Jump locations of jump-diffusion with state-dependent rates

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Setup

X_t = Markov process with **two** (coupled) noise sources

jump process

characterized by: operator $\mathbb{J}$; rate $\lambda(X_t)$

fixed jump size: $X_{t_+} = X_{t_-} + \Delta$

$\mathbb{J}q := q(x-\Delta)$

fixed jump location: $X_{t_+} = \eta$

$\mathbb{J}q := \delta(x-\eta)\int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x$

diffusion process

characterized by: operator $\mathbb{L}$ (forward operator)

Itô diffusion: $\mathrm{d}Y_t = A(Y_t)\mathrm{d}t + \sqrt{2D(Y_t)}\,\mathrm{d}W_t$

$\mathbb{L}q := \partial_y\{A(y)q\} + D(y)\partial_{yy}q$

Forward Kolmogorov (Fokker-Planck) equation

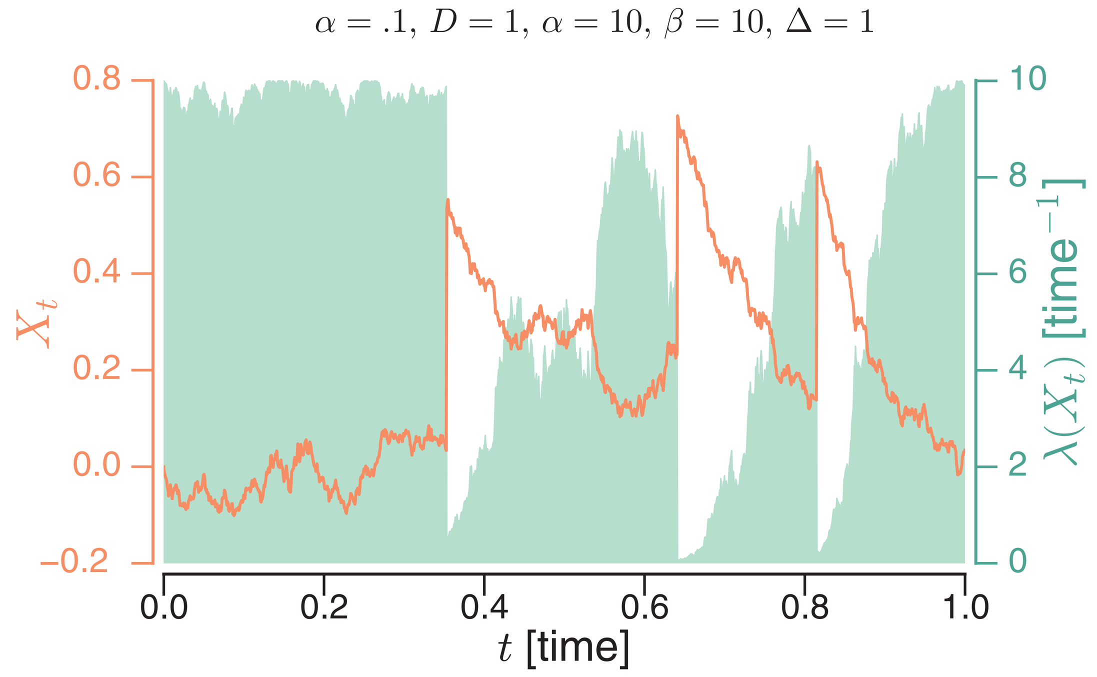
$$\partial_t\tilde{p}(x,t) = \mathbb{L}\tilde{p} - \lambda(x)\tilde{p} + \mathbb{J}\lambda\tilde{p}$$

$\tilde{p}(x,t)$:= density for X_t

Example

$\mathbb{L}p = \partial_x\{axp\} + Dp_{xx}$ $\quad \lambda = \alpha\exp\{-x^2/\beta\}$ $\quad \mathbb{J}p = p(x-\Delta)$

$a = .1,\ D = 1,\ \alpha = 10,\ \beta = 10,\ \Delta = 1$



Jump Locations

$\{t_1, t_2, \ldots\}$:= jump times

ith jump location := X_{t_i}

other quantities of interest

$\tau_i := t_i - t_{i-1}$:= interjump times

Survival formulation

$$\begin{cases}\partial_t p(x,t) = \mathbb{L}p - \lambda p\\ \partial_t q(x,t) = \lambda p.\end{cases}$$

next jump location $\quad p_j(x) = \int_0^{\infty}\lambda p\,\mathrm{d}t$

next jump time $\quad p_\tau(t) = \int_{-\infty}^{\infty}\lambda p\,\mathrm{d}x$

Results

Theorem 1 $p_i(x)$:= distribution of ith jump location, satisfies

$$\begin{cases}\partial_t\hat{p}_i(x,t) = \mathbb{L}\hat{p}_i - \lambda\hat{p}_i\\ \hat{p}_i(x,0) = \mathbb{J}p_{i-1}\\ p_{i+1}(x) = \int_0^{\infty}\lambda\hat{p}_i\,\mathrm{d}t.\end{cases} \quad (1)$$

more convenient to study $\quad u_i := p_i(x)/\lambda(x)$

Theorem 2 (1) is equivalent to the map

$$\mathbb{T}u_{i+1} = \mathbb{J}u_i \qquad \mathbb{T} := [\lambda(x) - \mathbb{L}]$$

importance

can construct sequence $\{u_1, u_2, \ldots\}$ and easily recover jump locations $\{p_1, p_2, \ldots\}$

Stationarity

Assuming X_t reaches stationarity

$p_\star$:= stationary jump distribution, $u_\star := p_\star/\lambda$

$\hat{p}_s$:= stationary distribution of full process

both satisfy

$$0 = \mathbb{L}u_\star - \lambda u_\star + \mathbb{J}\lambda u_\star, \qquad 0 = \mathbb{L}\hat{p}_s - \lambda\hat{p}_s + \mathbb{J}\lambda\hat{p}_s$$

explicit connection between jump locations and stationary density

but scaling is different $\int \hat{p}_s\,\mathrm{d}x = 1, \ \int u_\star\lambda\,\mathrm{d}x = 1$ → consequence

Theorem 3 stationary distribution $\hat{p}_s = p_\star$ jump location distribution

iff

$\lambda(x) = \lambda_0$

(no state dependence)

Interjump Times

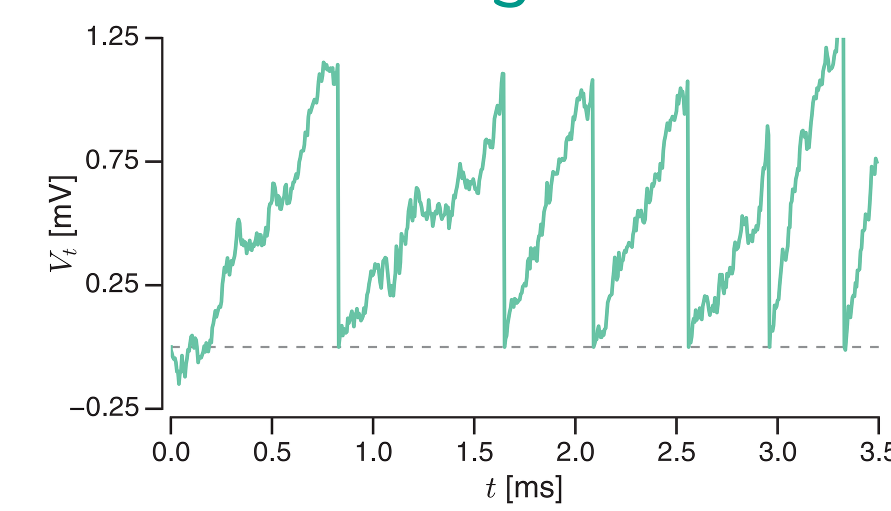
Assuming jump locations $\{p_1, p_2, \ldots\}$ known from previous results

mean interjump time $\quad \langle\tau_i\rangle = \int_{-\infty}^{\infty} u_i\,\mathrm{d}x$

higher order moments satisfy more complicated (but tractable) differential relationships

Applications

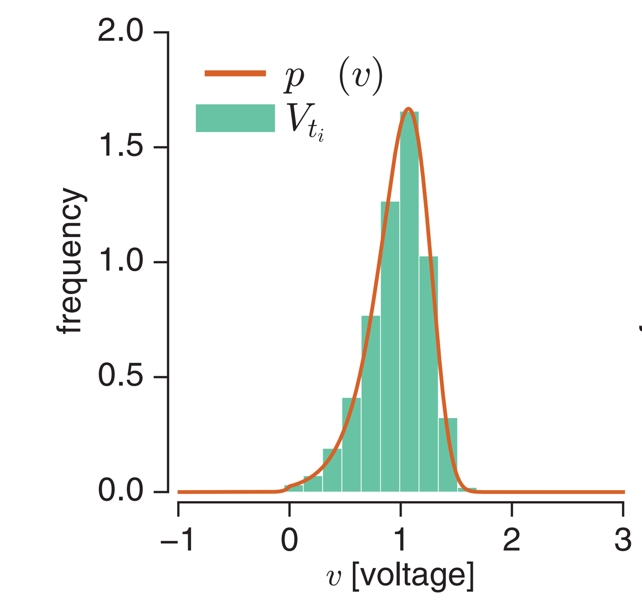
Neuronal Integrate-and-fire

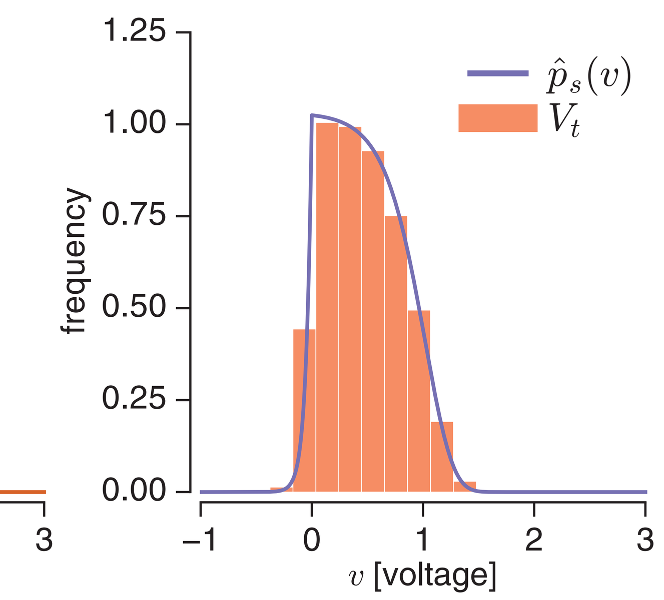


$\mathbb{L}p = -\partial_v\{\alpha p\} + Dp_{vv}$

$\lambda(v) = \gamma e^{v/\beta}$

$\mathbb{J}p = \delta(v)\int_{-\infty}^{\infty} p\,\mathrm{d}v$



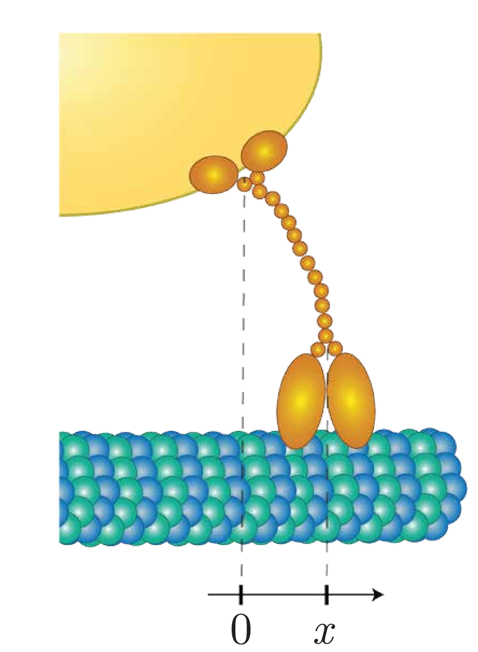


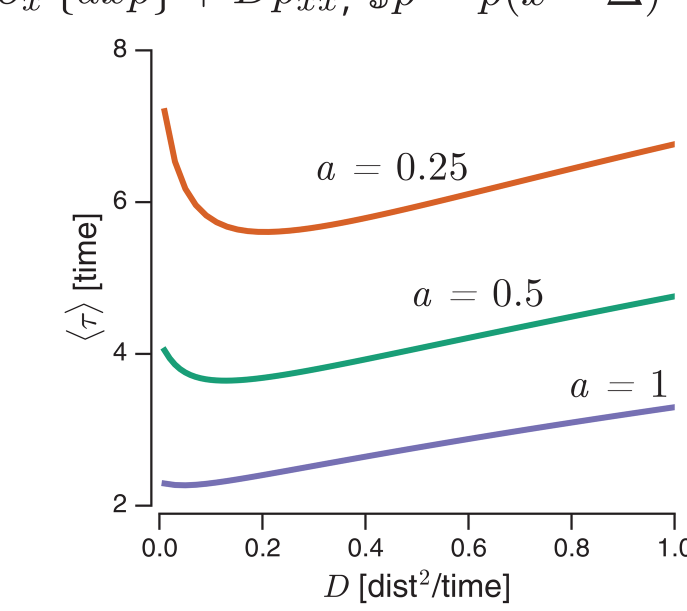
conclusion sharp firing threshold appears as a consequence of stochasticity from state-dependent rate

Molecular Motors?

$\lambda = \alpha\exp\{-x^2/\beta\}$

$\mathbb{L}p = \partial_x\{axp\} + Dp_{xx},\ \mathbb{J}p = p(x-\Delta)$





conclusion diffusion may have a non-monotonic effect on motor stepping rate

Future Work

use map formulation to study convergence to stationarity

find more applications (finance, anomalous diffusion?)

relate to state-dependent switched systems (stochastic hybrid systems)

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