6. \( f(x, y) = x^3y + 12x^2 - 8y \Rightarrow f_x = 3x^2y + 24x, \)
\[ f_y = x^3 - 8, \quad f_{xx} = 6xy + 24, \quad f_{xy} = 3x^2, \quad f_{yy} = 0. \]

Then \( f_y = 0 \) implies \( x = 2 \), and substitution into \( f_x = 0 \) gives
\[ 12y + 48 = 0 \Rightarrow y = -4. \]
Thus, the only critical point is \((2, -4)\).

\[ D(2, -4) = (-24)(0) - 12^2 = -144 < 0, \] so \((2, -4)\) is a saddle point.
12. \( f(x, y) = y \cos x \) \( \Rightarrow \) \( f_x = -y \sin x, f_y = \cos x, f_{xx} = -y \cos x, \)
\( f_{xy} = -\sin x, f_{yy} = 0. \) Then \( f_y = 0 \) if and only if \( x = \frac{\pi}{2} + n\pi \) for \( n \) an integer. But \( \sin \left( \frac{\pi}{2} + n\pi \right) \neq 0, \) so \( f_x = 0 \) \( \Rightarrow \) \( y = 0 \) and the critical points are \( \left( \frac{\pi}{2} + n\pi, 0 \right), \) \( n \) an integer.

\( D \left( \frac{\pi}{2} + n\pi, 0 \right) = (0)(0) - (\pm1)^2 = -1 < 0, \) so each critical point is a saddle point.
28. Since \( f \) is a polynomial it is continuous on \( D \), so an absolute maximum and minimum exist. \( f_x = y - 1, f_y = x - 2 \), and setting \( f_x = f_y = 0 \) gives \((2, 1)\) as the only critical point, where \( f(2, 1) = 1 \). Along \( L_1: x = 1 \) and \( f(1, y) = 2 - y \) for \( 0 \leq y \leq 4 \), a decreasing function in \( y \), so the maximum value is \( f(1, 0) = 2 \) and the minimum value is \( f(1, 4) = -2 \). Along \( L_2: y = 0 \) and \( f(x, 0) = 3 - x \) for \( 1 \leq x \leq 5 \), a decreasing function in \( x \), so the maximum value is \( f(1, 0) = 2 \) and the minimum value is \( f(5, 0) = -2 \). Along \( L_3: y = 5 - x \) and

\[
f(x, 5 - x) = -x^2 + 6x - 7 = -(x - 3)^2 + 2\]

for \( 1 \leq x \leq 5 \), which has a maximum at \( x = 3 \) where \( f(3, 2) = 2 \) and a minimum at both \( x = 1 \) and \( x = 5 \), where \( f(1, 4) = f(5, 0) = -2 \). Thus the absolute maximum of \( f \) on \( D \) is \( f(1, 0) = f(3, 2) = 2 \) and the absolute minimum is \( f(1, 4) = f(5, 0) = -2 \).
36. Here the distance $d$ from a point on the plane to the point $(1, 2, 3)$ is 

$$d = \sqrt{(x - 1)^2 + (y - 2)^2 + (z - 3)^2},$$

where $z = 4 - x + y$. We can minimize $d^2 = f(x, y) = (x - 1)^2 + (y - 2)^2 + (1 - x + y)^2$, so

$$f_x(x, y) = 2(x - 1) + 2(1 - x + y)(-1) = 4x - 2y - 4 \quad \text{and} \quad f_y(x, y) = 2(y - 2) + 2(1 - x + y) = 4y - 2x - 2.$$

Solving $4x - 2y - 4 = 0$ and $4y - 2x - 2 = 0$ simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$.

This point must correspond to the minimum distance, so the point on the plane closest to $(1, 2, 3)$ is $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$. 
4. \( f(x, y) = 4x + 6y, \ g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = (4, 6), \lambda \nabla g = (2\lambda x, 2\lambda y). \) Then \( 2\lambda x = 4 \) and \( 2\lambda y = 6 \) imply \( x = \frac{2}{\lambda} \) and \( y = \frac{3}{\lambda} \). But \( 13 = x^2 + y^2 = \left( \frac{2}{\lambda} \right)^2 + \left( \frac{3}{\lambda} \right)^2 \Rightarrow 13 = \frac{13}{\lambda^2} \Rightarrow \lambda = \pm 1, \) so \( f \) has possible extreme values at the points \((2, 3),(2, 3)\) and \((-2, -3),(-2, -3)\). We compute \( f(2, 3) = 26 \) and \( f(-2, -3) = -26 \), so the maximum value of \( f \) on \( x^2 + y^2 = 13 \) is \( f(2, 3) = 26 \) and the minimum value is \( f(-2, -3) = -26 \).
20. (a) \( f(x, y) = 2x + 3y \), \( g(x, y) = \sqrt{x} + \sqrt{y} = 5 \) \( \Rightarrow \) \( \nabla f = (2, 3) = \lambda \nabla g = \lambda \left( \frac{1}{2 \sqrt{x}}, \frac{1}{2 \sqrt{y}} \right) \). Then

\[
2 = \frac{\lambda}{2 \sqrt{x}} \quad \text{and} \quad 3 = \frac{\lambda}{2 \sqrt{y}} \quad \text{so} \quad 4\sqrt{x} = \lambda = 6\sqrt{y} \quad \Rightarrow \quad \sqrt{y} = \frac{2}{3} \sqrt{x}.
\]

With \( \sqrt{x} + \sqrt{y} = 5 \) we have \( \sqrt{x} + \frac{2}{3} \sqrt{x} = 5 \) \( \Rightarrow \)

\( \sqrt{x} = 3 \) \( \Rightarrow \) \( x = 9 \). Substituting into \( \sqrt{y} = \frac{2}{3} \sqrt{x} \) gives \( \sqrt{y} = 2 \) or \( y = 4 \). Thus the only possible extreme value subject to the constraint is \( f(9, 4) = 30 \). (The question remains whether this is indeed the maximum of \( f \).)

(b) \( f(25, 0) = 50 \) which is larger than the result of part (a).

(c) We can see from the level curves of \( f \) that the maximum occurs at the left endpoint \( (0, 25) \) of the constraint curve \( g \).

The maximum value is \( f(0, 25) = 75 \).

(d) Here \( \nabla g \) does not exist if \( x = 0 \) or \( y = 0 \), so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of \( f \) share a common tangent line with the constraint curve \( g \).

This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

(e) Here \( f(9, 4) \) is the absolute \emph{minimum} of \( f \) subject to \( g \).

We can find \( \lambda \) and \( (x, y) \) by solving the system of equations:

\[
\begin{align*}
2 &= \frac{\lambda}{2 \sqrt{x}} \\
3 &= \frac{\lambda}{2 \sqrt{y}} \\
\sqrt{x} + \sqrt{y} &= 5
\end{align*}
\]

This gives \( \lambda = 4 \sqrt{xy} \) and \( \sqrt{x} = 2 \), \( \sqrt{y} = 3 \) so \( x = 4 \) and \( y = 9 \) giving \( f(4, 9) = 30 \). Therefore, \( f(9, 4) = 30 \) is the absolute minimum.