44. This follows immediately from the result in Problem 43, because an invertible matrix is row-equivalent to the identity matrix.

45. One can simply photocopy the portion of the proof of Theorem 7 that follows Equation (20). Starting only with the assumption that A and B are square matrices with \( AB = I \), it is proved there that A and B are then invertible.

46. If \( C = AB \) is invertible, so \( C^{-1} \) exists, then \( A(BC^{-1}) = I \) and \( (C^{-1}A)B = I \). Hence the fact that A and B are invertible follows immediately from Problem 45.

SECTION 3.6

DETERMINANTS

1. \[
\begin{vmatrix}
0 & 0 & 3 \\
4 & 0 & 0 \\
0 & 5 & 0
\end{vmatrix} = +(3) \begin{vmatrix}
4 & 0 \\
0 & 5
\end{vmatrix} = 3 \cdot 4 \cdot 5 = 60
\]

2. \[
\begin{vmatrix}
2 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{vmatrix} = +(2) \begin{vmatrix}
2 & 1 \\
1 & 2
\end{vmatrix} - (1) \begin{vmatrix}
1 & 1 \\
0 & 2
\end{vmatrix} = 2(4 - 1) - (2 - 0) = 4
\]

3. \[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
2 & 0 & 5 & 0 \\
3 & 6 & 9 & 8 \\
4 & 0 & 10 & 7
\end{vmatrix} = +(1) \begin{vmatrix}
0 & 5 & 0 \\
6 & 9 & 8 \\
0 & 10 & 7
\end{vmatrix} = - (5) \begin{vmatrix}
6 & 8 \\
0 & 7
\end{vmatrix} = -5(42 - 0) = -210
\]

4. \[
\begin{vmatrix}
5 & 1 & 1 & 8 & 7 \\
3 & -2 & 6 & 23 & 0 \\
0 & 0 & 0 & -3 & 17
\end{vmatrix} = -(3) \begin{vmatrix}
5 & 11 & 8 \\
3 & -2 & 6 \\
0 & 4 & 0
\end{vmatrix} = 3(-4) \begin{vmatrix}
5 & 8 \\
3 & 6
\end{vmatrix} = -12(30 - 24) = -72
\]

5. \[
\begin{vmatrix}
0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 \\
0 & 5 & 0 & 0 & 0
\end{vmatrix} = +1 \begin{vmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 4 & 0 \\
0 & 5 & 0 & 0
\end{vmatrix} = +2 \begin{vmatrix}
0 & 3 & 0 \\
0 & 0 & 4 \\
5 & 0 & 0
\end{vmatrix} = 2(+5) \begin{vmatrix}
3 & 0 \\
0 & 4
\end{vmatrix} = 2 \cdot 5 \cdot 3 \cdot 4 = 120
\]
14. \[ \begin{vmatrix} 4 & 2 & -2 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & -5 \\ -5 & -4 & 3 \end{vmatrix} R_{1}^{3} R_{2}^{2} = 1 \begin{vmatrix} 1 & 1 & 3 \\ 0 & -2 & -14 \\ 0 & 1 & 18 \end{vmatrix} = 1 + 18 = -14 = -22 \]

15. \[ \begin{vmatrix} 2 & 4 & -2 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 13 & 14 \\ 5 & 3 & 1 \\ 1 & 4 & 5 \end{vmatrix} R_{1}^{1} R_{2}^{3} = 0 -17 -24 = 13 14 = -17 -24 = -74 \]

16. \[ \begin{vmatrix} 2 & 4 & -2 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 10 & 0 & -4 \\ -5 & -4 & -1 \\ -4 & 2 & 1 \end{vmatrix} R_{1}^{1} R_{2}^{3} = 10 0 1 = -2 10 -4 = -13 1 = 84 \]

17. \[ \begin{vmatrix} 2 & 3 & 3 \\ 0 & 4 & 3 \\ 2 & 3 & 3 \end{vmatrix} R_{3}^{3} R_{1} = 0 4 3 R_{3}^{1} = 2 4 3 -3 = 2 10 1 = 8 \]

18. \[ \begin{vmatrix} 1 & 4 & 4 \\ 0 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix} R_{3}^{3} R_{1} = 1 4 4 1 \begin{vmatrix} 1 & 4 & 4 \\ 0 & 1 & 2 \\ 3 & 3 & 1 \end{vmatrix} = 1 9 11 1 = 1 3 2 0 = 129 19 = 135 \]

19. \[ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -3 & 3 \end{vmatrix} R_{3}^{1} R_{1} = 1 0 0 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -3 & 3 \end{vmatrix} = C_{2}^{2} C_{1} = 3 4 9 = 39 \]

20. \[ \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & 4 & -2 \end{vmatrix} R_{2}^{2} R_{1}^{1} = 1 2 1 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & 4 & -2 \end{vmatrix} = 3 1 5 R_{3}^{1} R_{1} = 3 1 5 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & 4 & -2 \end{vmatrix} = 5 0 13 = 79 \]

21. \[ \Delta = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 1; \quad x = \frac{1 2 4}{\Delta} 1 7 = 10, \quad y = \frac{1 3 2}{\Delta} 5 1 = -7 \]

22. \[ \Delta = \begin{vmatrix} 5 & 8 \\ 8 & 13 \end{vmatrix} = 1; \quad x = \frac{1 3 8}{\Delta} 5 13 = -1, \quad y = \frac{1 5 3}{\Delta} 8 5 = 1 \]
5. \( \mathbf{v} = \frac{1}{2} \mathbf{u} \), so the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are linearly dependent.

6. \( a \mathbf{u} + b \mathbf{v} = a(0, 2) + b(3, 0) = (3b, 2a) = \mathbf{0} \) implies \( a = b = 0 \), so the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are linearly independent.

7. \( a \mathbf{u} + b \mathbf{v} = a(2, 2) + b(-2, 2) = (2a + 2b, 2a - 2b) = \mathbf{0} \) implies \( a = b = 0 \), so the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are linearly independent.

8. \( \mathbf{v} = -\mathbf{u} \), so the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are linearly dependent.

In each of Problems 9–14, we set up and solve (as in Example 2 of this section) the system

\[
\begin{align*}
\begin{bmatrix}
   u_1 \\
   u_2
\end{bmatrix}
\begin{bmatrix}
   v_1 \\
   v_2
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   w_1 \\
   w_2
\end{bmatrix}

\end{align*}
\]

so that \( \mathbf{w} = a \mathbf{u} + b \mathbf{v} \).

9. \[
\begin{bmatrix}
   1 \\
   -2
\end{bmatrix}
\begin{bmatrix}
   -1 \\
   3
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   1 \\
   0
\end{bmatrix}
\Rightarrow a = 3, b = 2 \text{ so } \mathbf{w} = 3 \mathbf{u} + 2 \mathbf{v}
\]

10. \[
\begin{bmatrix}
   3 \\
   4
\end{bmatrix}
\begin{bmatrix}
   2 \\
   3
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   0 \\
   -1
\end{bmatrix}
\Rightarrow a = 2, b = -3 \text{ so } \mathbf{w} = 2 \mathbf{u} - 3 \mathbf{v}
\]

11. \[
\begin{bmatrix}
   5 \\
   7
\end{bmatrix}
\begin{bmatrix}
   2 \\
   3
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   1 \\
   1
\end{bmatrix}
\Rightarrow a = 1, b = -2 \text{ so } \mathbf{w} = \mathbf{u} - 2 \mathbf{v}
\]

12. \[
\begin{bmatrix}
   4 \\
   1
\end{bmatrix}
\begin{bmatrix}
   -2 \\
   -1
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   2 \\
   -2
\end{bmatrix}
\Rightarrow a = 3, b = 5 \text{ so } \mathbf{w} = 3 \mathbf{u} + 5 \mathbf{v}
\]

13. \[
\begin{bmatrix}
   7 \\
   5
\end{bmatrix}
\begin{bmatrix}
   3 \\
   4
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   5 \\
   -2
\end{bmatrix}
\Rightarrow a = 2, b = -2 \text{ so } \mathbf{w} = 2 \mathbf{u} - 3 \mathbf{v}
\]

14. \[
\begin{bmatrix}
   5 \\
   -2
\end{bmatrix}
\begin{bmatrix}
   -6 \\
   4
\end{bmatrix}
\begin{bmatrix}
   a \\
   b
\end{bmatrix}

\begin{bmatrix}
   5 \\
   6
\end{bmatrix}
\Rightarrow a = 7, b = 5 \text{ so } \mathbf{w} = 7 \mathbf{u} + 5 \mathbf{v}
\]

In Problems 15–18, we calculate the determinant \( \det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \) so as to determine (using Theorem 4) whether the three vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are linearly dependent (\( \det = 0 \)) or linearly independent (\( \det \neq 0 \)).
The nontrivial solution \( a = 11, \ b = 4, \ c = -1 \) gives \( 11\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \mathbf{0} \), so the three vectors are linearly dependent.

22. \[ \mathbf{A} = \begin{bmatrix} 1 & 5 & 0 & \mathbf{1} & 0 & 0 \\ 1 & 1 & 1 & \mathbf{0} & 1 & 0 \\ 0 & 3 & 2 & \mathbf{0} & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \mathbf{E} \]

The system \( \mathbf{Ax} = \mathbf{0} \) has only the trivial solution \( a = b = c = 0 \), so the vectors \( \mathbf{u}, \ \mathbf{v}, \ \text{and} \ \mathbf{w} \) are linearly independent.

23. \[ \mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E} \]

The system \( \mathbf{Ax} = \mathbf{0} \) has only the trivial solution \( a = b = c = 0 \), so the vectors \( \mathbf{u}, \ \mathbf{v}, \ \text{and} \ \mathbf{w} \) are linearly independent.

24. \[ \mathbf{A} = \begin{bmatrix} 1 & 4 & -3 \\ 4 & 2 & 3 \\ 5 & 5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E} \]

The system \( \mathbf{Ax} = \mathbf{0} \) has only the trivial solution \( a = b = c = 0 \), so the vectors \( \mathbf{u}, \ \mathbf{v}, \ \text{and} \ \mathbf{w} \) are linearly independent.

In Problems 25–28, we solve the nonhomogeneous system \( \mathbf{Ax} = \mathbf{t} \) by reducing the augmented coefficient matrix \( \mathbf{A} = [\mathbf{u} \ \mathbf{v} \ \mathbf{w} \ \mathbf{t}] \) to echelon form \( \mathbf{E} \). The solution vector \( \mathbf{x} = [a \ b \ c]^T \) appears as the final column of \( \mathbf{E} \), and provides us with the desired linear combination \( \mathbf{t} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \).

25. \[ \mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & 0 & -1 & -7 \\ 2 & 1 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \mathbf{E} \]

Thus \( a = 2, \ b = -1, \ c = 3 \) so \( \mathbf{t} = 2\mathbf{u} - \mathbf{v} + 3\mathbf{w} \).

26. \[ \mathbf{A} = \begin{bmatrix} 5 & 1 & 5 & 5 \\ 2 & 5 & -3 & 30 \\ -2 & -3 & 4 & -21 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{E} \]

Thus \( a = 1, \ b = 5, \ c = -1 \) so \( \mathbf{t} = \mathbf{u} + 5\mathbf{v} - \mathbf{w} \).
so their sum \((x+u, y+v, z+w)\) is in \(V\). Similarly,

\[cz = c(2x+3y) = 2(cx)+3(cy),\]

so the scalar multiple \((cx, cy, cz)\) is in \(V\).

33. \((0,1,0)\) is in \(V\) but the sum \((0,1,0)+(0,1,0) = (0,2,0)\) is not in \(V\); thus \(V\) is not closed under addition. Alternatively, \(2(0,1,0) = (0,2,0)\) is not in \(V\), so \(V\) is not closed under multiplication by scalars.

34. \((1,1,1)\) is in \(V\), but

\[2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)\]

is not, so \(V\) is closed neither under addition of vectors nor under multiplication by scalars.

35. Evidently \(V\) is closed under addition of vectors. However, \((0,0,1)\) is in \(V\) but \((-1)(0,0,1) = (0,0,-1)\) is not, so \(V\) is not closed under multiplication by scalars.

36. \((1,1,1)\) is in \(V\), but

\[2(1,1,1) = (1,1,1)+(1,1,1) = (2,2,2)\]

is not, so \(V\) is closed neither under addition of vectors nor under multiplication by scalars.

37. Pick a fixed element \(u\) in the (nonempty) vector space \(V\). Then, with \(c = 0\), the scalar multiple \(cu = 0u = 0\) must be in \(V\). Thus \(V\) necessarily contains the zero vector \(0\).

38. Suppose \(u\) and \(v\) are vectors in the subspace \(V\) of \(R^3\) and \(a\) and \(b\) are scalars. Then \(au\) and \(bv\) are in \(V\) because \(V\) is closed under multiplication by scalars. But then it follows that the linear combination \(au+bv\) is in \(V\) because \(V\) is closed under addition of vectors.

39. It suffices to show that every vector \(v\) in \(V\) is a scalar multiple of the given nonzero vector \(u\) in \(V\). If \(u\) and \(v\) were linearly independent, then — as illustrated in Example 2 of this section — every vector in \(R^2\) could be expressed as a linear combination of \(u\) and \(v\). In this case it would follow that \(V\) is all of \(R^2\) (since, by Problem 38, \(V\) is closed under taking linear combinations). But we are given that \(V\) is a proper subspace of \(R^2\), so we must conclude that \(u\) and \(v\) are linearly dependent vectors. Since \(u \neq 0\), it follows that the arbitrary vector \(v\) in \(V\) is a scalar multiple of \(u\), and thus \(V\) is precisely the set of all scalar multiples of \(u\). In geometric language, the subspace \(V\) is then the straight line through the origin determined by the nonzero vector \(u\).