INTERTWINING FUNCTORS AND IRREDUCIBILITY OF STANDARD HARISH-CHANDRA SHEAVES

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Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and σ an involution of $\mathfrak{g}$. Denote by $\mathfrak{k}$ the fixed point set of this involution. Let K be a connected algebraic group and φ a morphism of K into the group $G = \mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ such that its differential is injective and identifies the Lie algebra of K with $\mathfrak{k}$. Let X be the flag variety of $\mathfrak{g}$, i.e. the variety of all Borel subalgebras in $\mathfrak{g}$. Then K acts algebraically on X, and it has finitely many orbits which are locally closed smooth subvarieties. The typical situation is the following: $\mathfrak{g}$ is the complexification of the Lie algebra of a connected real semisimple Lie group G_0 with finite center, K is the complexification of a maximal compact subgroup of G_0, and σ the corresponding Cartan involution.

Let $\mathfrak{h}$ be the (abstract) Cartan algebra of $\mathfrak{g}$, Σ the root system in $\mathfrak{h}^*$, and Σ^+ the set of positive roots determined by the condition that the homogeneous line bundles $\mathcal{O}(-\mu)$ on X corresponding to dominant weights μ are positive. For each $\lambda \in \mathfrak{h}^*$, A. Beilinson and J. Bernstein defined a G-homogeneous twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X (compare [1], [5]). For a detailed discussion of their construction see §2 in Schmid's lecture in this volume [11].

Let $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ be the category of coherent $\mathcal{D}_\lambda$-modules on X with algebraic K-action ([5], Appendix). The objects of this category are called *Harish-Chandra sheaves*. Every Harish-Chandra sheaf has finite length, and there is a simple geometric description of irreducible Harish-Chandra sheaves which we shall describe now. Let Q be a K-orbit in X and $i : Q \longrightarrow X$ the natural inclusion. Then $\mathcal{D}_\lambda$ induces a K-homogeneous twisted sheaf of differential operators $\mathcal{D}_\lambda^i$ on Q. Fix $x \in Q$, let $\mathfrak{b}_x$ be the Borel subalgebra corresponding to this point, and define $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra in $\mathfrak{b}_x$. Then the composition of the canonical maps $\mathfrak{c} \longrightarrow \mathfrak{b}_x/\mathfrak{n}_x \longrightarrow \mathfrak{h}$ is an isomorphism. It induces an isomorphism, called a *specialization*, of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ onto the triple $(\mathfrak{c}^*, R, R^+)$; here R is the root system of the pair $(\mathfrak{g}, \mathfrak{c})$ and R^+ the set of positive roots determined by $\mathfrak{b}_x$. Let ρ be the half-sum of positive roots in Σ^+. If the restriction of the specialization of $\lambda + \rho$ to $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{c}$ is the differential of a character of the identity

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component of the stabilizer S_x of x in K, there exist K-homogeneous $\mathcal{D}^i_\lambda$-connections on Q—we say that they are compatible with $\lambda+\rho$. Let τ be an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Then its direct image $R^0 i_+(\tau)$ is the *standard Harish-Chandra sheaf* $\mathcal{I}(Q,\tau)$. It is holonomic and therefore of finite length. Moreover, it has a unique irreducible $(\mathcal{D}_\lambda, K)$-submodule $\mathcal{L}(Q,\tau)$. The irreducible objects $\mathcal{L}(Q,\tau)$ exhaust the isomorphism classes of all irreducible objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. Therefore, the composition series of standard Harish-Chandra sheaves $\mathcal{I}(Q,\tau)$ consist of modules isomorphic to some $\mathcal{L}(Q',\tau')$ for orbits Q' in the closure of Q and irreducible K-homogeneous connections τ' on Q' compatible with $\lambda+\rho$.

For integral $\lambda \in \mathfrak{h}^*$, the structure of the composition series of these modules is determined by Vogan's version of the Kazhdan-Lusztig conjectures [13]. As in the Verma module case, one should expect that the necessary and sufficient condition for the irreducibility of standard Harish-Chandra sheaves must be a far less deep result than the Kazhdan-Lusztig conjectures. If K is the fixed point set of an involution acting on a covering group of G, such a result is equivalent to the irreducibility theorem of [12]. Although the final result in this case (as in the case of Verma modules) suggests that the irreducibility criterion is completely controlled by SL_2-phenomena, this is not so evident from the existing proofs. The purpose of this paper is to describe the irreducibility result for the general case, and to sketch a proof which is conceptually as simple as in the case of Verma modules. This result is a part of a joint work with Henryk Hecht, Wilfried Schmid and Joseph A. Wolf. The complete details will appear in [6].

1. The basic example

In this section we discuss the simplest case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. In this case the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ can be identified with $\mathrm{PSL}(2,\mathbb{C})$, and we can identify the flag variety X of $\mathfrak{g}$ with the one-dimensional projective space $\mathbb{P}^1$. If we denote by $[x_0, x_1]$ the projective coordinates of $x \in \mathbb{P}^1$, the corresponding Borel subalgebra $\mathfrak{b}_x$ is the Lie subalgebra of $\mathfrak{sl}(2,\mathbb{C})$ which leaves the line x invariant. Let σ be the conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathfrak{g}$. Then $\mathfrak{k}$ is the subalgebra of diagonal matrices in $\mathfrak{g}$.

Let T the one-dimensional torus which stabilizes both $0 = [1,0]$ and $\infty = [0,1]$. Its Lie algebra is $\mathfrak{k}$. Hence, K can be an arbitrary n-fold covering of T with covering map φ. The K-orbits in $\mathbb{P}^1$ are $\{0\}$, $\{\infty\}$, and $\mathbb{C}^*$.

First we want to construct a suitable trivializations of $\mathcal{D}_\lambda$ on the open cover of $\mathbb{P}^1$ consisting of $\mathbb{P}^1 - \{0\}$ and $\mathbb{P}^1 - \{\infty\}$. We denote by $\alpha \in \mathfrak{h}^*$ the positive root of $\mathfrak{g}$ and put $\rho = \frac{1}{2}\alpha$ and $t = \alpha\check{}(\lambda)$, where $\alpha\check{}$ is the dual root of α.

Let $\{E, F, H\}$ denote the standard basis of $\mathfrak{sl}(2,\mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation relations

$$[H,E] = 2E \qquad [H,F] = -2F \qquad [E,F] = H.$$

Also, H spans the Lie algebra $\mathfrak{k}$. Moreover, we remark that if we specialize at 0, H corresponds to the dual root $\alpha\check{}$, but if we specialize at ∞, H corresponds to the negative of $\alpha\check{}$.

First we discuss $\mathbb{P}^1 - \{\infty\}$. On this set we define the usual coordinate z by $z([1, x_1]) = x_1$. In this way one identifies $\mathbb{P}^1 - \{\infty\}$ with the complex plane $\mathbb{C}$. After a short calculation we get

$$E = -z^2\partial - (t+1)z, \quad F = \partial, \quad H = 2z\partial + (t+1)$$

in this coordinate system. Analogously, on $\mathbb{P}^1 - \{0\}$ with the natural coordinate $\zeta([x_0, 1]) = x_0$, we have

$$E = \partial, \quad F = -\zeta^2\partial - (t+1)\zeta, \quad H = -2\zeta\partial - (t+1).$$

On $\mathbb{C}^*$ these two coordinate systems are clearly related by $\zeta = \frac{1}{z}$. This implies that $\partial_\zeta = -z^2\partial_z$, i. e., on $\mathbb{C}^*$ the second trivialization gives

$$E = -z^2\partial, \quad F = \partial - \frac{1+t}{z} \quad H = 2z\partial - (t+1).$$

Therefore, the first and the second trivialization on $\mathbb{C}^*$ are related by the automorphism of $\mathcal{D}_{\mathbb{C}^*}$ induced by

$$\partial \longrightarrow \partial - \frac{1+t}{z} = z^{1+t}\,\partial\,z^{-(1+t)}.$$

Now we want to analyze the standard Harish-Chandra sheaves attached to the open K-orbit $\mathbb{C}^*$. If we identify K with another copy of $\mathbb{C}^*$, the stabilizer in K of any point in the orbit $\mathbb{C}^*$ is the group M of n^{th} roots of 1. Let η_0 be the trivial representation of M, η_1 the identity representation of M, and $\eta_k = (\eta_1)^k$, $2 \le k \le n-1$, the remaining irreducible representations of the cyclic group M. Denote by τ_k the irreducible K-equivariant connection on $\mathbb{C}^*$ corresponding to the representation η_k of M, and by $\mathcal{I}(\mathbb{C}^*, \tau_k)$ the corresponding standard Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. To analyze these $\mathcal{D}_\lambda$-modules it is convenient to introduce a trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^* = \mathbb{P}^1 - \{0, \infty\}$ such that H corresponds to the differential operator $2z\partial$ on the orbit $\mathbb{C}^*$ and $T \cong \mathbb{C}^*$ acts on it by multiplication. We obtain this trivialization by restricting the original z-trivialization to $\mathbb{C}^*$ and twisting it by the automorphism

$$\partial \longrightarrow \partial - \frac{1+t}{2z} = z^{\frac{1+t}{2}}\,\partial\,z^{-\frac{1+t}{2}}.$$

This gives a trivialization of $\mathcal{D}_\lambda|\mathbb{C}^*$ which satisfies

$$E = -z^2\partial - \frac{1+t}{2}z, \quad F = \partial - \frac{1+t}{2z}, \quad H = 2z\partial.$$

The global sections of τ_k on $\mathbb{C}^*$ form the linear space spanned by functions $z^{p+\frac{k}{n}}$, $p \in \mathbb{Z}$. To analyze irreducibility of the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(\mathbb{C}^*, \tau_k)$ we have to study its behavior at 0 and ∞. By the preceding discussion, if we use the z-trivialization of $\mathcal{D}_\lambda$ on $\mathbb{C}^*$, $\mathcal{I}(\mathbb{C}^*, \tau_k)$ looks like the $\mathcal{D}_{\mathbb{C}}$-module which is the direct image of the $\mathcal{D}_{\mathbb{C}^*}$-module generated by $z^{\frac{k}{n}-\frac{1+t}{2}}$. This module is clearly reducible if and only if it contains functions regular at the origin, i. e. if and only if $\frac{k}{n}-\frac{1+t}{2}$ is an integer. Analogously, $\mathcal{I}(\mathbb{C}^*,\tau_k)|\mathbb{P}^1-\{0\}$ is reducible if and only if $\frac{k}{n}+\frac{1+t}{2}$ is an integer. Therefore, $\mathcal{I}(\mathbb{C}^*,\tau_k)$ is irreducible if and only if neither $\frac{k}{n}-\frac{1+t}{2}$ nor $\frac{k}{n}+\frac{1+t}{2}$ is an integer.

We can summarize this as follows.

1.1. Lemma. *Let K be the n-fold covering of T, $k\in\{0,1,\dots,n-1\}$, and $\lambda\in\mathfrak{h}^*$. Then the following conditions are equivalent:*

(i) $\alpha\check{}(\lambda)\notin\left\{\frac{2k}{n},-\frac{2k}{n}\right\}+2\mathbb{Z}+1$;

(ii) *the standard module $\mathcal{I}(\mathbb{C}^*,\tau_k)$ is irreducible.*

2. The irreducibility theorem

First we shall formulate the irreducibility result precisely. To do this we must analyze in detail the parametrization of K-homogeneous connections compatible with $\lambda+\rho\in\mathfrak{h}^*$ on a K-orbit Q.

Fix $x\in Q$ and denote by B_x the Borel subgroup of G with Lie algebra $\mathfrak{b}_x$. Then $S_x=\varphi^{-1}(\varphi(K)\cap B_x)$ is the stabilizer of x in K. The Borel subalgebra $\mathfrak{b}_x$ contains a σ-stable Cartan subalgebra $\mathfrak{c}$ and all such Cartan subalgebras in $\mathfrak{b}_x$ are S_x-conjugate [7]. Therefore, Q determines a unique K-conjugacy class of σ-stable Cartan subalgebras of $\mathfrak{g}$.

The involution σ defines an involution on the root system R in $\mathfrak{c}^*$, and its pull-back by the specialization map is an involution σ_Q on the root system Σ which depends only on Q. Therefore we can divide the roots in Σ in the following groups: $\Sigma_{Q,I}=\{\alpha\in\Sigma\,|\,\sigma_Q\alpha=\alpha\}$ – Q-imaginary roots

$\Sigma_{Q,\mathbb{R}}=\{\alpha\in\Sigma\,|\,\sigma_Q\alpha=-\alpha\}$ – Q-real roots,

$\Sigma_{Q,\mathbb{C}}=\Sigma-(\Sigma_{Q,I}\cup\Sigma_{Q,\mathbb{R}})$ – Q-complex roots.

The Lie algebra $\mathfrak{s}_x=\mathfrak{k}\cap\mathfrak{b}_x$ of S_x is the semidirect product of $\mathfrak{t}=\mathfrak{k}\cap\mathfrak{c}$ with the nilpotent radical $\mathfrak{u}_x=\mathfrak{k}\cap\mathfrak{n}_x$ of $\mathfrak{s}_x$. Let U_x be the unipotent subgroup of K corresponding to $\mathfrak{u}_x$; it is the unipotent radical of S_x. Let C be the torus in G corresponding to $\mathfrak{c}$. Put $T=\varphi^{-1}(\varphi(K)\cap C)$. Then S_x is the semidirect product of T with U_x. A K-homogeneous connection τ on Q compatible with $\lambda+\rho$ determines a finite-dimensional algebraic representation ω of S_x on the geometric fibre $T_x(\tau)$ of τ at x. This representation is trivial on U_x, hence it can be viewed as a representation of the group T. The differential of the representation ω, considered as a representation of $\mathfrak{t}$, is a direct sum of a finite number of copies of the one dimensional representation defined by the restriction of the specialization of $\lambda+\rho$ to $\mathfrak{t}$. What remains to be described is the action of the other components of S_x. The information relevant for determination of irreducibility of standard Harish-Chandra sheaves is determined by the action of the elements which will be described now.

Let α be a Q-real root. Denote by $\mathfrak{s}_\alpha$ the three-dimensional simple algebra generated by the root subspaces corresponding to α and $-\alpha$. Let S_α be the connected subgroup of G with Lie algebra $\mathfrak{s}_\alpha$; it is isomorphic either to $\mathrm{SL}(2,\mathbb{C})$ or to $\mathrm{PSL}(2,\mathbb{C})$. Denote by H_α the element of $\mathfrak{s}_\alpha\cap\mathfrak{c}$ such that $\alpha(H_\alpha)=2$. Then $m_\alpha=\exp(\pi iH_\alpha)\in G$ satisfies $m_\alpha^2=1$. Moreover, $\sigma(m_\alpha)=\exp(-\pi iH_\alpha)=m_\alpha^{-1}=m_\alpha$. Clearly, $m_\alpha=1$ if $S_\alpha\cong\mathrm{PSL}(2,\mathbb{C})$, and

$m_\alpha \neq 1$ if $S_\alpha \cong \mathrm{SL}(2,\mathbb{C})$—in the latter case m_α corresponds to the negative of the identity matrix in $\mathrm{SL}(2,\mathbb{C})$. Let $\mathfrak{k}_\alpha = \mathfrak{s}_\alpha \cap \mathfrak{k}$; it is the Lie algebra of a one dimensional torus K_α in K. Its image $\varphi(K_\alpha)$ in G is a torus in S_α. Therefore, $m_\alpha \in \varphi(K_\alpha)$. The composition of $\varphi : K_\alpha \longrightarrow S_\alpha$ and the covering projection $S_\alpha \longrightarrow \mathrm{Int}(\mathfrak{s}_\alpha)$ is an n-fold covering map between two one dimensional tori. If we identify K_α with $\mathbb{C}^*$, the kernel of this map is isomorphic to $\{e^{\frac{2\pi i p}{n}} \mid 0 \leq p \leq n-1\}$. Let n_α correspond to $e^{\frac{2\pi i}{n}}$ under this isomorphism (there are two possible choices for n_α and they are inverses of each other). Then φ maps n_α to m_α, hence n_α lies in T.

Let

$$D_-(Q) = \{\beta \in \Sigma^+ \cap \Sigma_{Q,\mathbb{C}} \mid -\sigma_Q\beta \in \Sigma^+\}.$$

Then $D_-(Q)$ is the union of $-\sigma_Q$-orbits consisting pairs $\{\beta, -\sigma_Q\beta\}$. Let A be a set of representatives of $-\sigma_Q$-orbits in $D_-(Q)$. Then, for arbitrary Q-real root α, the number

$$\delta_Q(m_\alpha) = \prod_{\beta \in A} e^\beta(m_\alpha)$$

is independent of the choice of A and equal to ± 1.

Following B. Speh and D. Vogan [12][1], we say that τ satisfies the SL_2-*parity condition with respect to the Q-real root* α if the spectrum of the linear transformation $\omega(n_\alpha)$ does not contain $-e^{\pm i\pi\alpha\check{}(\lambda)}\delta_Q(\varphi(n_\alpha))$. Since n_α is determined up to inversion, this condition does not depend on the choice of n_α. Clearly, this condition specializes to the condition of 1.1.(i) in our basic example.

Let Σ_α be the smallest σ_Q-invariant closed root subsystem of Σ containing α. Then $\Sigma_\alpha \cap \Sigma^+$ is a set of positive roots in Σ_α which contains α and $-\sigma_Q\alpha$. Put

$$C_-(Q) = \{\alpha \in D_-(Q) \mid \alpha \text{ is minimal in } \{\alpha, -\sigma_Q\alpha\} \text{ with respect to the ordering of } \Sigma_\alpha\}.$$

Then $C_-(Q)$ contains at least one representative of each $-\sigma_Q$-orbit in $D_-(Q)$. Finally, let

$$\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha\check{}(\lambda) \in \mathbb{Z}\}$$

be the root subsystem of Σ consisting of all roots integral with respect to λ. We can now state the main result of this paper.

2.1. Theorem. *Let Q be a K-orbit in X, λ an element of $\mathfrak{h}^*$, and τ an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Then the following conditions are equivalent:*

(i) *$C_-(Q) \cap \Sigma_\lambda = \emptyset$, and τ satisfies the* SL_2*-parity condition with respect to every Q-real root in Σ; and*
(ii) *the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(Q,\tau)$ is irreducible.*

Let $\tilde{G}$ be a covering of G and σ the involution of $\tilde{G}$ determined by the involution σ of $\mathfrak{g}$. Assume that K is the fixed point set of σ in $\tilde{G}$. Then we say that the pair $(\mathfrak{g}, K)$ is *linear*. In this case we have a slightly simpler criterion, which is equivalent to [12].

[1] In fact, they consider the reducibility condition, while ours is the irreducibility condition.

2.2. Corollary. *Assume that $(\mathfrak{g}, K)$ is a linear pair. Let Q be a K-orbit in X, λ an element of $\mathfrak{h}^*$, and τ an irreducible K-homogeneous connection on Q compatible with $\lambda+\rho$. Then the following conditions are equivalent:*

(i) *$D_-(Q) \cap \Sigma_\lambda = \emptyset$, and τ satisfies the SL_2-parity condition with respect to every Q-real root in Σ; and*
(ii) *the standard $\mathcal{D}_\lambda$-module $\mathcal{I}(Q, \tau)$ is irreducible.*

In the next example we show that, for a pair $(\mathfrak{g}, K)$ which is not linear, the first condition of 2.2 can fail for an irreducible standard module.

2.3. Example. Let G_0 be the universal cover of $\mathrm{SL}(3, \mathbb{R})$, $\mathfrak{g}$ its complexified Lie algebra, K the complexification of a maximal compact subgroup of G_0, and σ the corresponding Cartan involution. Then σ acts on the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ of all 3×3 complex matrices of trace zero by $\sigma(A) = -A^t$, where A^t is the transpose of the matrix A.

There are two K-conjugacy classes of σ-stable Cartan subalgebras in $\mathfrak{g}$: (1) the "split" class consisting of Cartan subalgebras on which σ acts as -1, which is represented by the subalgebra of all diagonal matrices in $\mathfrak{g}$, and (2) the "fundamental" class consisting of Cartan subalgebras on which σ acts as a reflection, represented by the Cartan subalgebra

$$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} \,\middle|\, a, b \in \mathbb{C} \right\}.$$

The flag variety X of $\mathfrak{g}$ is three dimensional, and the only K-orbit attached to the "split" class of σ-stable Cartan subalgebras is the open orbit Q_o.

For any fundamental σ-stable Cartan subalgebra $\mathfrak{c}$, the root system of $(\mathfrak{g}, \mathfrak{c})$ consists of two imaginary roots and four complex roots. Let W_K be the subgroup of the Weyl group of $(\mathfrak{g}, \mathfrak{c})$ consisting of elements induced by the elements of K. Then the order of W_K is equal to 2, and the only nontrivial element of W_K acts as $-\sigma$ on $\mathfrak{c}$. It follows that there are three K-orbits attached to the "fundamental" class of σ-stable Cartan subalgebras. The closed orbit C, which is one dimensional, has the property that two simple roots in Σ^+ are permuted by σ_C. Therefore, both of them are C-complex and their sum is C-imaginary. The remaining two orbits are two dimensional. They correspond to the cases where one simple root is Q-imaginary.

Let x be a point in one of the K-orbits attached to the "fundamental" conjugacy class of σ-stable Cartan subalgebras. Then the stabilizer S_x of that point in K is connected. Therefore, every irreducible algebraic representation of S_x is one dimensional and completely determined by its differential.

Let Q be one of the two dimensional orbits. Let α be the Q-complex simple root and β the Q-imaginary simple root. Then $\gamma = \alpha + \beta$ is the other Q-complex positive root, and $\sigma_Q(\alpha) = -\gamma$. Therefore, $D_-(Q) = \{\alpha, \gamma\}$, but $C_-(Q) = \alpha$.

By the preceding discussion, there is at most one irreducible K-homogeneous connection on Q compatible with $\lambda + \rho \in \mathfrak{h}^*$. It exists if and only if $\beta\check{}(\lambda) \in \frac{1}{2}\mathbb{Z}$. We denote the corresponding standard module by $\mathcal{I}(Q, \lambda)$. Evidently, the quantity $\gamma\check{}(\lambda) = \alpha\check{}(\lambda) + \beta\check{}(\lambda)$, can be an integer without $\alpha\check{}(\lambda)$ being an integer, i.e., $\mathcal{I}(Q,\lambda)$ can be irreducible for λ integral with respect to γ.

3. Intertwining functors

Let θ be a Weyl group orbit in $\mathfrak{h}^*$. We consider the derived category $D^b(\mathcal{U}_\theta)$ of bounded complexes of $\mathcal{U}_\theta$-modules. For each $\lambda \in \mathfrak{h}^*$ we also consider the derived category $D^b(\mathcal{D}_\lambda)$ of bounded complexes of (quasi-coherent) $\mathcal{D}_\lambda$-modules. The derived functor $R\Gamma$ of the functor of global sections Γ maps $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{U}_\theta)$ since its right cohomological dimension is $\leq \dim X$. If λ is regular, this functor is an equivalence of categories [2]. On the other hand, the localization functor $L\Delta_\lambda$, defined by

$$L\Delta_\lambda(V^{\cdot}) = \mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\theta} V^{\cdot}, \quad V^{\cdot} \in D^b(\mathcal{U}_\theta),$$

maps $D^b(\mathcal{U}_\theta)$ into the derived category $D^-(\mathcal{D}_\lambda)$ of complexes bounded from below for arbitrary λ. If λ is regular, the left cohomological dimension of Δ_λ is finite, and $L\Delta_\lambda$ defines a quasi-inverse of $R\Gamma$. This implies, in particular, that for any two $\lambda, \mu \in \theta$, the categories $D^b(\mathcal{D}_\lambda)$ and $D^b(\mathcal{D}_\mu)$ are equivalent. This equivalence is given by the functor $L\Delta_\mu \circ R\Gamma$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_\mu)$. We now describe another functor, defined in geometric terms, which is (under certain conditions) isomorphic to $L\Delta_\mu \circ R\Gamma$. This is the intertwining functor of Beilinson and Bernstein ([2], [3]; for compete details see [9]).

Define the action of $G = \mathrm{Int}(\mathfrak{g})$ on $X \times X$ by

$$g \cdot (x, x') = (g \cdot x, g \cdot x')$$

for $g \in G$ and $(x, x') \in X \times X$. The G-orbits in $X \times X$ can be parametrized in the following way. First we introduce a relation between Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two Borel subalgebras in $\mathfrak{g}$. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \mathfrak{b}'$. Denote by R the root system of $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$ and by R^+ the set of positive roots determined by $\mathfrak{b}$. This determines a specialization of the Cartan triple $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R^+)$. On the other hand, $\mathfrak{b}'$ determines another set of positive roots in R, which corresponds via this specialization to $w(\Sigma^+)$ for some uniquely determined $w \in W$. The element $w \in W$ doesn't depend on the choice of $\mathfrak{c}$, and we say that $\mathfrak{b}'$ is in *relative position* w with respect to $\mathfrak{b}$.

Let

$$Z_w = \{(x, x') \in X \times X \mid \mathfrak{b}_{x'} \text{ is in the relative position } w \text{ with respect to } \mathfrak{b}_x\}$$

for $w \in W$. Then the map $w \longrightarrow Z_w$ is a bijection of W onto the set of G-orbits in $X \times X$, hence the sets Z_w, $w \in W$, are smooth subvarieties of $X \times X$.

Denote by p_1 and p_2 the projections of Z_w onto the first and second factor in $X \times X$, respectively. Let $\Omega_{Z_w|X}$ be the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \longrightarrow X$. Let $\mathcal{T}_w$ be its inverse. The twisted sheaves of differential operators $\mathcal{D}_{w\lambda}$ and $\mathcal{D}_\lambda$, "pulled back" to Z_w by the projections p_1 and p_2

respectively, determine twisted sheaves $(\mathcal{D}_{w\lambda})^{p_1}$ and $\mathcal{D}_\lambda^{p_2}$ on Z_w ([5], A.1). It is easy to check that they differ by a $\mathcal{T}_w$ twist, i.e.,

$$(\mathcal{D}_{w\lambda})^{p_1} = (\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}.$$

Since the morphism $p_2 : Z_w \longrightarrow X$ is a surjective submersion, the inverse image p_2^+ is an exact functor from the category $\mathcal{M}(\mathcal{D}_\lambda)$ of $\mathcal{D}_\lambda$-modules into $\mathcal{M}((\mathcal{D}_\lambda)^{p_2})$. Twisting by $\mathcal{T}_w$ defines an exact functor $\mathcal{V} \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V})$ from $\mathcal{M}(\mathcal{D}_\lambda)$ into $\mathcal{M}((\mathcal{D}_{w\lambda})^{p_1})$. Therefore, we have a functor $\mathcal{V}^\cdot \longrightarrow \mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot)$ from $D^b(\mathcal{D}_\lambda)$ into $D^b((\mathcal{D}_{w\lambda})^{p_1})$. By composing it with the direct image functor $Rp_{1+} : D^b((\mathcal{D}_{w\lambda})^{p_1}) \longrightarrow D^b(\mathcal{D}_{w\lambda})$, we get the functor from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_{w\lambda})$ given by the formula

$$LI_w(\mathcal{V}^\cdot) = Rp_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}^\cdot))$$

for any $\mathcal{V}^\cdot \in D^b(\mathcal{D}_\lambda)$. This is the left derived functor of the functor

$$I_w(\mathcal{V}) = R^0p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V}))$$

from $\mathcal{M}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{D}_{w\lambda})$. It is called *the intertwining functor* (attached to $w \in W$).

Moreover, we have the following basic fact.

3.1. Proposition. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then LI_w is an equivalence of the category $D^b(\mathcal{D}_\lambda)$ with $D^b(\mathcal{D}_{w\lambda})$.*

Intertwining functors satisfy a natural "product formula." It allows the reduction of the analysis of intertwining functors to the ones attached to simple reflections.

3.2. Proposition. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then, for any $\lambda \in \mathfrak{h}^*$, the functors $LI_{w'} \circ LI_w$ and $LI_{w'w}$ from $D^b(\mathcal{D}_\lambda)$ into $D^b(\mathcal{D}_{w'w\lambda})$ are isomorphic; in particular the functors $I_w \circ I_{w'}$ and $I_{w'w}$ from $\mathcal{M}(\mathcal{D}_\lambda)$ into $\mathcal{M}(\mathcal{D}_{w'w\lambda})$ are isomorphic.*

Let $\alpha \in \Sigma$. We say that $\lambda \in \mathfrak{h}^*$ is α-*antidominant* if $\alpha\check{}(\lambda)$ is not a strictly positive integer. For any $S \subset \Sigma^+$, we say that $\lambda \in \mathfrak{h}^*$ is S-*antidominant* if it is α-antidominant for all $\alpha \in S$. Put

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\alpha \in -\Sigma^+\} = \Sigma^+ \cap (-w^{-1}(\Sigma^+))$$

for any $w \in W$. The name of the functor LI_w comes from the following basic result of Beilinson and Bernstein to which we alluded before.

3.3. Theorem. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant and regular. Then LI_w is an equivalence of the category $D^b(\mathcal{D}_\lambda)$ with $D^b(\mathcal{D}_{w\lambda})$, isomorphic to $L\Delta_{w\lambda} \circ R\Gamma$.*

We also have the following estimate for the left cohomological dimension of the intertwining functors.

3.4. Proposition. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then the left cohomological dimension of I_w is less than or equal to* $\operatorname{Card}(\Sigma_w^+ \cap \Sigma_\lambda)$.

In particular, we have the following consequence which is critical for our argument.

3.5. Corollary. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be such that $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$. Then*

$$I_w : \mathcal{M}(\mathcal{D}_\lambda) \longrightarrow \mathcal{M}(\mathcal{D}_{w\lambda})$$

is an equivalence of categories and $I_{w^{-1}}$ is its quasi-inverse.

4. A sketch of the proof of the irreducibility theorem

The idea of the proof of the irreducibility theorem is simple. We shall show that, if a standard Harish-Chandra sheaf is reducible, then there exists an intertwining functor which is an equivalence of categories and which maps the original sheaf into a standard Harish-Chandra sheaf for which the reducibility is obvious.

The standard Harish-Chandra sheaves attached to irreducible K-homogeneous connections on closed K-orbits are obviously irreducible. The analogous remark is all we need in the Verma module case—in this case the orbits in question are just Bruhat cells $C(w)$, $w \in W$, in X. Since the stabilizers of the unipotent radical N of a Borel subgroup of G are always connected, then for each Bruhat cell $C(w)$ and $\lambda \in \mathfrak{h}^*$ there exists a unique standard $\mathcal{D}_\lambda$-module $\mathcal{I}(w, \lambda)$ supported on the closure of $C(w)$. By a direct calculation [9],

$$I_{w^{-1}}(\mathcal{I}(1, w^{-1}\lambda)) = \mathcal{I}(w, \lambda).$$

If I_w satisfies the conditions of 3.5, it is an equivalence of categories and $\mathcal{I}(w, \lambda)$ is irreducible. A slightly more careful argument also implies the necessity of this condition.

In the case of Harish-Chandra modules we cannot reduce the argument to the case of a closed K-orbit. However, we can do the next best thing: we can reduce the argument to the orbits of minimal dimension attached to a particular K-conjugacy class of σ-stable Cartan subalgebras. The orbit Q has the minimal dimension among all K-orbits attached to a particular conjugacy class of σ-stable Cartan subalgebras if and only if the set $D_-(Q)$ is empty.

Assume that $D_-(Q)$ is not empty. Then it contains a simple root α, and this root must be in $C_-(Q)$. Let X_α be the flag variety of all parabolic subalgebras of type α in $\mathfrak{g}$. Denote by p_α the natural projection of X onto X_α. Then $p_\alpha^{-1}(p_\alpha(Q))$ is a K-invariant subset of X which is the union of two K-orbits: the orbit Q, and another orbit Q' which satisfies $\dim Q' = \dim Q - 1$. The orbit Q' is attached to the same conjugacy class of σ-stable Cartan subalgebras as Q, but

$$D_-(Q') = s_\alpha(D_-(Q) - \{\alpha, -\sigma_Q\alpha\}),$$

i.e., $\operatorname{Card} D_-(Q') = \operatorname{Card} D_-(Q) - 2$.

Let τ be an irreducible K-homogeneous connection on Q compatible with $\lambda + \rho$. Then there exists an irreducible K-homogeneous connection τ' on Q', compatible with $s_\alpha\lambda + \rho$, such that the following result holds.[2]

4.1. Lemma.

$$I_{s_\alpha}(\mathcal{I}(Q', \tau')) = \mathcal{I}(Q, \tau).$$

In addition, τ satisfies the SL_2*-parity condition with respect to a Q-real root β if and only if τ' satisfies the* SL_2*-parity condition with respect to the Q'-real root $s_\alpha\beta$.*

As was the case for the previous formula in the Verma module situation, this result is a straightforward consequence of the geometry of K-orbits and the base change [4]. If $\alpha\check{}(\lambda)$

[2] Compare with §6. of Schmid's lecture in this volume [11].

is not an integer, I_{s_α} is an equivalence of categories by 3.5, and $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q',\tau')$ is irreducible. On the other hand, if $\alpha\check{}(\lambda)$ is an integer, $\mathcal{I}(Q,\tau)$ contains an obvious submodule of local sections "which extend over Q'"—hence $\mathcal{I}(Q,\tau)$ is reducible, and we are done. This inductive argument allows us to eliminate $D_-(Q)$ completely and reduce the discussion to the case of standard Harish-Chandra sheaves attached to orbits of minimal dimension for a given conjugacy class of σ-stable Cartan subalgebras. This procedure is not unique; in some situations we can choose several different simple complex roots to do the reduction of dimension. Also, in each step we "lose" a pair of complex roots from $D_-(Q)$, and the integrality of λ with respect to one of them doesn't necessarily imply the integrality with respect to the other (compare 2.3). Fortunately, the smaller set $C_-(Q)$ has the property that it contains all "relevant" roots, and if it contains a pair $\{\alpha, -\sigma_Q\alpha\}$, the integrality with respect to one of them implies integrality with respect to the other if the SL_2-parity condition holds for all Q-real roots. On the contrary, in the linear case, the integrality of λ with respect to any Q-complex root α implies the integrality with respect to $\sigma_Q\alpha$. This is the reason why we could use the full set $D_-(Q)$ in the corollary to the main theorem.

This reduces the proof to the case of orbits of minimal dimension attached to a particular conjugacy class of σ-stable Cartan subalgebras. Let Q be such K-orbit. Since $D_-(Q)$ is empty in this situation, the set of all positive Q-complex roots is σ_Q-invariant. Therefore, the union of positive roots and Q-real roots is a σ_Q-stable parabolic set of roots, and it determines a generalized flag variety X_Θ for some subset Θ of the set of simple roots in Σ. Let p_Θ be the corresponding natural projection from X onto X_Θ. The projection $p_\Theta(Q)$ of the orbit Q to X_Θ is a closed K-orbit in X_Θ. Therefore, the inverse image $p_\Theta^{-1}(p_\Theta(Q))$ is a smooth closed subvariety in X invariant under the action of K. Let $F = p_\Theta^{-1}(p_\Theta(x))$ be a fibre of the projection p_Θ passing through the point $x \in Q$. Let $\mathfrak{p}$ be the parabolic subalgebra determined by $p_\Theta(x)$ and $\mathfrak{g}^\Theta$ the Levi factor of $\mathfrak{p}$ which contains the σ-stable Cartan subalgebra $\mathfrak{c}$. Clearly $\mathfrak{g}^\Theta$ is σ-stable. Let K^Θ be the centralizer of the center of $\mathfrak{g}^\Theta$ in K. Then K^Θ acts on $\mathfrak{g}^\Theta$ by automorphisms and its Lie algebra is identified with $\mathfrak{k}^\Theta = \mathfrak{k} \cap \mathfrak{g}^\Theta$. The map $\mathfrak{b} \longmapsto \mathfrak{b} \cap \mathfrak{g}^\Theta$ defines an isomorphism of F with the flag variety X^Θ of $\mathfrak{g}^\Theta$. The map $Q' \longmapsto Q' \cap F$ defines a bijection between the K-orbits in $p_\Theta^{-1}(p_\Theta(Q))$ and K^Θ-orbits in X^Θ. A more careful analysis shows that an appropriate derived functor of the "restriction" to F defines an equivalence of the full subcategory of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ consisting of modules supported in $p_\Theta^{-1}(p_\Theta(Q))$ with the corresponding category of $\mathcal{D}$-modules on X^Θ and this equivalence maps standard Harish-Chandra sheaves into standard Harish-Chandra sheaves.

Since $Q \cap F$ is dense in F, this reduces the proof to the case of standard modules attached to the open K-orbit Q in the flag variety X. Also, Q is attached to a conjugacy class of Cartan subalgebras on which σ acts as -1, i.e., all roots in Σ are Q-real. Let τ be a K-homogeneous connection on Q. The stabilizer S_x of a point $x \in Q$ is finite. The intersection $\mathfrak{b}_x \cap \sigma(\mathfrak{b}_x)$ is equal to a σ-stable Cartan subalgebra $\mathfrak{c}$ on which σ acts as -1. Let α be a simple root. Then there exists the unique Borel subalgebra $\mathfrak{b}_y$, $y \in X$, containing $\mathfrak{c}$ and in relative position s_α with respect to $\mathfrak{b}_x$. The point y is contained in Q and its stabilizer S_y is equal to S_x. Therefore, the representation ω of S_x in $T_x(\tau)$

determines another K-homogeneous connection τ_α on Q such that the representation of $S_y = S_x$ in $T_y(\tau_\alpha)$ is equal to ω. The next lemma follows again from the analysis of the SL_2-situation and the base change.

4.2. Lemma. *Assume that τ satisfies the SL_2-parity condition with respect to α. Then*

$$I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau_\alpha).$$

If we assume in addition that $p = -\alpha\check{}(\lambda)$ is an integer,

$$I_{s_\alpha}(\mathcal{I}(Q,\tau)) = \mathcal{I}(Q,\tau)(p\alpha) = \mathcal{I}(Q,\tau) \otimes_{\mathcal{O}_X} \mathcal{O}(p\alpha).$$

Moreover, if τ satisfies the SL_2-parity condition with respect to a root β, τ_α satisfies the SL_2-parity condition with respect to $s_\alpha\beta$.

Therefore, if $\alpha\check{}(\lambda)$ is not an integer, I_{s_α} is an equivalence of categories by 3.5, and $\mathcal{I}(Q,\tau)$ is irreducible if and only if $\mathcal{I}(Q,\tau_\alpha)$ is irreducible. On the other hand, if $\alpha\check{}(\lambda)$ is an integer, $\mathcal{I}(Q,\tau_\alpha) = \mathcal{I}(Q,\tau)(p\alpha)$ and the same assertion is obvious. Therefore, to check irreducibility, we can freely "move around" λ by the action of the Weyl group.

Assume that the SL_2-parity condition fails for some root β. Then, by applying the intertwining functors, we can assume that it fails for a simple root. In this case the reducibility is obvious: the standard Harish-Chandra sheaf has a submodule of local sections which extend over a K-orbit of codimension one in X. This proves the necessity of the condition. The sufficiency is equally simple. If the parity condition holds for all roots and $\mathcal{I}(Q,\tau)$ is reducible, then $\mathcal{I}(Q,\tau)$ has a nontrivial quotient supported on a closed subvariety of X of codimension ≥ 1. By applying the intertwining functors we can decrease the codimension of the support of this quotient until it reaches 1 [2]. Let Q' be a K-orbit of codimension one in X contained in the support of this quotient. In this case there exists a simple root α such that $p_\alpha^{-1}(p_\alpha(Q'))$ contains the open orbit Q. For an arbitrary point $y \in Q'$, its fibre $F = p_\alpha^{-1}(p_\alpha(y))$ is isomorphic to $\mathbb{P}^1$—the flag variety of $\mathfrak{sl}(2,\mathbb{C})$—and $Q \cap F$ corresponds to $\mathbb{C}^*$. On the other hand, $Q' \cap F$ corresponds to either $\{0\}$ or $\{0,\infty\}$. Since the restriction to F of a standard Harish-Chandra sheaf attached to Q is a standard Harish-Chandra sheaf on $\mathbb{P}^1$ of the type we discussed in 1, it is irreducible by 1.1. But this contradicts the fact that it should have a quotient supported in $\{0,\infty\}$. This completes the proof of the main theorem.

Finally, we would like to make a remark about the relationship of our result with the main result of Speh and Vogan [12]. Their result gives a necessary and sufficient condition for irreducibility of the principal series representations for regular infinitesimal characters and leaves the singular case open. The reason for this is that the Beilinson-Bernstein equivalence of categories fails for singular infinitesimal characters—there exist $\mathcal{D}_\lambda$-modules with no cohomology at all! This allows global sections of reducible standard Harish-Chandra sheaves to be irreducible in some cases. In the case of irreducible unitary principal series representations, I. Mirković proved that they are always global sections of some irreducible standard Harish-Chandra sheaf [10]; this explains relative simplicity of the tempered spectrum of semisimple Lie groups.

A completely analogous argument for irreducibility of standard modules works for the category of generalized Verma modules [9] and the category of Whittaker modules [8].

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