.*

The proof of this result consists of several steps. First we prove the symbol map is an algebra morphism.

8.6. LEMMA. *Let $T, S \in D(n)$ of order $\leq p$ and $\leq q$ respectively. Then*

$$\mathrm{Symb}_{p+q}(TS) = \mathrm{Symb}_p(T)\ \mathrm{Symb}_q(S).$$

PROOF. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \to D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then

$$\tau_\xi(TS) = [TS, \ell_\xi] = TS\ell_\xi - \ell_\xi TS = [T, \ell_\xi]S + T[S, \ell_\xi] = \tau_\xi(T)S + T\tau_\xi(S).$$

Therefore, for any $k \in \mathbb{Z}_+$, we have

$$\tau_\xi^k(TS) = \sum_{i=0}^k \binom{k}{i} \tau_\xi^{k-i}(T)\ \tau_\xi^i(S).$$

This implies that

$$\begin{aligned} \mathrm{Symb}_{p+q}(TS) &= \frac{1}{(p+q)!} \sigma_{p+q}(T)(\ell_\xi, \ell_\xi, \dots, \ell_\xi) = \frac{1}{(p+q)!} \tau_\xi^{p+q}(TS) \\ &= \frac{1}{p!q!} \tau_\xi^p(T)\ \tau_\xi^q(S) = \mathrm{Symb}_p(T)\ \mathrm{Symb}_q(S). \quad \square \end{aligned}$$

Since $\mathrm{Symb}_0(X_i) = X_i$ and $\mathrm{Symb}_1(\partial_i) = \xi_i$, $1 \leq i \leq n$, the symbol map is surjective. It remains to show that the symbol map is injective.

8.7. Lemma. *Let* $T \in \mathrm{F}_p\, D(n)$. *Then* $\mathrm{Symb}_p(T) = 0$ *if and only if* T *is of order* $\leq p-1$.

Proof. We prove the statement by induction in p. It is evident if $p = 0$.

Therefore we can assume that $p > 0$. Let $\xi \in k^n$, and define the map $\tau_\xi : D(n) \to D(n)$ by $\tau_\xi(T) = [T, \ell_\xi]$. Then, for any $\lambda \in k$ and $\eta \in k^n$, we have

$$\tau_{\xi+\lambda\eta}(T) = [T, \ell_{\xi+\lambda\eta}] = [T, \ell_\xi] + \lambda[T, \ell_\eta] = \tau_\xi(T) + \lambda\tau_\eta(T).$$

Since τ_ξ and τ_η commute we see that, for any $k \in \mathbb{Z}_+$, we have

$$\tau^k_{\xi+\lambda\eta}(T) = \sum_{i=0}^{k} \binom{k}{i} \lambda^i\ \tau_\xi^{k-i}(\tau^i_\eta(T)).$$

By our assumption, $\tau^p_{\xi+\lambda\eta}(T) = 0$ for arbitrary $\lambda \in k$. Therefore, since k is infinite, $\tau_\xi^{p-i}(\tau^i_\eta(T)) = 0$ for $0 \leq i \leq p$. In particular, we see that $\tau_\xi^{p-1}(\tau_\eta(T)) = 0$ for any $\xi, \eta \in k^n$. This implies that $\mathrm{Symb}_{p-1}([T, \ell_\eta]) = 0$ for any $\eta \in k^n$, in particular

$$\mathrm{Symb}_{p-1}([T, X_i]) = 0$$

for $1 \leq i \leq n$, and by the induction assumption, $[T, X_i]$, $1 \leq i \leq n$, are of order $\leq p-2$. Let $P, Q \in k[X_1, X_2, \dots, X_n]$. Then

$$[T, PQ] = TPQ - PQT = [T, P]Q + P[T, Q],$$

hence the order of $[T, PQ]$ is less than or equal to the maximum of the orders of $[T, P]$ and $[T, Q]$. Since X_i, $1 \leq i \leq n$, generate $k[X_1, X_2, \dots, X_n]$ we conclude that the order of $[T, P]$ is $\leq p-2$ for any polynomial P. This implies that the order of T is $\leq p-1$. □

This also ends the proof of 5. In particular, we see that $D(n)$ satisfies properties (i)-(vii) from 3. From 3.4 we immediately deduce the following result.

8.8. Corollary. *The ring* $D(n)$ *is right and left nötherian.*

8.9. Corollary. $(X^I \partial^J\,;\, I, J \in \mathbb{Z}^n_+)$ *is a basis of* $D(n)$ *as a vector space over* k.

Proof. If $|J| = p$, the p-symbol of $X^I\partial^J$ is equal to $X^I\xi^J$ and $(X^I\xi^J\ ;\ I, J \in \mathbb{Z}^n_+)$ form a basis of $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ as a vector space over k. □

The following caracterization of $D(n)$ is frequently useful.

8.10. Theorem. $D(n)$ *is the* k*-algebra generated by* $X_1, X_2, \dots, X_n$ *and* $\partial_1, \partial_2, \dots, \partial_n$ *satisfying the defining relations* $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ *and* $[\partial_i, X_j] = \delta_{ij}$ *for all* $1 \leq i, j \leq n$.

Proof. Let B be the k-algebra generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ satisfying the defining relations $[X_i, X_j] = 0$, $[\partial_i, \partial_j] = 0$ and $[\partial_i, X_j] = \delta_{ij}$ for all $1 \leq i, j \leq n$. Since these relations hold in $D(n)$ and it is generated by $X_1, X_2, \dots, X_n$ and $\partial_1, \partial_2, \dots, \partial_n$ we conclude that there is a unique surjective morphism of B onto $D(n)$ which maps generators into the corresponding generators. Clearly, B is spanned by $(X^I\partial^J\,;\, I, J \in \mathbb{Z}^n_+)$. Therefore, by 9, this morphism is also injective. □

8.11. Proposition. *The center of $D(n)$ is equal to $k \cdot 1$.*

Proof. Let T be a central element of $D(n)$. Then, $[T, P] = 0$ for any polynomial P, and T is of order ≤ 0. Therefore, by 2, $T \in k[X_1, X_2, \ldots, X_n]$. On the other hand, $0 = [\partial_i, T] = \partial_i(T)$ for $1 \leq i \leq n$. This implies that T is a constant polynomial. $\square$

Let $D(n)^\circ$ be the opposite algebra of $D(n)$. Then, by 10, there exists a unique isomorphism $\phi : D(n)^\circ \longrightarrow D(n)$ which is defined by $\phi(X_i) = X_i$ and $\phi(\partial_i) = -\partial_i$ for $1 \leq i \leq n$. The morphism ϕ is called the *principal antiautomorphism* of $D(n)$. This proves the following result.

8.12. Proposition. *The algebra $D(n)^\circ$ is isomorphic to $D(n)$.*

Moreover, by 10, we can define an automorphism $\mathcal{F}$ of $D(n)$ by $\mathcal{F}(X_i) = \partial_i$ and $\mathcal{F}(\partial_i) = -X_i$ for $1 \leq i \leq n$. This automorphism is called the *Fourier automorphism* of $D(n)$. It satisfies $\mathcal{F}^2 = -1$.

In contrast to the filtration by the order of differential operators, $D(n)$ has another filtration compatible with its ring structure which is not defined on more general rings of differential operators. We put

$$D_p(n) = \{\sum a_{IJ} X^I \partial^J \mid |I| + |J| \leq p\}$$

for $p \in \mathbb{Z}$. Clearly, $(D_p(n)\,;\, p \in \mathbb{Z})$ is an increasing exhaustive filtration of $D(n)$ by finite-dimensional vector spaces over k.

8.13. Lemma. *For any $p, q \in \mathbb{Z}$ we have*

(i) $D_p(n) \circ D_q(n) \subset D_{p+q}(n)$;
(ii) $[D_p(n), D_q(n)] \subset D_{p+q-2}(n)$.

Proof. By 9. and the definition of the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$, it is enough to check that

$$[\partial^I, X^J] \in D_{|I|+|J|-2}(n).$$

We prove this statement by an induction in $|I|$. If $|I| = 1$, we have $\partial^I = \partial_i$ for some $1 \leq i \leq n$ and $[\partial_i, X^J] = \partial_i(X^J) \in D_{|J|-1}(n)$. If $|I| > 1$, we can write $\partial^I = \partial^{I'} \partial_i$ for some $I' \in \mathbb{Z}_+^n$ and $1 \leq i \leq n$. This leads to

$$\begin{aligned}
[\partial^I, X^J] &= [\partial^{I'} \partial_i, X^J] = \partial^{I'} \partial_i X^J - X^J \partial^{I'} \partial_i \\
&= \partial^{I'} [\partial_i, X^J] + [\partial^{I'}, X^J] \partial_i = [\partial^{I'}, [\partial_i, X^J]] + [\partial_i, X^J] \partial^{I'} + [\partial^{I'}, X^J] \partial_i,
\end{aligned}$$

hence, by the induction assumption, $[\partial^I, X^J] \in D_{|I|+|J|-2}(n)$. $\square$

This implies that $(D_p(n)\,;\, p \in \mathbb{Z})$ is a filtration compatible with the ring structure on $D(n)$. In addition, the graded ring $\operatorname{Gr} D(n)$ is a commutative k-algebra. If we define the linear map Ψ_p from $D_p(n)$ into $k[X_1, X_2, \ldots, X_n, \xi_1, \xi_2, \ldots, \xi_n]$ by

$$\Psi_p\Big(\sum_{|I|+|J| \leq p} a_{IJ} X^I \partial^J\Big) = \sum_{|I|+|J| = p} a_{IJ} X^I \xi^J$$

we see that it is a linear isomorphism of $\operatorname{Gr}_p D(n)$ into the homogeneous polynomials of degree p. Therefore, it extends to a linear isomorphism

$$\Psi : \operatorname{Gr} D(n) \longrightarrow k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n].$$

By 13. we see that this map is an isomorphism of k-algebras. Therefore, the ring $D(n)$ equipped with the filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ satisfies the properties (i)-(vii) from 3. The filtration $(D_p(n)\,;\, p \in \mathbb{Z})$ is called the *Bernstein filtration* of $D(n)$.

Evidently, the principal antiautomorphism and the Fourier automorphism of $D(n)$ preserve the Bernstein filtration.

9. Modules over rings of differential operators with polynomial coefficients

In this section we study the category of modules over the rings $D(n)$ of differential operators with polynomial coefficients. Denote by $\mathcal{M}^L(D(n))$, resp. $\mathcal{M}^R(D(n))$ the categories of left, resp. right, $D(n)$-modules. The principal antiautomorphism ϕ of $D(n)$ defines then an exact functor from the category $\mathcal{M}^R(D(n))$ into the category $\mathcal{M}^L(D(n))$ which maps the module M into its *transpose* M^t, which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \phi(T)m$ for $T \in D(n)$ and $m \in M$. An analogous functor is defined from $\mathcal{M}^L(D(n))$ into $\mathcal{M}^R(D(n))$. Clearly these functors are mutually inverse equivalencies of categories. If we denote by $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ the corresponding categories of finitely generated modules, we see that these functors also induce their equivalence. Therefore in the following we can restrict ourselves to the discussion of left modules and drop the superscript L from our notation (except in the cases when we want to stress that we deal with right modules).

First we consider $D(n)$ as a ring equipped with the Bernstein filtration. Since in this case $D_0(n) = k$ we can define the dimension of modules from $\mathcal{M}^L_{fg}(D(n))$ and $\mathcal{M}^R_{fg}(D(n))$ using the additive function $\dim_k$ on the category of finite-dimensional vector spaces over k. This dimension $d(M)$ and the corresponding multiplicity $e(M)$ of a module M are called the *Bernstein dimension* and the *Bernstein multiplicity* respectively. Since the principal antiautomorphism preserves the Bernstein filtration we see that $d(M) = d(M^t)$ for any finitely generated $D(n)$-module M.

For any finitely generated $D(n)$-module M we have an exact sequence $D(n)^p \longrightarrow M \longrightarrow 0$, hence $d(M) \leq d(D(n))$. In addition, from 8.5 we conclude the following result.

9.1. Lemma. *For any finitely generated $D(n)$-module M we have $d(M) \leq 2n$.*

The main result of the dimension theory of $D(n)$ is the following statement.

9.2. Theorem. *Let M be a finitely generated $D(n)$-module and $M \neq 0$. Then $d(M) \geq n$.*

Proof. Since M is a finitely generated $D(n)$-module, by 3.3, we can equip it with a good filtration. Also, we can clearly assume that $\mathrm{F}_n M = 0$ for $n < 0$ and $\mathrm{F}_0 M \neq 0$.

For any $p \in \mathbb{Z}_+$ we can consider the linear map $D_p(n) \longrightarrow \operatorname{Hom}_k(\mathrm{F}_p M, \mathrm{F}_{2p} M)$ which attaches to $T \in D_p(n)$ the linear map $m \longmapsto Tm$. We claim that this map is injective. For

$p \leq 0$ this is evident. Assume that it holds for $p-1$ and that $T \in D_p(n)$ satisfies $Tm = 0$ for all $m \in F_p M$. Then, for any $v \in \mathrm{F}_{p-1} M$ and $1 \leq i \leq n$ we have $X_i v \in \mathrm{F}_p M$ and $\partial_i v \in \mathrm{F}_p M$, hence

$$[X_i, T]v = X_i T v - T X_i v = 0$$

and

$$[\partial_i, T]v = \partial_i T v - T \partial_i v = 0$$

and $[X_i, T], [\partial_i, T] \in D_{p-1}(n)$ by 8.13. By the induction assumption this implies that $[X_i, T] = 0$ and $[\partial_i, T] = 0$ for $1 \leq i \leq n$, and T is in the center of $D(n)$. Since the center of $D(n)$ is equal to k by 8.11, we conclude that $T = 0$. Therefore,

$$\dim_k(D_p(n)) \leq \dim_k(\mathrm{Hom}_k(\mathrm{F}_p M, \mathrm{F}_{2p} M)) = \dim_k(\mathrm{F}_p M) \cdot \dim_k(\mathrm{F}_{2p} M)$$

for any $p \in \mathbb{Z}$. On the other hand, for large $p \in \mathbb{Z}_+$ the left side is equal to a polynomial in p of degree $2n$ with positive leading coefficient and the right side is equal to a polynomial in p of degree $2d(M)$ with positive leading coefficient. This is possible only if $d(M) \geq n$. □

We say that a finitely generated $D(n)$-module is *holonomic* if $d(M) \leq n$. In other words, M is holonomic if and only if either $M = 0$ or $d(M) = n$.

9.3. Remark. There exist nonzero holonomic $D(n)$-modules. For example, the ring $k[X_1, X_2, \dots, X_n]$ has a natural structure of a finitely generated $D(n)$-module, and its filtration by degree of polynomials is a good filtration. It follows that $d(k[X_1, X_2, \dots, X_n]) = n$, and it is a holonomic $D(n)$-module.

Now we want to study some homological properties of the ring $D(n)$. Since the graded ring $\mathrm{Gr}\, D(n)$ with respect to the Bernstein filtration is equal to $k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ we conclude from 5.13 and 7.8 that $\mathrm{hd}(D(n)) \leq \mathrm{hd}(\mathrm{Gr}\, D(n)) = 2n$. In particular, we see that the homological dimension of $D(n)$ is finite.

To get a more precise result we need to study the Bernstein dimension of finitely generated right $D(n)$-modules $\mathrm{Ext}^j_{D(n)}(M, D(n))$, $0 \leq j \leq 2n$, for a finitely generated left $D(n)$-module M. First, from 5.18 we know that if $M \neq 0$, there exists at least one j such that $\mathrm{Ext}^j_{D(n)}(M, D(n)) \neq 0$. Therefore, we can define as usual

$$j(M) = \min\{j \in \mathbb{Z}_+ \mid \mathrm{Ext}^j_{D(n)}(M, D(n)) \neq 0\}.$$

The following result gives a homological interpretation of the Bernstein dimension.

9.4. Theorem. *Let M be a finitely generated left $D(n)$-module. Then*

(i) $\mathrm{Ext}^j_{D(n)}(M, D(n)) = 0$ *for* $j < 2n - d(M)$;

(ii) $d(\mathrm{Ext}^j_{D(n)}(M, D(n))) \leq 2n - j$ *for all* $0 \leq j \leq 2n$;

(iii) $d(\mathrm{Ext}^{2n-d(M)}_{D(n)}(M, D(n))) = d(M)$.

In particular, if $M \neq 0$, $\operatorname{Ext}^{2n-d(M)}_{D(n)}(M, D(n)) \neq 0$ *, i.e.,*

$$d(M) + j(M) = 2n.$$

Proof. To prove this we use the fact that $\operatorname{Gr} D(n) = k[X_1, X_2, \dots, X_n, \xi_1, \xi_2, \dots, \xi_n]$ and the results from the end of 7. There we constructed, for a finitely generated left $D(n)$-module M with a good filtration $\mathrm{F}\, M$, a spectral sequence with E_1-term given by $E_1^{pq} = \operatorname{Gr}_p \operatorname{Ext}^{p+q}_{\operatorname{Gr} D(n)}(\operatorname{Gr} M, \operatorname{Gr} D(n))$ which converges to $\operatorname{Ext}^{p+q}_{D(n)}(M, D(n))$. More precisely, if we equip $\operatorname{Ext}^j_{D(n)}(M, D(n))$ with good filtrations induced by the good filtrations on the modules in the complex $K^\cdot$, we have $E_\infty^{pq} = \operatorname{Gr}_p \operatorname{Ext}^{p+q}_{D(n)}(M, D(n))$. By 6.3.(i) we know that $\operatorname{Ext}^j_{\operatorname{Gr} D(n)}(\operatorname{Gr} M, \operatorname{Gr} D(n)) = 0$ for $j < 2n - d(\operatorname{Gr} M) = 2n - d(M)$. Therefore, $E_1^{pq} = 0$ for $p + q < 2n - d(M)$ and $E_\infty^{pq} = 0$ for $p + q < 2n - d(M)$. This implies (i).

Since $\oplus_{p+q=j} E_r^{pq}$ are finitely generated $\operatorname{Gr} D(n)$-modules, and $\oplus_{p+q=j} E_{r+1}^{pq}$ is a quotient of a submodule of $\oplus_{p+q=j} E_r^{pq}$, by 3.8 we conclude that $d(\oplus_{p+q=j} E_{r+1}^{pq}) \leq d(\oplus_{p+q=j} E_r^{pq})$ for any $r \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$. This implies in particular that

$$\begin{aligned} d(\operatorname{Ext}^j_{\operatorname{Gr} D(n)}(\operatorname{Gr} M, \operatorname{Gr} D(n))) &= d(\oplus_{p+q=j} E_1^{pq}) \geq d(\oplus_{p+q=j} E_\infty^{pq}) \\ &= d(\operatorname{Gr} \operatorname{Ext}^j_{D(n)}(M, D(n))) = d(\operatorname{Ext}^j_{D(n)}(M, D(n))) \end{aligned}$$

for any $j \in \mathbb{Z}_+$. Therefore, by 6.3.(ii), we see that (ii) holds.

Moreover, the differentials d_r induce morphisms $d_r : \oplus_{p+q=j} E_r^{pq} \longrightarrow \oplus_{p+q=j+1} E_r^{pq}$ of $\operatorname{Gr} D$-modules for any $j \in \mathbb{Z}$. If $j = 2n - d(M)$, the differential $d_r : \oplus_{p+q=j-1} E_r^{pq} \longrightarrow \oplus_{p+q=j} E_r^{pq}$ is always zero since, by the preceding discussion, $\oplus_{p+q=j-1} E_r^{pq} = 0$ for all $r \in \mathbb{N}$. Therefore, $E_{r+1}^{pq} \subset E_r^{pq}$ for any $p + q = j$ and $r \in \mathbb{N}$. On the other hand, we also remarked that $d(\oplus_{p+q=j+1} E_r^{pq}) \leq 2n - j - 1$ for any $r \in \mathbb{N}$. Since, by 6.3.(iii),

$$d(\oplus_{p+q=j} E_1^{pq}) = d(\operatorname{Ext}^j_{\operatorname{Gr} D(n)}(\operatorname{Gr} M, \operatorname{Gr} D(n))) = d(\operatorname{Gr} M) = d(M) = 2n - j,$$

we see by induction in r that $d(\oplus_{p+q=j} E_r^{pq}) = 2n - j$ for any $r \in \mathbb{N}$. This implies that $d(\operatorname{Ext}^j_{D(n)}(M, D(n))) = 2n - j = d(M)$ and proves (iii). □

9.5. Corollary. *Let M be a finitely generated left $D(n)$-module. Then*

(i) $\operatorname{Ext}^j_{D(n)}(M, D(n)) = 0$ *for* $j > n$.
(ii) $\operatorname{Ext}^n_{D(n)}(M, D(n))$ *is a holonomic module.*

Proof. (i) By 4. we know that $d(\operatorname{Ext}^j_{D(n)}(M, D(n))) \leq 2n - j < n$ for $j > n$. On the other hand, 2. then implies that $\operatorname{Ext}^j_{D(n)}(M, D(n)) = 0$ for $j > n$.

(ii) By 4, $d(\operatorname{Ext}^n_{D(n)}(M, D(n))) \leq 2n - n = n$. □

9.6. Corollary. *Let $M \neq 0$ be a finitely generated left $D(n)$-module. Then the following conditions are equivalent:*

(i) *M is holonomic;*

(ii) $\operatorname{Ext}^j_{D(n)}(M, D(n)) = 0$ *for $j \neq n$.*

Proof. The condition (ii) is, by 5, equivalent to $j(M) = n$. The statement follows immediately from 4. $\square$

9.7. Theorem. $\operatorname{hd}(D(n)) = n$.

Proof. From 5. and 5.15 we conclude that $\operatorname{hd}(D(n)) \leq n$. In addition, from 3. we know that there exists a nonzero holonomic left $D(n)$-module M, for which by 6. we have $\operatorname{Ext}^n_{D(n)}(M, D(n)) \neq 0$. $\square$

The next result will be useful later. It shows that equivalencies of categories preserve the dimension function on $\mathcal{M}_{fg}(D(n))$.

9.8. Proposition. *Let $\Psi : \mathcal{M}_{fg}(D(n)) \to \mathcal{M}_{fg}(D(n))$ be an equivalence of additive categories. Then $d(\Psi(M)) = d(M)$ for any $M \in \mathcal{M}_{fg}(D(n))$.*

Proof. If Ψ is an equivalence of categories, $M \neq 0$ is equivalent to $\Psi(M) \neq 0$. Therefore, we can consider $j(M)$ and $j(\Psi(M))$, and by 4.(iii) it is enough to show that $j(M) = j(\Psi(M))$.

First we claim that the projective objects in $\mathcal{M}_{fg}(D(n))$ are exactly the finitely generated projective $D(n)$-modules. Clearly, any finitely generated projective $D(n)$-module is a projective object in $\mathcal{M}_{fg}(D(n))$. On the other hand, a projective object in $\mathcal{M}_{fg}(D(n))$ is a direct summand of a free $D(n)$-module of finite rank, hence is a projective $D(n)$-module.

Therefore, for any $M \in \mathcal{M}_{fg}(D(n))$,

$$\begin{aligned}
j(M) &= \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(M, D(n)) \neq 0\} \\
&= \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(M, F) \neq 0,\ F \text{ free } D(n)\text{-module of finite rank}\} \\
&= \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(M, P) \neq 0,\ P \text{ finitely generated projective } D(n)\text{-module}\}.
\end{aligned}$$

Let $P.$ be a left resolution of M by finitely generated projective $D(n)$-modules. Then, since Ψ preserves projective objects in $\mathcal{M}_{fg}(D(n))$, $\Psi(P.)$ is a left resolution of $\Psi(M)$ by finitely generated projective $D(n)$-modules. This implies that for any $D(n)$-module N,

$$\begin{aligned}
\operatorname{Ext}^j_{D(n)}(\Psi(M), \Psi(N)) &= H^j(\operatorname{Hom}_{D(n)}(\Psi(P.), \Psi(N))) \\
&\cong H^j(\operatorname{Hom}_{D(n)}(P., N)) = \operatorname{Ext}^j_{D(n)}(M, N)
\end{aligned}$$

for any $j \in \mathbb{Z}_+$. In particular,

$$\begin{aligned}
j(M) &= \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(M, P) \neq 0,\ P \text{ finitely generated projective } D(n)\text{-module}\} \\
&= \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(\Psi(M), \Psi(P)) \neq 0,\ P \text{ finitely generated projective } D(n)\text{-module}\} \\
&\geq \min\{j \in \mathbb{Z}_+ \mid \operatorname{Ext}^j_{D(n)}(\Psi(M), Q) \neq 0,\ Q \text{ finitely generated projective } D(n)\text{-module}\} \\
&= j(\Psi(M)).
\end{aligned}$$

Since Ψ is an equivalence of categories, we analogously conclude that $j(M) \leq j(\Psi(M))$. $\square$

Now we want to prove some basic properties of holonomic modules.

9.9. THEOREM.

(i) *Holonomic modules are of finite length.*
(ii) *Submodules, quotient modules and extensions of holonomic modules are holonomic.*

PROOF. (ii) follows immediately from 3.8.

(i) Let M be a holonomic $D(n)$-module different from zero. Then, by definition, $d(M) = n$ and $e(M) \in \mathbb{N}$. Since M is finitely generated and $D(n)$ is a nötherian ring, there exists a maximal $D(n)$-submodule M' of M different from M. Therefore we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0.$$

By (ii) M' and M/M' are holonomic and M/M' is an irreducible $D(n)$-module. If $M' \neq 0$, we conclude from 3.8 that $e(M') < e(M)$. Therefore, by induction in $e(M)$, it follows that M has finite length. $\square$

Therefore, if we denote by $\mathcal{M}_{fl}(D(n))$, resp. $\mathcal{H}ol(D(n))$, the full subcategories of $\mathcal{M}_{fg}(D(n))$ consisting of $D(n)$-modules of finite length, resp. holonomic $D(n)$-modules, we see that $\mathcal{H}ol(D(n))$ is a full subcategory of $\mathcal{M}_{fl}(D(n))$. One can show that $\mathcal{H}ol(D(n))$ is strictly smaller than $\mathcal{M}_{fl}(D(n))$ for $n > 1$.

From 6. and the long exact sequence of $\operatorname{Ext}^{\cdot}_{D(n)}(-, D(n))$ we see that the map $M \longrightarrow M^* = \operatorname{Ext}^n_{D(n)}(M, D(n))$ is an exact contravariant functor from the category $\mathcal{H}ol^L(D(n))$ into the category $\mathcal{H}ol^R(D(n))$. By abuse of notation denote by the same symbol the analogous functor from $\mathcal{H}ol^R(D(n))$ into $\mathcal{H}ol^L(D(n))$. Therefore, $M \longrightarrow M^{**}$ is an exact covariant functor from $\mathcal{H}ol^L(D(n))$ into itself.

9.10. THEOREM. *The functor $M \longrightarrow M^{**}$ is isomorphic to the identity functor on $\mathcal{H}ol^L(D(n))$.*

PROOF. If F is a free left $D(n)$-module of finite rank, $\operatorname{Hom}_{D(n)}(F, D(n))$ is a free right $D(n)$-module of finite rank. Moreover, since any finitely generated projective left $D(n)$-module P is a direct summand of a free left $D(n)$-module of finite rank, $\operatorname{Hom}_{D(n)}(P, D(n))$ is a finitely generated projective right $D(n)$-module.

For any left $D(n)$-module M we have the natural morphism

$$D_M : M \longrightarrow \operatorname{Hom}_{D(n)}(\operatorname{Hom}_{D(n)}(M, D(n)), D(n))$$

defined by $D_M(m)(\phi) = \phi(m)$ for $\phi \in \operatorname{Hom}_{D(n)}(M, D(n))$, $m \in M$. Clearly, it induces an isomorphism for any free $D(n)$-module F of finite rank. Using again the fact that any finitely generated projective $D(n)$-module P is a direct summand of a free $D(n)$-module of finite rank, we see that $D_P : P \longrightarrow \operatorname{Hom}_{D(n)}(\operatorname{Hom}_{D(n)}(P, D(n)), D(n))$ is an isomorphism.

Let M be a finitely generated $D(n)$-module. By 7. there exists a left resolution $P.$ of length n of M by finitely generated projective left $D(n)$-modules. Therefore, the complex $C^{\cdot} = \operatorname{Hom}_{D(n)}(P., D(n))$ satisfies the following properties:

(i) C^j are finitely generated projective right $D(n)$-modules,
(ii) $C^j = 0$ for $j < 0$ and $j > n$,
(iii) $H^j(C^{\cdot}) = \operatorname{Ext}^j_{D(n)}(M, D(n))$ for any $0 \leq j \leq n$.

By 6. we know that $H^j(C^{\cdot}) = 0$ for $j \neq n$ and $H^n(C^{\cdot}) = \operatorname{Ext}^n_{D(n)}(M, D(n))$. Therefore, $C^{\cdot}$ is a left resolution of M^* shifted to the right by n. Applying the same argument again, we see that the complex $D^{\cdot} = \operatorname{Hom}_{D(n)}(C^{\cdot}, D(n))$ is a left resolution of M^{**}. On the other hand, D induces an isomorphism of this complex with $P.$. Hence, $M^{**} \cong M$. □

Finally, if M is a $D(n)$-module, we can define its *Fourier transform* $\mathcal{F}(M)$ as the module which is equal to M as additive group and the action of $D(n)$ is given by the map $(T, m) \longmapsto \mathcal{F}(T)m$ for $T \in D(n)$ and $m \in M$. Clearly the Fourier transform is an autoequivalence of the category $\mathcal{M}(D(n))$. It also induces an autoequivalence of the category $\mathcal{M}_{fg}(D(n))$. From the fact that the Fourier automorphism $\mathcal{F}$ preserves the Bernstein filtration (or from 3.9) we conclude the following result.

9.11. Lemma. *Let M be a finitely generated $D(n)$-module. Then $d(\mathcal{F}(M)) = d(M)$.*

In particular, Fourier transform preserves holonomic modules.

10. Characteristic variety

Now we want to study properties of $D(n)$-modules in more geometric terms. In particular, we want to consider the filtration $\mathrm{F}\,D(n)$ of $D(n)$ by the degree of differential operators instead of the Bernstein filtration, since the first one makes sense for rings of differential operators on smooth affine varieties.

First, since any $D(n)$-module M can be viewed as a $k[X_1, X_2, \dots, X_n]$-module, we can consider its support $\operatorname{supp}(M) \subset k^n$.

10.1. Proposition. *Let M be a finitely generated $D(n)$-module. Then* $\operatorname{supp}(M)$ *is a closed subvariety of k^n.*

Proof. Fix a good filtration $\mathrm{F}\,M$ on M. Then, for $x \in k^n$, $M_x = 0$ is equivalent to $(\mathrm{F}_p\,M)_x = 0$ for all $p \in \mathbb{Z}$. Therefore, by the exactness of localization, it is equivalent to $(\operatorname{Gr} M)_x = 0$. Let I_p be the annihilator of the $k[X_1, X_2, \dots, X_n]$-module $\operatorname{Gr}_p M$, $p \in \mathbb{Z}$. Since $\operatorname{Gr}_p M$ are finitely generated $k[X_1, X_2, \dots, X_n]$-modules, by 4.2 their supports $\operatorname{supp}(\operatorname{Gr}_p M)$ are equal to $V(I_p)$. This implies that $\operatorname{supp}(M) = \cup_{p\in\mathbb{Z}} V(I_p)$. Let $m_1, m_2, \dots, m_s$ be a set of homogeneous generators of $\operatorname{Gr} D(n)$-module $\operatorname{Gr} M$. Then the annihilator I of $m_1, m_2, \dots, m_s$ in $k[X_1, X_2, \dots, X_n]$ annihilates whole $\operatorname{Gr} M$. Therefore, there is a finite subset $\mathcal{S}$ of $\mathbb{Z}$ such that $\cap_{p\in\mathcal{S}} I_p = I \subset I_q$ for all $q \in \mathbb{Z}$. This implies that $\cup_{p\in\mathcal{S}} V(I_p) = V(I) \supset V(I_q)$ for all $q \in \mathbb{Z}$, and $\operatorname{supp}(M) = V(I)$. □

Let D be a filtered ring with a filtration $\mathrm{F}\,D$ satisfying the properties (i)-(vii) from the beginning of 3. Let M be a finitely generated D-module and $\mathrm{F}\,M$ a good filtration of M.

Then $\operatorname{Gr} M$ is a graded $\operatorname{Gr} D$-module. Let I be the annihilator of $\operatorname{Gr} M$ in $\operatorname{Gr} D$. This is clearly a graded ideal in $\operatorname{Gr} D$. Hence, its radical $r(I)$ is also a graded ideal. In general, I depends on the choice of the good filtration on M, but we also have the following result.

10.2. Lemma. *Let M be a finitely generated D-module and $\mathrm{F}\, M$ and $\mathrm{F}'\, M$ two good filtrations on M. Let I, resp. I' be the annihilators of the corresponding graded $\operatorname{Gr} D$-modules $\operatorname{Gr} M$ and $\operatorname{Gr}' M$. Then $r(I) = r(I')$.*

Proof. Let $T \in r(I) \cap \operatorname{Gr}_p D$. Then there exists $s \in \mathbb{Z}_+$ such that $T^s \in I$. If we take $Y \in \mathrm{F}_p\, D$ such that $Y + \mathrm{F}_{p-1}\, D = T$, we get $Y^s\, \mathrm{F}_q\, M \subset \mathrm{F}_{q+sp-1}\, M$ for all $q \in \mathbb{Z}$. Hence, by induction we get

$$Y^{ms}\, \mathrm{F}_q\, M \subset \mathrm{F}_{q+msp-m}\, M$$

for all $m \in \mathbb{N}$ and $q \in \mathbb{Z}$. On the other hand, by 3.6, we know that $\mathrm{F}\, M$ and $\mathrm{F}'\, M$ are equivalent. Hence there exists $l \in \mathbb{Z}_+$ such that $\mathrm{F}_q\, M \subset \mathrm{F}'_{q+l}\, M \subset \mathrm{F}_{q+2l}\, M$ for all $q \in \mathbb{Z}$. This leads to

$$Y^{ms}\, \mathrm{F}'_q\, M \subset Y^{ms}\, \mathrm{F}_{q+l}\, M \subset \mathrm{F}_{q+l+msp-m}\, M \subset \mathrm{F}'_{q+2l+msp-m}\, M$$

for all $q \in \mathbb{Z}$ and $m \in \mathbb{N}$. If we take $m > 2l$, it follows that $Y^{ms}\, \mathrm{F}'_q\, M \subset \mathrm{F}'_{q+msp-1}\, M$ for any $q \in \mathbb{Z}$, i. e. $T^{ms} \in I'$. Therefore, $T \in r(I')$ and we have $r(I) \subset r(I')$. Since the roles of I and I' are symmetric we conclude that $r(I) = r(I')$. $\square$

Therefore the radical of the annihilator of $\operatorname{Gr} M$ is independent of the choice of a good filtration on $\operatorname{Gr} M$. We call it the *characteristic ideal* of M and denote by $J(M)$.

Now we can apply this construction to $D(n)$. Since $\operatorname{Gr} D(n) = k[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$ by 8.5, we can define the closed algebraic set

$$Ch(M) = V(J(M)) \subset k^{2n}$$

which we call the *characteristic variety* of M.

Since $J(M)$ is a homogeneous ideal in last n variables, we immediately conclude the following result.

10.3. Lemma. *The characteristic variety $Ch(M)$ of a finitely generated $D(n)$-module M has the following property: if $(x, \xi) \in Ch(M)$ then $(x, \lambda\xi) \in Ch(M)$ for any $\lambda \in k$.*

We say that $Ch(M)$ is a *conical* variety.

10.4. Proposition. *Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated $D(n)$-modules. Then

$$Ch(M) = Ch(M') \cup Ch(M'').$$

PROOF. Let $\mathrm{F}\, M$ be a good filtration on M. Then it induces a filtration $\mathrm{F}\, M'$ on M' and $\mathrm{F}\, M''$ on M''. By 3.7 we know that these filtrations are also good. Moreover, we have the exact sequence

$$0 \longrightarrow \mathrm{Gr}\, M' \longrightarrow \mathrm{Gr}\, M \longrightarrow \mathrm{Gr}\, M'' \longrightarrow 0$$

of finitely generated $k[X_1, \ldots, X_n, \xi_1, \ldots, \xi_n]$-modules, and their supports are, by 4.2, the characteristic varieties of $D(n)$-modules M, M' and M'' respectively. Therefore the assertion follows from 4.1. $\square$

The fundamental result about characteristic varieties is the following theorem. It also gives a geometric description of the Bernstein dimension.

10.5. THEOREM. *Let M be a finitely generated $D(n)$-module. Then*

$$\dim Ch(M) = d(M).$$

PROOF. Clearly we can assume that $M \neq 0$. The proof of 9.4 applies, word by word, to the situation where Bernstein filtration on $D(n)$ is replaced with the filtration by the order of differential operators and Bernstein dimension by the dimension of $\mathrm{Gr}\, M$ considered as a module over $\mathrm{Gr}\, D(n)$. Hence, we see that

$$d(M) + j(M) = 2n = d(\mathrm{Gr}\, M) + j(M).$$

Therefore, $d(\mathrm{Gr}\, M) = d(M)$. On the other hand, by 4.2, $\dim Ch(M) = d(\mathrm{Gr}\, M)$. $\square$

In particular, by combining 9.2 and 5, we get the following result.

10.6. THEOREM. *Let M be a finitely generated $D(n)$-module, $M \neq 0$, and $Ch(M)$ its characteristic variety. Then* $\dim Ch(M) \geq n$.

Let $\pi : k^{2n} \longrightarrow k^n$ be the map defined by $\pi(x, \xi) = x$ for any $x, \xi \in k^n$.

10.7. PROPOSITION. *Let M be a finitely generated $D(n)$-module. Then* $\mathrm{supp}(M) = \pi(Ch(M))$.

PROOF. Denote by $m_1, m_2, \ldots, m_s$ a set of homogeneous generators of $\mathrm{Gr}\, M$. Then, as in the proof of 1, the annihilator I of $m_1, m_2, \ldots, m_s$ in $k[X_1, X_2, \ldots, X_n]$ satisfies $\mathrm{supp}(M) = V(I)$. On the other hand, if J is the annihilator of $m_1, m_2, \ldots, m_s$ in $k[X_1, X_2, \ldots, X_n, \xi_1, \xi_2, \ldots, \xi_n]$, it is a homogeneous ideal in $\xi_1, \xi_2, \ldots, \xi_n$ which satisfies $I = k[X_1, X_2, \ldots, X_n] \cap J$, and $Ch(M) = V(J)$. This implies that $x \in V(I) = \mathrm{supp}(M)$ is equivalent with $(x, 0) \in V(J) = Ch(M)$. Since $Ch(M)$ is conical this implies the assertion. $\square$

Let M be a finitely generated $D(n)$-module. Define the *singular support* of M as

$$\mathrm{sing\ supp}(M) = \{x \in k^n \mid (x, \xi) \in Ch(M) \text{ for some } \xi \neq 0\}.$$

Then $\mathrm{sing\ supp}(M) \subset \mathrm{supp}(M)$.

10.8. Lemma. *Let M be a finitely generated $D(n)$-module. Then* sing supp(M) *is a closed subvariety of* supp(M).

Proof. Let $p : k^n - \{0\} \longrightarrow \mathbb{P}^{n-1}(k)$ be the natural projection. Then

$$1 \times p : k^n \times (k^n - \{0\}) \longrightarrow k^n \times \mathbb{P}^{n-1}(k)$$

projects $Ch(M) - (k^n \times \{0\})$ onto the closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$ corresponding to the ideal $J(M)$ which is homogeneous in $\xi_1, \xi_2, \dots, \xi_n$. Finally, the projection to the first factor $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ maps it onto sing supp(M). Since $\mathbb{P}^{n-1}(k)$ is a complete variety, the projection $k^n \times \mathbb{P}^{n-1}(k) \longrightarrow k^n$ is a closed map. Therefore, sing supp(M) is closed. $\square$

10.9. Corollary. *Let M be a holonomic $D(n)$-module. Then* $\dim \operatorname{sing\,supp}(M) \leq n - 1$.

Proof. We can assume that $M \neq 0$. By 5. we see that $J(M)$ defines an n-dimensional subvariety of k^{2n}. Therefore, since $J(M)$ is homogeneous in last n variables it defines an $(n-1)$-dimensional variety in $k^n \times \mathbb{P}^{n-1}(k)$ and its projection into k^n is at most $(n-1)$-dimensional. $\square$

11. Exterior tensor products

Let $X = k^n$ and $Y = k^m$ in the following, and denote by D_X and D_Y the corresponding algebras of differential operators with polynomial coefficients. Then we can consider the algebra $D_X \boxtimes D_Y$ which is equal to $D_X \otimes_k D_Y$ as a vector space over k, and the multiplication is defined by $(T \otimes S)(T' \otimes S') = TT' \otimes SS'$ for $T, T' \in D_X$ and $S, S' \in D_Y$. We call $D_X \boxtimes D_Y$ the *exterior tensor product* of D_X and D_Y.

The following result is evident.

11.1. Lemma. $D_X \boxtimes D_Y = D_{X \times Y}$.

If M and N are D_X-, resp. D_Y-modules, we can define $D_{X\times Y}$-module $M \boxtimes N$ which is equal to $M \otimes_k N$ as a vector space over k, and the action of $D_X \boxtimes D_Y = D_{X\times Y}$ is given by $(T \otimes S)(m \otimes n) = Tm \otimes Sn$ for any $T \in D_X$, $S \in D_Y$, $m \in M$ and $n \in N$.

11.2. Lemma. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $M \boxtimes N$ is a finitely generated $D_{X\times Y}$-module.*

Proof. Let $e_1, e_2, \dots, e_p$ and $f_1, f_2, \dots, f_q$ be generators of M and N respectively. Then for any $m \in M$ and $n \in N$, we have $m = \sum T_i e_i$, $T_i \in D_X$, and $n = \sum S_j f_j$, $S_j \in D_Y$. This implies that $m \otimes n = \sum\sum T_i e_i \otimes S_j f_j = \sum\sum (T_i \otimes S_j)(e_i \otimes f_j)$, and $e_i \otimes f_j$, $1 \leq i \leq p$, $1 \leq j \leq q$, generate $M \boxtimes N$. $\square$

Our main goal in this section is to prove the following result.

11.3. Theorem. *Let M be a finitely generated D_X-module and N a finitely generated D_Y-module. Then $d(M \boxtimes N) = d(M) + d(N)$.*

This result has the following important consequence.

11.4. Corollary. *Let M be a holonomic D_X-module and N a holonomic D_Y-module. Then $M \boxtimes N$ is a holonomic $D_{X\times Y}$-module.*

Let D_X and D_Y be equipped with the Bernstein filtration. Let M and N be finitely generated D_X-, resp. D_Y-modules with good filtrations $\mathrm{F}\,M$ and $\mathrm{F}\,N$ respectively. Define the *product filtration* on $M \boxtimes N$ by

$$\mathrm{F}_j(M \boxtimes N) = \sum_{p+q=j} \mathrm{F}_p\,M \otimes_k \mathrm{F}_q\,N$$

for any $j \in \mathbb{Z}$. Clearly the product filtration on $D_X \boxtimes D_Y = D_{X\times Y}$ agrees with the Bernstein filtration. Therefore, $\mathrm{F}(M \boxtimes N)$ is an exhaustive hausdorff $D_{X\times Y}$-module filtration.

Now we want to describe $\mathrm{Gr}(M \boxtimes N)$. Let $j \in \mathbb{Z}$. If $p+q=j$ we have a well-defined k-linear map $\mathrm{F}_p\,M \otimes_k \mathrm{F}_q\,N \longrightarrow \mathrm{Gr}_p\,M \otimes_k \mathrm{Gr}_q\,N$ with kernel $\mathrm{F}_{p-1}\,M \otimes_k \mathrm{F}_q\,N + \mathrm{F}_p\,M \otimes_k \mathrm{F}_{q-1}\,N$. On the other hand,

$$\mathrm{F}_{p-1}\,M \otimes_k \mathrm{F}_q\,N + \mathrm{F}_p\,M \otimes_k \mathrm{F}_{q-1}\,N = (\mathrm{F}_p\,M \otimes_k \mathrm{F}_q\,N) \cap \left(\sum_{p'+q'=j,\ p'\neq p,\ q'\neq q} \mathrm{F}_{p'}\,M \otimes_k \mathrm{F}_{q'}\,N \right),$$

hence the natural linear surjection of $\oplus_{p+q=j} \mathrm{F}_p\,M \otimes_k \mathrm{F}_q\,N \longrightarrow \oplus_{p+q=j} \mathrm{Gr}_p\,M \otimes_k \mathrm{Gr}_q\,N$ factors through $\mathrm{F}_j(M \boxtimes N)$. The kernel of the linear map

$$\mathrm{F}_j(M \boxtimes N) \longrightarrow \oplus_{p+q=j} \mathrm{Gr}_p\,M \otimes_k \mathrm{Gr}_q\,N$$

is equal to the image of $\oplus_{p+q=j}(\mathrm{F}_{p-1}\,M \otimes_k \mathrm{F}_q\,N + \mathrm{F}_p\,M \otimes_k \mathrm{F}_{q-1}\,N)$ in $\mathrm{F}_j(M \boxtimes N)$ which is equal to $\mathrm{F}_{j-1}(M \boxtimes N)$. This implies that $\mathrm{Gr}_j(M \boxtimes N) = \oplus_{p+q=j} \mathrm{Gr}_p\,M \otimes_k \mathrm{Gr}_q\,N$ for any $j \in \mathbb{Z}$.

If we define analogously the algebra $\mathrm{Gr}\,D_X \boxtimes \mathrm{Gr}\,D_Y$ with grading given by the total degree, we see that $\mathrm{Gr}\,D_X \boxtimes \mathrm{Gr}\,D_Y = \mathrm{Gr}\,D_{X\times Y}$. In addition, $\mathrm{Gr}\,M \boxtimes \mathrm{Gr}\,N$ becomes a graded $\mathrm{Gr}\,D_{X\times Y}$-module isomorphic to $\mathrm{Gr}(M \boxtimes N)$ by the preceding discussion. Since the filtrations $\mathrm{F}\,M$ and $\mathrm{F}\,N$ are good, $\mathrm{Gr}\,M$ and $\mathrm{Gr}\,N$ are finitely generated $\mathrm{Gr}\,D_X$-, resp. $\mathrm{Gr}\,D_Y$-modules by 3.1. By an analogue of 2, $\mathrm{Gr}(M \boxtimes N)$ is a finitely generated $\mathrm{Gr}\,D_{X\times Y}$-module. This implies that the product filtration is a good filtration on $M \boxtimes N$.

Let

$$P(M,t) = \sum_{p\in\mathbb{Z}} \dim_k(\mathrm{Gr}_p\,M)\,t^p$$

and

$$P(N,t) = \sum_{q\in\mathbb{Z}} \dim_k(\mathrm{Gr}_q\,N)\,t^q$$

be the Poincaré series of $\operatorname{Gr} M$ and $\operatorname{Gr} N$. Then

$$\begin{aligned}
P(M,t)\,P(N,t) &= \sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} \dim_k(\operatorname{Gr}_p M)\dim_k(\operatorname{Gr}_q N)\,t^{p+q} \\
&= \sum_{j\in\mathbb{Z}}\left(\sum_{p+q=j}\dim_k(\operatorname{Gr}_p M)\ \dim_k(\operatorname{Gr}_q N)\right)\,t^j \\
&= \sum_{j\in\mathbb{Z}}\left(\sum_{p+q=j}\dim_k(\operatorname{Gr}_p M\otimes_k \operatorname{Gr}_q N)\right)\,t^j \\
&= \sum_{j\in\mathbb{Z}}\dim_k \operatorname{Gr}_j(M\boxtimes N)\,t^j = P(M\boxtimes N,t)
\end{aligned}$$

is the Poincaré series of $M\boxtimes N$. Therefore, the order of the pole at 1 of $P(M\boxtimes N,t)$ is the sum of the orders of poles of $P(M,t)$ and $P(N,t)$. From 1.5, we see that this immediately implies 3.

12. Inverse and direct images

Let $X=k^n$ and $Y=k^m$ and denote by $x_1,x_2,\dots,x_n$ and $y_1,y_2,\dots,y_m$ the canonical coordinate functions on X and Y respectively. Let $R(X)=k[x_1,x_2,\dots,x_n]$ and $R(Y)=k[y_1,y_2,\dots,y_m]$ denote the rings of regular functions on X and Y respectively.

Let $F:X\longrightarrow Y$ be a polynomial map, i.e.,

$$F(x_1,x_2,\dots,x_n)=(F_1(x_1,x_2,\dots,x_n),F_2(x_1,x_2,\dots,x_n),\dots,F_m(x_1,x_2,\dots,x_n))$$

with $F_i\in R(X)$. Then F defines a ring homomorphism $\phi_F:R(Y)\longrightarrow R(X)$ by $\phi_F(P)=P\circ F$ for $P\in R(Y)$. Therefore we can view $R(X)$ as an $R(Y)$-module. Hence, we can define functor F^* from the category $\mathcal{M}(R(Y))$ of $R(Y)$-modules into the category $\mathcal{M}(R(X))$ of $R(X)$-modules given by the following formula

$$F^*(M)=R(X)\otimes_{R(Y)}M$$

for any $R(Y)$-module M. Clearly $F^*:\mathcal{M}(R(Y))\longrightarrow\mathcal{M}(R(X))$ is a right exact functor. We call it the *inverse image* functor.

Denote now by D_X and D_Y the algebras of differential operators with polynomial coefficients on X and Y respectively. If M is a left D_Y-module, we want to define a D_X-module structure on the inverse image $F^*(M)$. (As we remarked at the beginning of 9, the transposition functor is an equivalence of the category of left modules with the category of right modules, hence we can analogously treat right modules.) First we consider the bilinear map

$$(P,v)\longmapsto \frac{\partial P}{\partial x_i}\otimes v+\sum_{j=1}^{n}P\frac{\partial F_j}{\partial x_i}\otimes\frac{\partial}{\partial y_j}v,$$

from $R(X) \times M$ into $R(X) \otimes_{R(Y)} M$. Since

$$\begin{aligned}
&\frac{\partial P(Q\circ F)}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q\circ F)\frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P\left(\frac{\partial Q}{\partial y_j}\circ F\right)\frac{\partial F_j}{\partial x_i} \otimes v + \sum_{j=1}^{n} P(Q\circ F)\frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} v \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P\frac{\partial F_j}{\partial x_i} \otimes (\frac{\partial Q}{\partial y_j} v + Q\frac{\partial}{\partial y_j} v) \\
&\quad = \frac{\partial P}{\partial x_i} \otimes Qv + \sum_{j=1}^{n} P\frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j} (Qv)
\end{aligned}$$

for any $Q \in R(Y)$, this map factors through a linear endomorphism of $F^*(M)$ which we denote by $\frac{\partial}{\partial x_i}$. By direct calculation we get

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right](P \otimes v) = 0$$

and

$$\left[\frac{\partial}{\partial x_i}, x_j\right](P \otimes v) = \delta_{ij}(P \otimes v),$$

hence, by 8.10, we see that $F^*(M)$ has a natural structure of a left D_X-module.

This structure can be described in another way. Let

$$D_{X\to Y} = F^*(D_Y) = R(X) \otimes_{R(Y)} D_Y.$$

Then, as we just described, $D_{X\to Y}$ has the structure of a left D_X-module. But it also has a structure of a right D_Y-module given by the right multiplication on D_Y. These two actions clearly commute, hence $D_{X\to Y}$ is a (left D_X, right D_Y)-bimodule. Moreover, for any D_Y-module M we have

$$F^*(M) = R(X) \otimes_{R(Y)} M = (R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} M = D_{X\to Y} \otimes_{D_Y} M$$

and the action of D_X on $F^*(M)$ is given by the action on the first factor in the last expression. From this relation it is evident that the inverse image functor is a right exact functor from $\mathcal{M}^L(D_Y)$ into $\mathcal{M}^L(D_X)$. Since $\mathrm{hd}(D_Y) = m$ by 9.7, the left cohomological dimension of F^* is $\leq m$. Therefore, the left derived functors $L^i F^*$ of F^*, given by

$$L^{-j} F^*(M) = \mathrm{Tor}_j^{D_Y}(D_{X\to Y}, M)$$

for a left D_Y-module M, vanish for $j > m$.

12.1. Lemma. *Let P be a projective left D_Y-module. Then $F^*(P)$ is a projective $R(X)$-module.*

Proof. Let P be a projective D_Y-module. Then it is a direct summand of a free D_Y-module $(D_Y)^{(I)}$. This implies that $F^*(P)$ is a direct summand of $F^*(D_Y^{(I)})$. Since D_Y is a free $R(Y)$-module, $F^*(D_Y^{(I)})$ is a free $R(X)$-module. □

12.2. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the the inverse image functor $(G \circ F)^*$ from $\mathcal{M}^L(D_Z)$ into $\mathcal{M}^L(D_X)$ is isomorphic to $F^* \circ G^*$;*

(ii) *for any left D_Z-module M there exist a spectral sequence with E_2-term $E_2^{pq} = L^p F^*(L^q G^*(M))$ which converges to $L^{p+q}(G \circ F)^*(M)$.*

Proof. (i) We consider first the polynomial ring structures. In this case

$$(G \circ F)^*(M) = R(X) \otimes_{R(Z)} M = R(X) \otimes_{R(Y)} (R(Y) \otimes_{R(Z)} M) = F^*(G^*(M))$$

for any D_Z-module M.

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial x_i}(P \otimes v) &= \frac{\partial}{\partial x_i}(P \otimes (1 \otimes v)) \\
&= \frac{\partial P}{\partial x_i} \otimes (1 \otimes v) + \sum_{j=1}^{m} P \frac{\partial F_j}{\partial x_i} \otimes \frac{\partial}{\partial y_j}(1 \otimes v) \\
&= \frac{\partial P}{\partial x_i} \otimes v + \sum_{j=1}^{m} P \frac{\partial F_j}{\partial x_i} \otimes \left(\sum_{k=1}^{p} \frac{\partial G_k}{\partial y_j} \otimes \frac{\partial}{\partial z_k} v \right) \\
&= \frac{\partial P}{\partial x_i} \otimes v + \sum_{k=1}^{p} P \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i} \left(\frac{\partial G_k}{\partial y_j} \circ F \right) \otimes \frac{\partial}{\partial z_k} v \\
&= \frac{\partial P}{\partial x_i} \otimes v + \sum_{k=1}^{p} P \frac{\partial (G_k \circ F)}{\partial x_i} \otimes \frac{\partial}{\partial z_k} v
\end{aligned}$$

for any $P \in R(X)$ and $v \in M$. Hence the D_X-actions agree.

(ii) By 1, for any projective D_Z-module P, the inverse image $G^*(P)$ is F^*-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

Now we consider an especially simple example. Let p be the projection of $X \times Y$ to the second factor. Then

$$p^*(M) = R(X \times Y) \otimes_{R(Y)} M = (R(X) \boxtimes R(Y)) \otimes_{R(Y)} M = R(X) \boxtimes M$$

as a module over $R(X \times Y) = R(X) \boxtimes R(Y)$. On the other hand, from the definition it follows immediately that the actions $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_j}$ also agree, i.e., $p^*(M) = R(X) \boxtimes M$. From 11.3 and 11.4 we immediately get the following result.

12.3. PROPOSITION. *Let $p : X \times Y \longrightarrow Y$ be the projection defined by $p(x, y) = y$ for $x \in X$, $y \in Y$. Then,*

(i) *p^* is an exact functor from $\mathcal{M}^L(D_Y)$ into $\mathcal{M}^L(D_{X\times Y})$;*
(ii) *$p^*(M) = R(X) \boxtimes M$ for any left D_Y-module M;*
(iii) *$p^*(M)$ is a finitely generated $D_{X\times Y}$-module if M is a finitely generated*
(iv) *$d(p^*(M)) = d(M) + n$ for any finitely generated left D_Y-module M.*

In particular, a finitely generated D_Y-module M is holonomic if and only if $p^*(M)$ is holonomic.

If we apply the transposition to the both actions on $D_{X\to Y}$ we get the (left D_Y, right D_X)-bimodule $D_{Y\leftarrow X}$. This allows the definition of the left D_Y-module

$$F_+(M) = D_{Y\leftarrow X} \otimes_{D_X} M$$

for any left D_X-module M. Clearly, F_+ is a right exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Y)$. We call it the *direct image* functor. Since $\mathrm{hd}(D_X) = n$ by 9.7, the left cohomological dimension of F_+ is $\leq n$. Therefore, the left derived functors L^iF_+ of F_+, given by

$$L^{-j}F_+(M) = \mathrm{Tor}_j^{D_X}(D_{Y\leftarrow X}, M)$$

for a left D_X-module M, vanish for $j > n$.

12.4. LEMMA. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *$D_{X\to Z} = D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}$;*
(ii) *$\mathrm{Tor}_j^{D_Y}(D_{X\to Y}, D_{Y\to Z}) = 0$ for $j \in \mathbb{N}$.*

PROOF. (i) By 2.(i) we have

$$D_{X\to Z} = (G \circ F)^*(D_Z) = F^*(G^*(D_Z)) = F^*(D_{Y\to Z}) = D_{X\to Y} \otimes_{D_Y} D_{Y\to Z}.$$

(ii) Let M be a left D_Y-module and $F_\cdot$ its left resolution by free D_Y-modules. Since D_Y is a free $R(Y)$-module for left multiplication, we can also view it as a resolution by free $R(Y)$-modules. This implies that

$$\begin{aligned}\mathrm{Tor}_j^{R(Y)}(R(X), M) &= H_j(R(X) \otimes_{R(Y)} F_\cdot) \\ &= H_j((R(X) \otimes_{R(Y)} D_Y) \otimes_{D_Y} F_\cdot) = \mathrm{Tor}_j^{D_Y}(D_{X\to Y}, M).\end{aligned}$$

Since $D_{Y\to Z}$ is a free $R(Y)$-module, $\mathrm{Tor}_j^{R(Y)}(R(X), D_{Y\to Z}) = 0$ for $j \in \mathbb{N}$, what implies our assertion. $\square$

This implies, by transposition of actions, the following statements

$$D_{Z\leftarrow X} = D_{Z\leftarrow Y} \otimes_{D_Y} D_{Y\leftarrow X}$$

and

$$\operatorname{Tor}_j^{D_Y}(D_{Z\leftarrow Y}, D_{Y\leftarrow X}) = 0$$

for $j \in \mathbb{N}$.

If P is a projective left D_X-module, $P \oplus Q = D_X^{(I)}$ for some left D_X-module Q and some I. Therefore, $F_+(P) \oplus F_+(Q) = F_+(D_X^{(I)}) = (D_{Y\leftarrow X})^{(I)}$. This implies the following result.

12.5. Lemma. *Let P be a projective left D_X-module. Then*

$$\operatorname{Tor}_j^{D_Y}(D_{Z\leftarrow Y}, F_+(P)) = 0$$

for $j \in \mathbb{N}$.

12.6. Theorem. *Let $X = k^n$, $Y = k^m$ and $Z = k^p$, and $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ polynomial maps. Then*

(i) *the direct image functor $(G \circ F)_+$ from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_Z)$ is isomorphic to $G_+ \circ F_+$;*
(ii) *for any left D_X-module M there exist a spectral sequence with E_2-term $E_2^{pq} = L^pG_+(L^qF_+(M))$ which converges to $L^{p+q}(G \circ F)_+(M)$.*

Proof. (i) For any left D_X-module M by 4.(i) we have

$$\begin{aligned}(G \circ F)_+(M) &= D_{Z\leftarrow X} \otimes_{D_X} M = (D_{Z\leftarrow Y} \otimes_{D_Y} D_{Y\leftarrow X}) \otimes_{D_X} M \\ &= D_{Z\leftarrow Y} \otimes_{D_Y} (D_{Y\leftarrow X} \otimes_{D_X} M) = D_{Z\leftarrow Y} \otimes_{D_Y} F_+(M) = G_+(F_+(M)).\end{aligned}$$

(ii) By 5, for any projective D_X-module P, the direct image $F_+(P)$ is G_+-acyclic. Therefore, the statement follows from the Grothendieck spectral sequence. □

Now we consider a simple example. Let i be the canonical injection of X into $X \times Y$ given by $i(x) = (x, 0)$ for any $x \in X$. Then

$$D_{X\to X\times Y} = i^*(D_{X\times Y}) = i^*(D_X \boxtimes D_Y) = D_X \boxtimes D_Y/((y_1, y_2, \dots, y_m)D_Y)$$

and

$$D_{X\times Y\leftarrow X} = D_X \boxtimes D_Y/(D_Y(y_1, y_2, \dots, y_m)).$$

This implies that

$$i_+(M) = M \boxtimes D_Y/(D_Y(y_1, y_2, \dots, y_m))$$

for any left D_X-module M.

12.7. Proposition. *Let $i : X \longrightarrow X \times Y$ be the injection defined by $i(x) = (x, 0)$ for $x \in X$. Then,*

(i) *i_+ is an exact functor from $\mathcal{M}^L(D_X)$ into $\mathcal{M}^L(D_{X\times Y})$;*
(ii) *$i_+(M) = M \boxtimes D_Y/(D_Y(y_1, y_2, \dots, y_m))$ for any left D_X-module M;*
(iii) *$i_+(M)$ is finitely generated $D_{X\times Y}$-module if M is a finitely generated D_X-module;*
(iv) *$d(i_+(M)) = d(M) + m$ for any finitely generated left D_X-module M.*

In particular, a finitely generated D_X-module M is holonomic if and only if $i_+(M)$ is holonomic.

Proof. We already proved (ii), and it immediately implies (i). Clearly, the quotient $D_Y/(D_Y(y_1, y_2, \dots, y_m))$ is a finitely generated D_Y-module, hence (iii) follows from 11.2. To prove (iv) we first remark that by 11.3, we have

$$d(i_+(M)) = d(M) + d(D_Y/(D_Y(y_1, y_2, \dots, y_m))).$$

It remains to determine the dimension of $N = D_Y/(D_Y(y_1, y_2, \dots, y_m))$.

The Fourier transform $\mathcal{F}(N)$ of this module is equal to $D_Y/(D_Y(\partial_1, \partial_2, \dots, \partial_m)) = R(Y)$. Since the Fourier transform preserves the dimension of modules by 9.1, we have $d(N) = d(R(Y)) = m$. $\square$

13. Kashiwara's theorem

Let $X = k^n$ and $Y = \{x_n = 0\} \subset X$. We put also $Z = \{x_1 = x_2 = \dots = x_{n-1} = 0\} \cong k$. Hence $X = Y \times Z$. This also implies that $D_X = D_Y \boxtimes D_Z$. Let M be a D_X-module and put

$$\Gamma_{[Y]}(M) = \{m \in M \mid x_n^p m = 0 \text{ for some } p \in \mathbb{N}\}.$$

13.1. Lemma. *Let M be a D_X-module. Then:*

(i) *$\Gamma_{[Y]}(M)$ is a D_X-submodule of M;*
(ii) *$\operatorname{supp}(\Gamma_{[Y]}(M)) \subset Y$;*
(iii) *if N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$, then $N \subset \Gamma_{[Y]}(M)$.*

Proof. (i) Let $m \in \Gamma_{[Y]}(M)$. Then $x_i m \in \Gamma_{[Y]}(M)$ and $\partial_j m \in \Gamma_{[Y]}(M)$ for $1 \le i \le n$ and $1 \le j < n$. It remains to check that $\partial_n m \in \Gamma_{[Y]}(M)$. We have

$$x_n^{j+1}\partial_n m = [x_n^{j+1}, \partial_n]m + \partial_n x_n^{j+1} m = -(j+1)x_n^j m + \partial_n x_n^{j+1} m$$

for any $j \in \mathbb{N}$. Hence, if $x_n^j m = 0$, we see that $x_n^{j+1}\partial_n m = 0$.

(ii) If $x \notin Y$, $x_n \notin \mathbf{m}_x$ and the localization $\Gamma_{[Y]}(M)_x = 0$.

(iii) Assume that N is a D_X-submodule of M with $\operatorname{supp}(N) \subset Y$. Let $m \in N$ and denote by N' the $R(X)$-submodule generated by m. Then $\operatorname{supp}(N') \subset Y$. Since N' is finitely generated, by 4.2, its support is equal to the variety determined by its annihilator I in $R(X)$. By Nullstelensatz we see that $r(I) \supset (x_n)$. This implies that x_n^j annihilates N' for some $j \in \mathbb{N}$, i.e., $m \in \Gamma_{[Y]}(M)$. $\square$

Therefore $\Gamma_{[Y]}(M)$ is the largest D_X-submodule of M supported in Y.

The multiplication by x_n defines an endomorphism of M as D_Y-module. Let

$$M_0 = \ker x_n \subset \Gamma_{[Y]}(M)$$

and

$$M_1 = \operatorname{coker} x_n = M/x_n M.$$

Denote by i the natural inclusion of Y into X.

13.2. Lemma. *Let M be a D_X-module. Then:*
(i) $L^j i^*(M) = M_{j+1}$ *for* $j = 0, -1$;
(ii) $L^j i^*(M) = 0$ *for* $j \neq 0, -1$.

Proof. By definition, $L^j i^*(M) = \operatorname{Tor}_{-j}^{D_X}(D_{Y\to X}, M)$. On the other hand, we established in the last section that

$$D_{Y\to X} = D_Y \boxtimes D_Z/(x_n)D_Z = D_X/(x_n)D_X.$$

Therefore, we have the exact sequence

$$0 \longrightarrow D_X \xrightarrow{x_n} D_X \longrightarrow D_{Y\to X} \longrightarrow 0$$

where the map from D_X into itself is by left multiplication by x_n. This is a short exact sequence of (left D_Y, right D_X)-bimodules and the first two terms are free D_X-modules. Therefore we constructed a resolution of the (left D_Y, right D_X)-bimodule by free right D_X-modules. This implies that $\operatorname{Tor}_{-j}^{D_X}(D_{Y\to X}, M)$ are the cohomology groups of the complex

$$\ldots \longrightarrow 0 \longrightarrow M \xrightarrow{x_n} M \longrightarrow 0 \longrightarrow \ldots . \quad \square$$

From the preceding argument we see that

$$D_{X\leftarrow Y} = D_X/D_X(x_n) = D_Y \boxtimes D_Z/D_Z(x_n) = \bigoplus_{j=0}^{\infty} \partial_n^j D_Y.$$

Clearly, we have a natural D_X-module morphism

$$i_+(M_0) = D_{X\leftarrow Y} \otimes_{D_Y} M_0 \longrightarrow \Gamma_{[Y]}(M).$$

By 2. this is actually a morphism of the functor $i_+ \circ L^{-1}i^*$ into $\Gamma_{[Y]}$.
The critical result of this section is the next lemma.

13.3. Lemma. *The morphism $i_+(M_0) \longrightarrow \Gamma_{[Y]}(M)$ is an isomorphism of D_X-modules.*

Proof. We first show that the morphism is surjective. We claim that

$$\{m \in M \mid x_n^p m = 0\} \subset D_X \cdot M_0$$

for any $p \in \mathbb{N}$. This is evident for $p = 1$. If $p > 1$ and $x_n^p m = 0$ we see that

$$0 = \partial_n(x_n^p m) = x_n^{p-1}(pm + x_n \partial_n m),$$

and by the induction hypothesis,

$$pm + x_n \partial_n m \in D_X \cdot M_0.$$

Also, by the induction hypothesis, $x_n m \in D_X \cdot M_0$. This implies that

$$(p-1)m = pm + [x_n, \partial_n]m = pm + x_n\partial_n m - \partial_n x_n m \in D_X \cdot M_0$$

and $m \in D_X \cdot M_0$. Hence the map is surjective.

Now we prove injectivity. By the preceding discussion

$$i_+(M_0) = D_{X\leftarrow Y} \otimes_{D_Y} M_0 = \bigoplus_{j=0}^{\infty} \partial_n^j M_0.$$

Let $(m_0, \partial_n m_1, \dots, \partial_n^q m_q, 0, \dots)$ be a nonzero element of this direct sum which maps into 0, i.e.,

$$m_0 + \partial_n m_1 + \dots + \partial_n^q m_q = 0,$$

with minimal possible q. Then

$$0 = x_n(\sum_{j=0}^{q} \partial_n^j m_j) = \sum_{j=1}^{q} [x_n, \partial_n^j] m_j = -\sum_{j=1}^{q} j\partial_n^{j-1} m_j$$

and we have a contradiction. Therefore, the kernel of the map is zero. □

13.4. Corollary. *$x_n \Gamma_{[Y]}(M) = \Gamma_{[Y]}(M)$.*

Proof. By 3. any element of $\Gamma_{[Y]}(M)$ has the form $\sum_{j\in\mathbb{Z}_+} \partial_n^j m_j$ with $m_j \in M_0$. On the other hand,

$$x_n \sum_{j\in\mathbb{Z}_+} \frac{1}{j+1} \partial_n^{j+1} m_j = -\sum_{j\in\mathbb{Z}_+} \partial_n^j m_j. \quad \square$$

13.5. Corollary. *Let M be a D_X-module. Then*

(i) *$\Gamma_{[Y]}(M)$ is a finitely generated D_X-module if and only if M_0 is a finitely generated D_Y-module;*
(ii) *$d(\Gamma_{[Y]}(M)) = d(M_0) + 1$.*

In particular, $\Gamma_{[Y]}(M)$ is holonomic if and only if M_0 is holonomic.

Proof. (i) From 3. and 12.7.(iii) we see that $\Gamma_{[Y]}(M)$ is finitely generated if M_0 is finitely generated. Assume that $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module. Let N_j, $j \in \mathbb{N}$, be an increasing sequence of D_Y-submodules of M_0. Then they generate D_X-submodules $i_+(N_j) = \oplus_{p=0}^{\infty} \partial_n^p N_j$ of $\Gamma_{[Y]}(M)$. Since $\Gamma_{[Y]}(M)$ is a finitely generated D_X-module, the increasing sequence $i_+(N_j)$, $j \in \mathbb{N}$, stabilizes. Moreover, N_j is the kernel of x_n in $i_+(N_j)$ and the sequence N_j, $j \in \mathbb{N}$, must also stabilize. Therefore, M_0 is finitely generated.

(ii) Follows from 3. and 12.7.(iv). □

13.6. Corollary. *Let M be a holonomic D_X-module. Then M_0 is a holonomic D_Y-module.*

Proof. If M is holonomic, $\Gamma_{[Y]}(M)$ is also holonomic. Therefore, the assertion follows from 5. $\square$

Let $\mathcal{M}_Y(D_X)$ be the full subcategory of $\mathcal{M}(D_X)$ consisting of D_X-modules supported in Y. Denote by $\mathcal{M}_{fg,Y}(D_X)$ and $\mathcal{H}ol_Y(D_X)$ the corresponding subcategories of finitely generated, resp. holonomic, D_X-modules supported in Y. Then, by 1, we have $M = \Gamma_{[Y]}(M)$ for any M in $\mathcal{M}_Y(D_X)$. By 2. and 4. we see that $i^*(M) = 0$ for any M in $\mathcal{M}_Y(D_X)$, hence $L^{-1}i^*$ is an exact functor from $\mathcal{M}_Y(D_X)$ into $\mathcal{M}(D_Y)$. On the other hand, i_+ defines an exact functor in the opposite direction, and by 3. the composition $i_+ \circ L^{-1}i^*$ is isomorphic to the identity functor on $\mathcal{M}_Y(D_X)$. Also it is evident that $L^{-1}i^* \circ i_+$ is isomorphic to the identity functor on $\mathcal{M}(D_Y)$.

This leads us to the following basic result due to Kashiwara.

13.7. Theorem. *The direct image functor i_+ defines an equivalence of the category $\mathcal{M}(D_Y)$ (resp. $\mathcal{M}_{fg}(D_Y)$, $\mathcal{H}ol(D_Y)$) with the category $\mathcal{M}_Y(D_X)$ (resp. $\mathcal{M}_{fg,Y}(D_X)$, $\mathcal{H}ol_Y(D_X)$). Its inverse is the functor $L^{-1}i^*$.*

Proof. It remains to show only the statements in parentheses. They follow immediately from 5. $\square$

14. Preservation of holonomicity

In this section we prove that direct and inverse images preserve holonomic modules. We start with a simple criterion for holonomicity.

14.1. Lemma. *Let $D(n)$ be equipped with the Bernstein filtration. Let M be a $D(n)$-module and $\mathrm{F}\,M$ an exhaustive $D(n)$-module filtration on M. If*

$$\dim_k \mathrm{F}_p\, M \le \frac{c}{n!}\, p^n + (\textit{lower order terms in } p)$$

for all $p \in \mathbb{Z}_+$, M is a holonomic $D(n)$-module and its length is $\le c$.

In particular, M is a finitely generated $D(n)$-module.

Proof. Let N be a finitely generated $D(n)$-submodule of M. Then $\mathrm{F}\,M$ induces an exhaustive $D(n)$-module filtration on N. By 3.5 there exists a good filtration $\mathrm{F}'\,N$ of N and $s \in \mathbb{Z}_+$ such that $\mathrm{F}'_p\,N \subset \mathrm{F}_{p+s}\,N$ for any $p \in \mathbb{Z}$. It follows that

$$\dim_k \mathrm{F}'_p\,N \le \dim_k \mathrm{F}_{p+s}\,N \le \dim_k \mathrm{F}_{p+s}\,M \le \frac{c}{n!}\, p^n + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$. Therefore, $d(N) \le n$ and N is holonomic. If $N \neq 0$, we have $e(N) \le c$. Clearly this implies that the length of N is $\le e(N) \le c$. It follows that any increasing sequence of finitely generated $D(n)$-submodules of M stabilizes, and that M itself is finitely generated. $\square$

Let M be a $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then on the localization M_P of M we can define k-linear maps $\partial_i : M_P \longrightarrow M_P$ by

$$\partial_i(\frac{m}{P^p}) = -p\partial_i(P)\frac{m}{P^{p+1}} + \frac{\partial_i m}{P^p}$$

for any $m \in M$ and $p \in \mathbb{Z}_+$. By direct calculation we can check that

$$[\partial_i, \partial_j](\frac{m}{P^p}) = 0$$

and

$$[\partial_i, x_j](\frac{m}{P^p}) = \delta_{ij}\frac{m}{P^p}$$

for any $1 \leq i, j \leq n$ and $p \in \mathbb{Z}_+$. By 8.10 this defines a structure of $D(n)$-module on M_P.

14.2. PROPOSITION. *Let M be a holonomic $D(n)$-module and $P \in k[X_1, X_2, \dots, X_n]$. Then M_P is a holonomic $D(n)$-module.*

PROOF. We can clearly assume that $P \neq 0$. Let $\mathrm{F}\,M$ be a good filtration on M such that $\mathrm{F}_p\,M = 0$ for $p \leq 0$ and $m = \deg P$. Define $\mathrm{F}_p\,M_P = 0$ for $p < 0$ and

$$\mathrm{F}_p\,M_P = \left\{ \frac{v}{P^p} \;\middle|\; v \in F_{(m+1)p}M \right\}$$

for $p \in \mathbb{Z}_+$. Clearly $\mathrm{F}_p\,M_P$, $p \in \mathbb{Z}$, are vector subspaces of M_P.

Let $w \in \mathrm{F}_p\,M_P$, $p > 0$. Then $w = \frac{v}{P^p} = \frac{Pv}{P^{p+1}}$ for some $v \in \mathrm{F}_{(m+1)p}\,M$. Since $Pv \in \mathrm{F}_{(m+1)p+m}\,M \subset \mathrm{F}_{(m+1)(p+1)}\,M$, we see that $w \in \mathrm{F}_{p+1}\,M_P$. This proves that the filtration $\mathrm{F}\,M_P$ is increasing.

Let $v \in \mathrm{F}_q\,M$. Then $\frac{v}{P^p} = \frac{P^s v}{P^{p+s}}$ for any $s \in \mathbb{Z}_+$. Also, $P^s v \in \mathrm{F}_{q+sm}\,M$ for any $s \in \mathbb{Z}_+$. Moreover, $(m+1)(p+s) - (q+sm) = s + (m+1)p - q \geq 0$ for $s \geq q - (m+1)p$. Hence

$$P^s v \in \mathrm{F}_{q+sm}\,M \subset \mathrm{F}_{(m+1)(p+s)}\,M$$

and $\frac{v}{P^p} \in \mathrm{F}_{p+s}\,M_P$. Therefore, the filtration $\mathrm{F}\,M_P$ is exhaustive.

It remains to show that it is a $D(n)$-module filtration. First, for $v \in \mathrm{F}_{(m+1)p}\,M$, $x_i P v \in \mathrm{F}_{(m+1)(p+1)}\,M$, hence $x_i\frac{v}{P^p} = \frac{x_i P v}{P^{p+1}} \in \mathrm{F}_{p+1}\,M_P$. Also,

$$\partial_i\left(\frac{v}{P^p}\right) = \frac{-p\partial_i(P)v + P\partial_i v}{P^{p+1}}$$

and $-p\partial_i(P)v + P\partial_i v \in \mathrm{F}_{(m+1)(p+1)}\,M$; hence $\partial_i\left(\frac{v}{P^p}\right) \in \mathrm{F}_{p+1}\,M_P$.

Therefore, we constructed an exhaustive $D(n)$-module filtration on M_P. Since

$$\dim_k \mathrm{F}_p\,M_P \leq \dim_k \mathrm{F}_{(m+1)p}\,M \leq e(M)\frac{((m+1)p)^n}{n!} + (\text{lower order terms in } p)$$

for $p \in \mathbb{Z}_+$, M_P is holonomic by 1. $\square$

14.3. Corollary. *Let $P \in k[X_1, X_2, \dots, X_n]$. Then $k[X_1, X_2, \dots, X_n]_P$ is a holonomic $D(n)$-module.*

Now we put $X = k^n$ and $Y = k^m$. Let $F : X \longrightarrow Y$ be a polynomial map. We want to study the behavior of holonomic modules under the action of inverse and direct image functors. First we use the graph construction to reduce the problem to special maps. We have the following diagram:

$$\begin{tikzcd} X \arrow[r, "1\times F"] \arrow[d, equal] & X \times Y \arrow[d, "pr_2"'] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

where $(1 \times F)(x) = (x, F(x))$, for $x \in X$, is an immersion, and $pr_2(x,y) = y$ for $x \in X$ and $y \in Y$, is a projection.

By 12.3 we know that pr_2^* is exact and maps holonomic modules into holonomic modules. Also, since the spectral sequence from 12.2 collapses in this case, we see immediately that $L^p(1 \times F)^* \circ pr_2^* = L^pF^*$ for any $p \in \mathbb{Z}$. This reduces the analysis to the case of the immersion $j = 1 \times F$.

We first discuss the immersion $i : X \longrightarrow X \times Y$ given by $i(x) = (x, 0)$ for $x \in X$.

14.4. Lemma. *Let M be a $D_{X\times Y}$-module. Then $L^{-p}i^*(M) = 0$ for $p > \dim Y$.*
If M is a holonomic $D_{X\times Y}$-module, the D_X-modules $L^pi^*(M)$, $p \in \mathbb{Z}$, are holonomic.

We prove this statement by induction in $\dim Y$. If $\dim Y = 1$, this corresponds to the situation studied (with different notation) in 13.2. If we denote by y_1 the natural coordinate on Y, and consider the D_X-module morphism $M \xrightarrow{y_1} M$, we have $i^*(M) = \operatorname{coker} y_1$ and $L^{-1}i^*(M) = \ker y_1$ and all other derived inverse images vanish. Moreover, if M is holonomic $L^{-1}i^*(M)$ is holonomic by 13.6.

14.5. Lemma. *Let M be a holonomic $D_{X\times Y}$-module. Then $i^*(M)$ is holonomic.*

Proof. Let $\bar{M} = M/\Gamma_{[X]}(M)$, i.e., we have the short exact sequence

$$0 \longrightarrow \Gamma_{[X]}(M) \longrightarrow M \longrightarrow \bar{M} \longrightarrow 0.$$

Since i^* is right exact functor, this leads to the exact sequence

$$i^*(\Gamma_{[X]}(M)) \longrightarrow i^*(M) \longrightarrow i^*(\bar{M}) \longrightarrow 0.$$

On the other hand, by 13.4, we see that $i^*(\Gamma_{[X]}(M)) = 0$. Therefore, the natural map $i^*(M) \longrightarrow i^*(\bar{M})$ is an isomorphism.

Let $\bar{m} \in \Gamma_{[X]}(\bar{M}) \subset \bar{M}$ and denote by $m \in M$ the representative of $\bar{m}$. Then $y_1^p\bar{m} = 0$ for sufficiently large $p \in \mathbb{Z}_+$. Therefore, $y_1^pm \in \Gamma_{[X]}(M)$. This in turn implies that $y^{p+q}m = y^q(y^pm) = 0$ for sufficiently large $q \in \mathbb{Z}_+$. Hence, $m \in \Gamma_{[X]}(M)$ and $\bar{m} = 0$. It follows that $\Gamma_{[X]}(\bar{M}) = 0$.

If M is a holonomic $D_{X\times Y}$-module, $\bar{M}$ is a holonomic $D_{X\times Y}$-module.

Therefore, we can assume from the beginning that $\Gamma_{[X]}(M) = 0$. This means that the multiplication by y_1 is injective on M, and M imbeds into its localization M_{y_1}. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow M_{y_1} \xrightarrow{\phi} N \longrightarrow 0.$$

Let $n \in N$. Then $n = \phi\left(\frac{m}{y_1^p}\right)$ for some $m \in M$ and $p \in \mathbb{Z}_+$. Therefore, $y_1^p n = \phi(m) = 0$ and $n \in \Gamma_{[X]}(N)$. Hence, we have $\Gamma_{[X]}(N) = N$.

Since M is a holonomic $D_{X\times Y}$-module, from 2. we know that M_{y_1} is a holonomic. Hence, N is a holonomic $D_{X\times Y}$-module. By 13.7 this implies that $N = i_+(L^{-1}i^*(N))$ and $L^{-1}i^*(N)$ is a holonomic D_X-module.

Applying the long exact sequence of inverse images of i to our short exact sequence, we get

$$\ldots \longrightarrow L^{-1}i^*(M_{y_1}) \longrightarrow L^{-1}i^*(N) \longrightarrow i^*(M) \longrightarrow i^*(M_{y_1}) \longrightarrow i^*(N) \longrightarrow 0.$$

Since the multiplication by y_1 on M_{y_1} is invertible, by 13.2 we see that

$$i^*(M_{y_1}) = L^{-1}i^*(M_{y_1}) = 0.$$

Hence, it follows that $i^*(M) \cong L^{-1}i^*(N)$. By the preceding discussion we conclude that $i^*(M)$ is a holonomic D_X-module. $\square$

Now we can finish the proof of 4. Let $Y' = \{y_m = 0\} \subset Y$. Denote by $i' : X \longrightarrow X \times Y'$ the morphism given by $i'(x) = (x, 0)$ and by i'' the natural inclusion of $X \times Y'$ into $X \times Y$. By induction assumption 4. holds for i' and i''. Hence, by 12.2, it holds for their composition $i = i'' \circ i'$.

Now we use 4. to prove the corresponding statement for $j = 1 \times F$. Define the morphism $G : X \times Y \longrightarrow X \times Y$ by

$$G(x, y) = (x, y_1 + F_1(x), y_2 + F_2(x), \ldots, y_m + F_m(x))$$

for $x \in X$ and $y \in Y$. Then $H : X \times Y \longrightarrow X \times Y$ defined by

$$H(x, y) = (x, y_1 - F_1(x), y_2 - F_2(x), \ldots, y_m - F_m(x))$$

for $x \in X$ and $y \in Y$, is the inverse of G, i.e., G is an isomorphism of $X \times Y$ onto itself. This implies that G^* is an equivalence of the category $\mathcal{M}(D_{X\times Y})$ with itself. Also it preserves finitely generated $D_{X\times Y}$-modules. By 9.8, G^* preserves the dimension of finitely generated $D_{X\times Y}$-modules; in particular, it preserves holonomic modules.

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{i} & X \times Y \\ {\scriptstyle 1_X}\downarrow & & {\scriptstyle G}\downarrow \\ X & \xrightarrow{j} & X \times Y \end{array}$$

Therefore, by 12.2, we have

$$L^p j^*(M) = L^p(G \circ i)^*(M) = L^p i^*(G^*(M))$$

for any $p \in \mathbb{Z}$ and $D_{X\times Y}$-module M. By 4, we conclude that $L^p j^*(M)$, $p \in \mathbb{Z}$, are holonomic D_X-modules for any holonomic $D_{X\times Y}$-module M.

This finally ends the proof of the following result.

14.6. THEOREM. *Let $F : X \to Y$ be a polynomial map and M a holonomic D_Y-module. Then $L^p F^*(M)$, $p \in \mathbb{Z}$, are holonomic D_X-modules.*

Now we want to study analogous properties of the direct image functors. As in the preceding argument, we conclude that G_+ is an equivalence of the category $\mathcal{M}(D_{X\times Y})$ with itself, and it preserves holonomic modules. Therefore, since the spectral sequence from 12.6 collapses, we conclude that

$$L^p j_+(M) = L^p(G \circ i)_+(M) = G_+(L^p i_+(M))$$

for any D_X-module M. By 12.7, i_+ is an exact functor from $\mathcal{M}(D_X)$ into $\mathcal{M}(D_{X\times Y})$ and it maps holonomic modules into holonomic modules. Therefore, $j_+ = G_+ \circ i_+$ is an exact functor from $\mathcal{M}(D_X)$ into $\mathcal{M}(D_{X\times Y})$ and it maps holonomic modules into holonomic modules.

Applying now the graph decomposition to F, we see from 12.6 that

$$L^p F_+(M) = L^p(pr_2)_+(j_+(M))$$

for any D_X-module M. Therefore, it remains to analyze the direct images of pr_2.

Consider first the case of $\dim X = 1$. Then

$$D_{X\times Y\to Y} = R(X \times Y) \otimes_{R(Y)} D_Y = R(X) \boxtimes D_Y = D_X/D_X(\partial_1) \boxtimes D_Y.$$

Hence, $D_{Y\leftarrow X\times Y} = D_X/(\partial_1)D_X \boxtimes D_Y$. We have an exact sequence

$$0 \longrightarrow D_{X\times Y} \xrightarrow{\partial_1} D_{X\times Y} \longrightarrow D_{Y\leftarrow X\times Y} \longrightarrow 0$$

of (left D_Y, right $D_{X\times Y}$)-bimodules, where the second arrow represents left multiplication by ∂_1. Clearly, this is a left resolution of $D_{Y\leftarrow X\times Y}$ by free right $D_{X\times Y}$-modules, hence the cohomology of the complex

$$\ldots \longrightarrow 0 \longrightarrow M \xrightarrow{\partial_1} M \longrightarrow 0 \longrightarrow \ldots$$

is $\operatorname{Tor}_{-\cdot}^{D_{X\times Y}}(D_{Y\leftarrow X\times Y}, M) = L^{\cdot}(pr_2)_+(M)$ for any $D_{X\times Y}$-module M. By applying the Fourier transform we get the complex

$$\ldots \longrightarrow 0 \longrightarrow \mathcal{F}(M) \xrightarrow{x_1} \mathcal{F}(M) \longrightarrow 0 \longrightarrow \ldots$$

which calculates $\mathcal{F}(L^q(pr_2)_+(M))$. Hence, by 9.11, 13.2, 13.6 and 14.5, we conclude that $L^q(pr_2)_+(M) = 0$ for $q \neq 0, -1$, and $L^q(pr_2)_+(M)$ are holonomic D_Y-modules.

Consider now the general case. Let $X' = \{x_n = 0\} \subset X$, and denote by pr_2' the canonical projection of $X' \times Y$ onto Y. Also, denote by p the canonical projection of X onto X'. Then $pr_2 = pr_2' \circ (p \times 1_Y)$. Hence, by 12.7 and the induction assumption we conclude that $L^p(pr_2)_+(M)$, $p \in \mathbb{Z}$, are holonomic D_Y-modules for any holonomic $D_{X\times Y}$-module M.

This ends the proof of the following result.

14.7. Theorem. *Let $F : X \to Y$ be a polynomial map and M a holonomic D_X-module. Then $L^p F_+(M)$, $p \in \mathbb{Z}$, are holonomic D_Y-modules.*

14.8. Remark. The statements analogous to 6. and 7. for finitely generated modules are false. For example, if we put $X = \{0\}$, $Y = k$ and denote by $i : X \to Y$ the natural inclusion, the inverse image $i^*(D_Y)$ is an infinite-dimensional vector space over k. Analogously, if p is the projection of Y into a point, $p_+(D_Y)$ is an infinite-dimensional vector space over k.

15. b-functions

Let k be an algebraically closed field and D the algebra of differential operators with polynomial coefficients on k. Let A be a free $k[x,\lambda]$-module with basis $(x^{\lambda+j}\,;\, j \in \mathbb{Z})$. Denote by B its submodule generated by the elements $x \cdot x^{\lambda+j} - x^{\lambda+j+1}$, and put $M = A/B$. Clearly, we can view M as a free $k[\lambda]$-module with basis $(x^{\lambda+j}; j \in \mathbb{Z})$. On A we can define a $k[\lambda]$-linear map T by

$$T(Px^{\lambda+j}) = (\partial P) \cdot x^{\lambda+j} + (\lambda + j)P \cdot x^{\lambda+j-1}.$$

Now

$$\begin{aligned} T(P(x \cdot x^{\lambda+j} - x^{\lambda+j+1})) &= T(Px \cdot x^{\lambda+j} - P \cdot x^{\lambda+j+1}) \\ &= \partial(Px) \cdot x^{\lambda+j} + (\lambda+j)Px \cdot x^{\lambda+j-1} - \partial P \cdot x^{\lambda+j+1} - (\lambda+j+1)P \cdot x^{\lambda+j} \\ &= (\partial P)(x \cdot x^{\lambda+j} - x^{\lambda+j+1}) + (\lambda+j)P(x \cdot x^{\lambda+j-1} - x^{\lambda+j}) \in B, \end{aligned}$$

so T induces a $k[\lambda]$-linear map ∂ on M given by

$$\partial\, x^{\lambda+j} = (\lambda+j)x^{\lambda+j-1}.$$

Hence, it extends to $\tilde{M} = k(\lambda) \otimes_{k[\lambda]} M$ where $k(\lambda)$ denotes the field of rational functions in λ. Clearly, $\tilde{M}$ is a linear space with basis $\{x^{\lambda+j}, j \in \mathbb{Z}\}$ over the field $k(\lambda)$. Also

$$\partial(xx^{\lambda+j}) = \partial(x^{\lambda+j+1}) = (\lambda+j+1)x^{\lambda+j} = x\partial\, x^{\lambda+j} + x^{\lambda+j},$$

hence $[\partial, x] = 1$. This implies that $\tilde{M}$ has a natural structure of a D-module. If we put $D(\lambda) = k(\lambda) \otimes_k D$, $\tilde{M}$ becomes a $D(\lambda)$-module. From the definition of the action of $D(\lambda)$ it is evident that every $x^{\lambda+j}$, $j \in \mathbb{Z}$, generates $\tilde{M}$.

We can define a filtration $(\mathrm{F}_m \tilde{M}\,;\, m \in \mathbb{Z})$ of $\tilde{M}$ where $\mathrm{F}_m \tilde{M}$ is the linear span of $x^{\lambda+k}$, $|k| \leq m$. The filtration is evidently increasing and it is exhaustive. Finally, $x\,\mathrm{F}_m \tilde{M} \subset F_{m+1}\tilde{M}$ and $\partial\,\mathrm{F}_m \tilde{M} \subset \mathrm{F}_{m+1} \tilde{M}$, hence this is a $D(\lambda)$-module filtration. Clearly, $\dim_{k(\lambda)} \mathrm{F}_m \tilde{M} = 2m+1$ for any $m \in \mathbb{Z}_+$.

Let K be the algebraic closure of $k(\lambda)$. Put $D_K = K \otimes_{k(\lambda)} D(\lambda)$ and $M_K = K \otimes_{k(\lambda)} \tilde{M}$. Also, put $\mathrm{F}_m M_K = K \otimes_{k(\lambda)} \mathrm{F}_m \tilde{M}$ for $m \in \mathbb{Z}$. This defines a D_K-module filtration of M_K. Moreover, $\dim_K \mathrm{F}_m M_K = 2m+1$ for all $m \in \mathbb{Z}$. By 14.1, M_K is a holonomic D_K-module.

Let $D(n)$ be the algebra of all differential operators with polynomial coefficients on k^n. Denote by $D_K(n)$ corresponding algebra of differential operators on K^n. Let $f \in k[X_1, X_2, \dots, X_n]$. We can view f as a polynomial on K^n with coefficients in k.

We can view

$$f^*(M_K) = K[X_1, X_2, \dots, X_n] \otimes_{K[X]} M_K = K \otimes_{k(\lambda)} \left(k(\lambda)[X_1, X_2, \dots, X_n] \otimes_{k(\lambda)[X]} \tilde{M}\right)$$

as a $K[X_1, X_2, \dots, X_n]$-module spanned by $f^{\lambda+j} = 1 \otimes x^{\lambda+j}$ for $j \in \mathbb{Z}$. If we denote by $M_{(f)}$ the $k(\lambda)$-linear space $k(\lambda)[X_1, X_2, \dots, X_n] \otimes_{k(\lambda)[X]} \tilde{M}$, we have $f^*(M_K) = K \otimes_{k(\lambda)} M_{(f)}$.

First, we see that

$$f \cdot f^{\lambda+j} = f \cdot (1 \otimes x^{\lambda+j}) = f \otimes x^{\lambda+j} = 1 \otimes x \cdot x^{\lambda+j} = 1 \otimes x^{\lambda+j+1} = f^{\lambda+j+1}$$

for any $j \in \mathbb{Z}$.

Moreover, the short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0$$

by localization leads to the short exact sequence

$$0 \longrightarrow \tilde{B} \longrightarrow \tilde{A} \longrightarrow \tilde{M} \longrightarrow 0$$

where $\tilde{B} = k(\lambda) \otimes_{k[\lambda]} B$ and $\tilde{A} = k(\lambda) \otimes_{k[\lambda]} A$ is a free $k(\lambda)[X]$-module. By tensoring this short exact sequence with $k(\lambda)[X_1, X_2, \dots, X_n]$ we get the exact sequence

$$k(\lambda)[X_1, X_2, \dots, X_n] \otimes_{k(\lambda)[X]} \tilde{B} \longrightarrow k(\lambda)[X_1, X_2, \dots, X_n] \otimes_{k(\lambda)[X]} \tilde{A} \longrightarrow M_{(f)} \longrightarrow 0.$$

Therefore, the module $M_{(f)}$ is the quotient of the free $k(\lambda)[X_1, X_2, \dots, X_n]$-module with basis $\{f^{\lambda+j}; j \in \mathbb{Z}\}$ by the submodule generated by the elements $f \cdot f^{\lambda+j} - f^{\lambda+j+1}$, $j \in \mathbb{Z}$.

The $D_K(n)$-action on $f^*(M_K)$ is given by

$$\partial_i \, f^{\lambda+j} = \partial_i(1 \otimes x^{\lambda+j}) = \partial_i f \otimes \partial \, x^{\lambda+j} = (\lambda + j)\,(\partial_i f)\,(1 \otimes x^{\lambda+j-1}) = (\lambda + j)\,(\partial_i f)\, f^{\lambda+j-1}.$$

for any $1 \le j \le n$.

Put $D(\lambda, n) = k(\lambda) \otimes_k D(n)$. Then $M_{(f)}$ is a $D(\lambda, n)$-submodule of $f^*(M_K)$.

Since $f^*(M_K)$ is holonomic $D_K(n)$-module by 14.6, it is also of finite length. This in turn implies the following result.

15.1. Lemma. *The $D(\lambda, n)$-module $M_{(f)}$ is a module of finite length.*

Now, denote by M_p the $D(\lambda, n)$-submodule of $M_{(f)}$ generated by $f^{\lambda+p}$. Then

$$\cdots \supset M_p \supset M_{p+1} \supset \cdots$$

is an decreasing exhaustive filtration of $M_{(f)}$ by $D(\lambda, n)$-submodules. Since $M_{(f)}$ is of finite length, there exists $p_0 \in \mathbb{Z}$ such that $M_p = M_{(f)}$ for $p \leq p_0$.

Let $\eta_\pm$ be the automorphisms of $k(\lambda)$ defined by $\eta_\pm(g)(\lambda) = g(\lambda \pm 1)$ for $g \in k(\lambda)$. They extend to an automorphism of $D(\lambda, n)$ which we also denote by $\eta_\pm$. On the other hand,

$$\tau_\pm(P \cdot f^{\lambda+j}) = \eta_\pm(P) \cdot f^{\lambda+j\pm1}$$

defines k-linear endomorphisms of $M_{(f)}$. Also

$$\begin{aligned}\tau_\pm(\partial_i(P \cdot f^{\lambda+j})) &= \eta_\pm(\partial_i P) \cdot f^{\lambda+j\pm1} + (\lambda + j \pm 1)(\partial_i f)\eta_\pm(P) f^{\lambda+j-1\pm1} \\ &= \partial_i \eta_\pm(P) \cdot f^{\lambda+j\pm1} + (\lambda + j \pm 1)(\partial_i f)\eta_\pm(P) \cdot f^{\lambda+j-1\pm1} = \partial_i \tau_\pm(P \cdot f^{\lambda+j});\end{aligned}$$

hence, for any $T \in D(\lambda, n)$ and $m \in M_{(f)}$,

$$\tau_\pm(Tm) = \eta_\pm(T)\tau_\pm(m).$$

Evidently, τ_- is the inverse of τ_+, and

$$\tau_\pm(M_p) = \tau_\pm(D(\lambda, n) f^{\lambda+p}) = D(\lambda, n)\tau_\pm(f^{\lambda+p}) = D(\lambda, n) f^{\lambda+p\pm1} = M_{p\pm1}$$

for any $p \in \mathbb{Z}$. Hence, if $M_p = M_{(f)}$ we have

$$M_{p+1} = \tau_+(M_p) = \tau_+(M_{(f)}) = M_{(f)}.$$

Therefore, by the induction in p, the above remark implies $M_p = M_{(f)}$ for all $p \in \mathbb{Z}$.

In particular, there exists $\Delta(\lambda) \in D(\lambda, n)$ such that

$$\Delta(\lambda) f^{\lambda+1} = f^\lambda.$$

This implies that there exists $\Delta[\lambda] \in k[\lambda] \otimes_k D(n)$ and $b \in k[\lambda]$ such $b \neq 0$ and

$$\Delta[\lambda] f^{\lambda+1} = b(\lambda) f^\lambda$$

in $M_{(f)}$. The polynomial b is not unique, but all such polynomials form an ideal in $k[\lambda]$. Since $k[\lambda]$ is a principal ideal domain, there exists a unique polynomial of lowest degree with leading coefficient 1 with this property. This polynomial is called the *Bernstein-Sato polynomial*. One can show that it has rational zeros.

We can generalize this construction in the following way. First we need to construct the inner tensor product of two D-modules on X.

Let $X = k^n$ and M, N two D_X-modules. Then we can form the exterior tensor product $M \boxtimes N$, which is a D-module on $X \times X$. If M and N are holonomic D_X-modules, $M \boxtimes N$ is a holonomic $D_{X \times X}$-module by 11.4. Let $\Delta : X \longrightarrow X \times X$ be the diagonal map $\Delta(x) = (x, x)$. Then we can consider the D_X-module

$$\Delta^*(M \boxtimes N) = R(X) \otimes_{R(X \times X)} (M \boxtimes N) = (M \boxtimes N)/I(M \boxtimes N),$$

where we denoted by I the ideal of functions in $R(X \times X)$ which vanish on the diagonal $\Delta(X)$ in $X \times X$. This ideal is generated by functions $X_i \otimes 1 - 1 \otimes X_i$, $1 \leq i \leq n$. Therefore, $\Delta^*(M \boxtimes N)$ is the quotient of $M \boxtimes N$ by the subspace spanned by the elements of the form $X_i u \otimes v - u \otimes X_i v$, $1 \leq i \leq n$ and $u \in M$, $v \in N$. This implies that

$$\Delta^*(M \boxtimes N) = M \otimes_{R(X)} N$$

as an $R(X)$-module. Moreover,

$$\partial_i(u \otimes v) = \partial_i u \otimes v + u \otimes \partial_i v$$

for $1 \leq i \leq n$, $u \in M$ and $v \in N$. Therefore, we defined a structure of a D_X-module on $M \otimes_{R(X)} N$. We call this D_X-module the *inner tensor product* of M and N. By 14.6. we conclude that the following result holds.

15.2. PROPOSITION. *Let M and N be two holonomic D_X-modules. Then $M \otimes_{R(X)} N$ is a holonomic D_X-module.*

Now we can apply this to the previous situation. Let V be a $D(n)$-module. We can extend the field of scalars from k to K and define $V_K = K \otimes_k V$ as a $D_K(n)$-module. Next, we can construct the inner tensor product $V_K \otimes_{R(X_K)} M_K$. Let $V_{(f)}$ be the $k(\lambda)$-linear subspace of $V_K \otimes_{R(X_K)} M_K$ spanned by $v \otimes f^{\lambda+j}$, $v \in V$ and $j \in \mathbb{Z}$. Then $V_{(f)}$ is a $D(\lambda, n)$-module with the action given by

$$\partial_i(v \otimes f^{\lambda+j}) = \partial_i v \otimes f^{\lambda+j} + (\lambda + j)\,(\partial_i f)\,(v \otimes f^{\lambda+j-1})$$

for $j \in \mathbb{Z}$ and $v \in V$. Clearly, $V_K \otimes_{R(X_K)} M_K = K \otimes_{k(\lambda)} V_{(f)}$.

If V is a holonomic $D(n)$-module, V_K is a holonomic $D_K(n)$-module. This implies, by 2, that the inner tensor product $V_K \otimes_{R(X_K)} M_K$ is a holonomic $D_K(n)$-module, and therefore of finite length. Hence, we have the following generalization of 1.

15.3. LEMMA. *Let V be a holonomic $D(n)$-module. Then the $D(\lambda, n)$-module $V_{(f)}$ is a module of finite length.*

As before, denote by V_p the $D(\lambda, n)$-submodule of $V_{(f)}$ generated by $v \otimes f^{\lambda+p}$, $v \in V$. Then we get an decreasing exhaustive filtration of $D(\lambda, n)$-module $V_{(f)}$, and

$$\cdots \supset V_p \supset V_{p+1} \supset \cdots.$$

Since $V_{(f)}$ is of finite length, $V_p = V_{(f)}$ for sufficiently small p.

Define the k-linear endomorphisms $\omega_\pm$ of $V_{(f)}$ by

$$\omega_\pm(P(v \otimes f^{\lambda+j})) = \eta_\pm(P)(v \otimes f^{\lambda+j\pm1})$$

for any $v \in V$, $P \in k(\lambda) \otimes_k R(X)$ and $j \in \mathbb{Z}$. Then, as in the preceding argument, we show that ω_- is the inverse of ω_+, and $\omega_\pm(V_p) = V_{p\pm1}$ for any $p \in \mathbb{Z}$. This implies that $V_p = V_{(f)}$ for any $p \in \mathbb{Z}$. Therefore, we conclude that the following result holds.

15.4. LEMMA. *Let V be a holonomic $D(n)$-module. Then for any $j \in \mathbb{Z}$ we can find vectors $v_1, v_2, \ldots, v_p \in V$ such that the $D(\lambda, n)$-module $V_{(f)}$ is generated by the vectors $v_1 \otimes f^{\lambda+j}, v_2 \otimes f^{\lambda+j}, \ldots, v_p \otimes f^{\lambda+j}$.*

16. Meromorphic continuation of some distributions

Let $C_0^\infty(\mathbb{R}^n)$ be the space of smooth compactly supported complex-valued functions on $\mathbb{R}^n$ with the usual Schwartz topology. For any function $f \in L^1_{loc}(\mathbb{R}^n)$ the expression

$$\epsilon_f(\varphi) = \int_{\mathbb{R}^n} f(x)\,\varphi(x)\,dx$$

defines a continuous linear form on $C_0^\infty(\mathbb{R}^n)$, i.e., a distribution on $\mathbb{R}^n$.

Let f be a continuous real-valued function on $\mathbb{R}^n$. Then $U_+ = \{x \in \mathbb{R}^n \mid f(x) > 0\}$ is an open set in $\mathbb{R}^n$. The function $\sup(f, 0)$ is a positive continuous function on $\mathbb{R}^n$, therefore the function $x \longrightarrow \sup(f(x)^\lambda, 0)$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, is a continuous function on $\mathbb{R}^n$. This implies that for any union U of connected components of U_+ and for any $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$,

$$f^\lambda : \varphi \longrightarrow \int_U f(x)^\lambda \varphi(x)\,dx$$

defines a distribution on $\mathbb{R}^n$ which depends holomorphically on λ.

In his address to the International Congress of Mathematicians in Amsterdam in 1954, I.M. Gelfand posed the following problem:

If f is a polynomial on $\mathbb{R}^n$, does $\lambda \longrightarrow f^\lambda$ extend to a meromorphic function from $\mathbb{C}$ into distributions on $\mathbb{R}^n$?

Assume that there exists a differential operator $\Delta[\lambda]$ depending polynomially in λ and a polynomial $b \in \mathbb{C}[\lambda]$ on $\mathbb{R}^n$ such that

$$\Delta[\lambda] f^{\lambda+1} = b(\lambda) f^\lambda$$

for for $\operatorname{Re}\lambda > 0$. Assume that f^λ has a meromorphic extension to $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -k\}$ for $k \in \mathbb{Z}_+$ with the set of poles S_k. Then we have, on $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -k\}$,

$$f^\lambda(\varphi) = \frac{1}{b(\lambda)} (\Delta[\lambda] f^{\lambda+1})(\varphi) = \frac{1}{b(\lambda)} f^{\lambda+1}(\Delta[\lambda]^* \varphi),$$

where $\Delta[\lambda]^*$ denotes the formal adjoint of $\Delta[\lambda]$. Therefore, for $\operatorname{Re}\lambda > -k$ the two meromorphic functions

$$\lambda \longmapsto f^\lambda$$

and

$$\lambda \longmapsto \frac{1}{b(\lambda)} \Delta[\lambda] f^{\lambda+1}$$

agree. On the other hand, the latter one is meromorphic on $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -k-1\}$. This implies that $\lambda \mapsto f^\lambda$ extends to a meromorphic distribution on $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -k-1\}$ with poles in the union of $S_k - 1$ and the set of zeros of b. By induction in k, one proves the existence of a meromorphic continuation of f^λ to the whole complex plane with poles in the set $\{\alpha - k \mid \alpha \text{ a zero of } b,\ k \in \mathbb{Z}_+\}$.

It remains to show the existence of the differential operator $\Delta[\lambda]$ and a polynomial b with the required properties. We can view the space $(C_0^\infty(\mathbb{R}^n))'$ of all distributions on $\mathbb{R}^n$ as a module for the algebra of all differential operators on $\mathbb{R}^n$ with polynomial coefficients. Therefore, the spaces of all holomorphic maps from $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > k\}$, $k \in \mathbb{N}$, into $(C_0^\infty(\mathbb{R}^n))'$ are $D(n)$-modules. Hence, their direct limit $\mathcal{H}$ as $k \longrightarrow \infty$ is a $D(n)$-module. Since any function in $\mathbb{C}(\lambda)$ is holomorphic in the region $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > k\}$ for k large enough, the direct limit $\mathcal{H}$ is also a linear space over $\mathbb{C}(\lambda)$. Hence we can view it as a $D(\lambda, n)$-module.

Clearly, the functions $\lambda \mapsto f^{\lambda+j}$ are in $\mathcal{H}$ for arbitrary $j \in \mathbb{Z}$, and they satisfy the relations $f \cdot f^{\lambda+j} = f^{\lambda+j+1}$ and

$$\partial_i f^{\lambda+j} = (\lambda + j)(\partial_i f) f^{\lambda+j-1},$$

for any $i, j \in \mathbb{Z}$. This implies that the $D(\lambda, n)$-submodule of $\mathcal{H}$ generated by elements $f^{\lambda+j}$, $j \in \mathbb{Z}$, is a quotient of the module $M_{(f)}$ considered in 15. Hence, the existence of $\Delta[\lambda]$ and b follows from the results proved there, which completes the proof of the next result.

16.1. Theorem. *For any polynomial f with real coefficients on $\mathbb{R}^n$ the map $\lambda \mapsto f^\lambda$ extends to a meromorphic function with values in the space of distributions on $\mathbb{R}^n$. Its poles are of the form $\alpha - k$, where α is a zero of the Bernstein-Sato polynomial of f and $k \in \mathbb{Z}_+$.*

16.2. Remark. By inspecting the proof of 1. one easily checks that $\lambda \mapsto f^\lambda$ is actually a meromorphic function with values in the space of *tempered* distributions on $\mathbb{R}^n$.

16.3. Corollary. *Let f be a nonzero polynomial on $\mathbb{R}^n$. Then there exists a tempered distribution T on $\mathbb{R}^n$ such that $fT = 1$.*

Proof. Assume first that $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Since f is nonzero, the set of zeros of f has measure zero. Therefore, if we put

$$f^\lambda(\varphi) = \int_{\mathbb{R}^n} f(x)^\lambda \varphi(x)\, dx$$

for $\operatorname{Re}\lambda > 0$, by 2, f^λ extends to a meromorphic function with values in tempered distributions.

First we claim that f^λ is regular at 0 with value equal to 1. Clearly, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^n} f(x)^t \varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, dx.$$

Hence, the meromorphic function $\lambda \mapsto f^\lambda(\varphi)$ is regular at 0 for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let

$$f^\lambda = \sum_{n=-\infty}^{\infty} S_n \lambda^n$$

be the Laurent expansion of $\lambda \mapsto f^\lambda$ at 0. Then,

$$f^\lambda(\varphi) = \sum_{n=-\infty}^{\infty} S_n(\varphi)\lambda^n;$$

hence $S_n(\varphi) = 0$ vanish for $n < 0$ and $S_0(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\,dx$, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. This implies that tempered distributions S_n vanish for $n < 0$, and $S_0 = 1$.

Consider now the Laurent expansion

$$f^\lambda = \sum_{n=-\infty}^{\infty} T_n(\lambda+1)^n$$

at -1. Then T_n, $n \in \mathbb{Z}$, are tempered distributions and $T_n = 0$ for sufficiently negative $n \in \mathbb{Z}$. Moreover, since $f \cdot f^\lambda = f^{\lambda+1}$, the product $f \cdot f^\lambda$ is regular at -1 and has value 1. This implies that $fT_n = 0$ for all $n < 0$, and $fT_0 = 1$.

Assume now that f is arbitrary. Then we can put $g = f\bar{f}$, and g has non-negative values on $\mathbb{R}^n$. Hence, there exists a tempered distribution T such that $gT = 1$. But now

$$f(\bar{f}T) = gT = 1,$$

hence $\bar{f}T$ has the required property. □

17. Differential equations with constant coefficients

Let $P \in \mathbb{C}[X_1, X_2, \dots, X_n]$, i.e.,

$$P(X) = \sum_{I \in \mathbb{Z}_+^n} c_I X^I.$$

Then we can define the differential operator

$$P(\partial) = \sum_{I \in \mathbb{Z}_+^n} c_I \partial^I$$

with constant coefficients on $\mathbb{R}^n$. Let δ be the distribution $\varphi \mapsto \varphi(0)$ on $\mathbb{R}^n$. A distribution T on $\mathbb{R}^n$ is called a *fundamental solution* for $P(\partial)$ if it satisfies $P(\partial)T = \delta$.

As an application of the results of 16. we prove the following result about the existence of fundamental solutions.

17.1. THEOREM. *Let P be a nonzero polynomial on $\mathbb{R}^n$. Then there exists a tempered distribution T on $\mathbb{R}^n$ such that $P(\partial)T = \delta$.*

PROOF. We define the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ via

$$(\mathcal{F}\varphi)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ixy}\,dx$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$; and the inverse Fourier transform by

$$(\bar{\mathcal{F}}\psi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \psi(y) e^{ixy}\, dy$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$. This defines the Fourier transform on the space $(\mathcal{S}(\mathbb{R}^n))'$ of tempered distributions on $\mathbb{R}^n$ via

$$(\mathcal{F}T)(\varphi) = T(\mathcal{F}\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^n);$$

and

$$(\bar{\mathcal{F}}T)(\psi) = T(\bar{\mathcal{F}}\psi), \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

Clearly,

$$(\bar{\mathcal{F}}\delta)(\psi) = \delta(\bar{\mathcal{F}}\psi) = (\bar{\mathcal{F}}\psi)(0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \psi(y)\, dy$$

for arbitrary $\psi \in \mathcal{S}(\mathbb{R}^n)$, hence

$$\bar{\mathcal{F}}\delta = (2\pi)^{-\frac{n}{2}}$$

and

$$\delta = (2\pi)^{-\frac{n}{2}} \mathcal{F}(1).$$

One easily checks that

$$\partial^I \mathcal{F}T = \mathcal{F}((-i)^{|I|} x^I T).$$

for any tempered distribution T. Let $Q(x) = P(-ix)$. By 16.3 there exists a tempered distribution S on $\mathbb{R}^n$ such that $QS = 1$. Let

$$T = (2\pi)^{-\frac{n}{2}} \mathcal{F}(S).$$

Then

$$P(\partial)T = (2\pi)^{-\frac{n}{2}} P(\partial)\mathcal{F}(S) = (2\pi)^{-\frac{n}{2}} \mathcal{F}(QS) = (2\pi)^{-\frac{n}{2}} \mathcal{F}(1) = \delta. \quad \square$$