2.5.4

(1) Although it isn't 100% clear from the graph, we can assume the graph stays at \( y = 2 \) as \( x \to \infty \).

Thus, \( \lim_{x \to \infty} g(x) = 2 \).

b.) Similarly, \( \lim_{x \to \infty} g(x) = -2 \) as the graph seems to flatten out as we continue left (and beyond) of the graph.

c.) As \( x \to 3 \), the graph blows up, but matches on both sides, so

\[ \lim_{x \to 3} g(x) = +\infty. \]
d.) Similarly, as $x \to 0$, the graph goes to $-\infty$ from both sides, meaning we can conclude

$$\lim_{x \to 0} g(x) = -\infty.$$  

e.) From the right side of $-2$, the graph seems to become very negative. Therefore

$$\lim_{x \to -2^+} = -\infty.$$  

f.) There are only 4 asymptotes here, two vertical and two horizontal.

Vertical:

- $x = 3$
- $x = -2$

Notice $-2$ may confuse some people but we defined an asymptote to be if the limit on either side.
Since (a) (b) are true,

\[ y = 2 \quad \text{must be horizontal asymptote}, \]
\[ y = -2 \]

2.5 (b)

2.5. 16) \( \lim_{x \to -3^-} \frac{x+2}{x+3} \). This clearly describes a vertical asymptote at \( x = -3 \). Notice, to the left, it is cos. to the right, it is neg.

Thus, \( \lim_{x \to -3^-} \frac{x+2}{x+3} = +\infty \).
2.5.34) \[ \lim_{{x \to \infty}} \frac{{e^{{3x}} - e^{{-3x}}}}{{e^{{3x}} + e^{{-3x}}}} \]

Let \( u = e^{{3x}} \).

We now have:

\[ \lim_{{x \to \infty}} \frac{{e^{{3x}} - e^{{-3x}}}}{{e^{{3x}} + e^{{-3x}}}} = \lim_{{u \to \infty}} \frac{{u - \frac{1}{u}}}{{u + \frac{1}{u}}} \]

\[ = \lim_{{u \to \infty}} \frac{{u^2 - 1}}{{u^2 + 1}} \]

divide by \( u^2 \)

\[ = \lim_{{u \to \infty}} \frac{{1 - \frac{1}{{u^2}}}}{{1 + \frac{1}{{u^2}}}} \]

\[ \to 1 \]

Note, when we had

\[ \lim_{{u \to \infty}} \frac{{u^2 - 1}}{{u^2 + 1}} \]

we can read off the limit by noticing the leading forms have the same power and coefficient \( \frac{1}{1} = 1 \).
First, let's consider horizontal asymptotes by examining the behavior of \( f(x) \) as \( x \to \infty \) and \( x \to -\infty \).

Note, the leading terms have the same power of \( x \) meaning:

\[
\lim_{x \to \infty} f(x) = \text{a number}
\]

Since they grow at roughly the same rate, specifically:

\[
\lim_{x \to \infty} \frac{f(x)}{x} = \frac{(\text{big #})^2 + \text{small stuff}}{2(\text{big #})^2 + \text{small stuff}}
\]

\[x \to \frac{1}{2}.
\]

Note: a negative number squared!
is also positive so we can also conclude
\[ \lim_{x \to 2} f(x) = \frac{1}{2} , \]

This gives us a horizontal asymptote at \( y = \frac{1}{2} \).

What about vertical? Occurs when denominator = 0.

Using the quadratic equation or factoring we find
\[ f(x) = \frac{x^2 + 1}{2x - 3x - 2} = \frac{x^2 + 1}{(x - 2)(x + \frac{1}{2})} . \]

If one of these looked like it cancelled, we would have a removable discontinuity, so clearly \( x = 2 \) and \( x = -\frac{1}{2} \) are instead vertical asymptotes.
The plot of this graph is roughly:

2.4.4) The graph is continuous on any interval that the limit equals the function evaluation. We can include end points if it is left or right continuous.

Thus: \[ [-4, -2) \cup (-2, 2) \cup [2, 4) \cup (4, 6) \cup (6, 8) \].
2.4. 14.) $2\sqrt{3-x}$ continuous on $(-\infty, 3]$?

Consider that we said $\sqrt{\cdot}$ is continuous on the domain it is defined. Here, when is the function defined?

When: $3-x \geq 0 \Rightarrow 3 \geq x \Rightarrow x \leq 3$.

$$= x \in (-\infty, 3]$$

2.4. 16.) Is $$\begin{cases} \frac{x^2-x}{x^2-1} & x \neq 1 \\ 1 & x = 1 \end{cases}$$ continuous?

Obviously, the only point we need check is $x = 1$.

Particularly, for this to be continuous at $x = 1$, we need:

$$\lim_{x \to 1} f(x) = f(1) = 1.$$
What is this limit?

\[\lim_{x \to 1} \frac{x^2 - x}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x - 1)(x + 1)}\]

\[= \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}\]

Clearly, \(\lim_{x \to 1} f(x) \neq f(1)\), so this function is discontinuous at \(x = 1\).

2.4.22) \(h(x) = \frac{\sin(x)}{x + 1}\)

We know \(\sin(x)\) is continuous everywhere.

Dividing by 0 is a problem.

But our theorem says if have \(h = \frac{f(x)}{g(x)}\) and \(g(x) \to 0\), then it
f, g are each continuous, h is too.

Is x+1 continuous? Yes,

How do we deal with dividing by 0?

Is h(-1) even defined? No!

Therefore, x=-1 is not even in the domain of h(x) (No!)

Therefore, h(x) is continuous for all
x in its domain, where the domain is:

\[ D = \{ x \in \mathbb{R} : x \neq -1 \} \]

2.4.28) \[ \lim_{x \to \pi} \left( x \sin(x) + \sin(x) \right) \]

We have a theorem that says basically, if \( f(x) \) is continuous and \( g(x) \) is continuous, then \( f(g(x)) \) is continuous.
So here, consider
\[ f(x) = \sin(x), \]
\[ g(x) = x + \sin(x). \]

Clearly, \( f(x), g(x) \) are continuous since there are no jumps/gaps/asymptotes.
Thus, \( h(x) = f(g(x)) = \sin(x + \sin(x)) \) is continuous, meaning
\[ \lim_{x \to \pi} h(x) = h(\pi) \]
\[ = \sin(\pi + \sin(\pi)) \]
\[ = \sin(\pi + 0) \]
\[ = \sin(\pi) \]
\[ = 0. \]