\[ \sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!} (x - \frac{\pi}{2})^k \]

\[ = 1 - \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} - \frac{(x - \pi/2)^6}{6!} + \ldots \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{(x - \pi/2)^{2n}}{(2n)!} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left[ \frac{|x - \pi/2|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x - \pi/2|^{2n}} \right] \]

\[ = \lim_{n \to \infty} \frac{|x - \pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x, \text{ so } R = \infty. \]
20. If \( f(x) = \sin x \), then \( f^{(n+1)}(x) = \pm \sin x \) or \( \pm \cos x \). In each case, \( |f^{(n+1)}(x)| \leq 1 \), so by Formula 9 with \( a = 0 \) and \( M = 1 \), \( |R_n(x)| \leq \frac{1}{(n + 1)!} \left| x - \frac{\pi}{2} \right|^{n+1} \). Thus, \( |R_n(x)| \to 0 \) as \( n \to \infty \) by Equation 10. So \( \lim_{n \to \infty} R_n(x) = 0 \) and, by Theorem 8, the series in Exercise 16 represents \( \sin x \) for all \( x \).
44. \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) \( \Rightarrow \) \( e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \) \( \Rightarrow \) \( \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \) \( \Rightarrow \) \( \int \frac{e^x - 1}{x} \, dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \),

with \( R = \infty \).
\[
\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{1 - \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots\right)}{1 + x - \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \frac{1}{6!} x^6 + \ldots\right)}
\]

\[
= \lim_{x \to 0} \frac{\frac{1}{2!} x^2 - \frac{1}{4!} x^4 + \frac{1}{6!} x^6 - \ldots}{\frac{1}{2!} x^2 - \frac{1}{3!} x^3 - \frac{1}{4!} x^4 - \frac{1}{5!} x^5 - \frac{1}{6!} x^6 - \ldots}
\]

\[
= \lim_{x \to 0} \frac{\frac{1}{2!} - \frac{1}{4!} x^2 + \frac{1}{6!} x^4 - \ldots}{\frac{1}{2!} - \frac{1}{3!} x - \frac{1}{4!} x^2 - \frac{1}{5!} x^3 - \frac{1}{6!} x^4 - \ldots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1
\]

since power series are continuous functions.
(a) \( f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

so \( f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{x \to 0} \frac{1}{e^{1/x^2}} = \lim_{x \to 0} \frac{x}{2e^{1/x^2}} = 0 \)

(using l’Hospital’s Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l’Hospital’s Rule to show that \( f''(0) = 0, f^{(3)}(0) = 0, \ldots, f^{(n)}(0) = 0 \), so that the Maclaurin series for \( f \) consists entirely of zero terms. But since \( f(x) \equiv 0 \) except for \( x = 0 \), we see that \( f \) cannot equal its Maclaurin series except at \( x = 0 \).
From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be “infinitely flat” at $x = 0$, since all of its derivatives are 0 there.
24. \( \cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots \). By the Alternating Series Estimation Theorem, the error is less than \( \left| -\frac{1}{6!} x^6 \right| < 0.005 \) \( \iff \)

\[ x^6 < 720(0.005) \iff |x| < (3.6)^{1/6} \approx 1.238. \] The curves

\( y = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 \) and \( y = \cos x + 0.005 \) intersect at \( x \approx 1.244 \), so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for \( x > 0 \). Thus, the desired range of values for \( x \) is \( -1.238 < x < 1.238 \).
The linear approximation is

$$T_1(t) = \rho(20) + \rho'(20)(t - 20) = \rho_{20}[1 + \alpha(t - 20)]$$

The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t - 20) + \frac{\rho''(20)}{2} (t - 20)^2$$

$$= \rho_{20}[1 + \alpha(t - 20) + \frac{1}{2} \alpha^2(t - 20)^2]$$

From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \leq T_1(t) \leq 1.01\rho(t)$, for $-14^\circ C \leq t \leq 58^\circ C$. 