AN ISOPERIMETRIC INEQUALITY FOR RIESZ CAPACITIES

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Abstract. Let $A$ be a compact set of $\mathbb{R}^n$, and $A^*$ be the ball centered at the origin with the same measure as $A$. We prove that, if $C_\alpha$ is the $\alpha$-Riesz capacity with $0 < \alpha < 2$, then $C_\alpha(A) \geq C_\alpha(A^*)$. We also prove an isoperimetric inequality for the expected measure of the stable sausage generated by $A$. Our results also yield isoperimetric inequalities for the relativistic $\alpha$-stable processes, and other Lévy processes.

1. Introduction

It is well known that, among all compact sets of equal measure, the ball has the smallest Newtonian Capacity. This is one of the classical generalized isoperimetric inequalities of G. Pólya and G. Szegö [11]. In [10], J. M. Luttinger provided a new method, based on multiple integrals inequalities, to prove this and many other isoperimetric inequalities. In this paper we adapt the method of Luttinger [10] to obtain isoperimetric inequalities for Riesz capacities. The M. Riesz kernel is

$$k_\alpha(x - y) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(\frac{n}{2}\right)} \frac{1}{|x - y|^{n-\alpha}},$$

where $n \geq 2$ and $0 < \alpha < n$. Let $A$ be a compact set in $\mathbb{R}^n$, the $\alpha$-Riesz capacity of $A$ is defined by

$$C_\alpha(A) = \left[ \inf_\mu \iint k_\alpha(x - y) \, d\mu(x) \, d\mu(y) \right]^{-1},$$

where the infimum is taken over all probability Borel measures supported in $A$. If $\alpha = 2$ and $n \geq 3$, then this is the classic Newtonian capacity. Let $|A|$ be the Lebesgue measure of $A$, and let $A^*$ be the ball in $\mathbb{R}^n$ centered at the origin such that $|A^*| = |A|$. Then Pólya and Szegö’s classical result is:

$$C_2(A) \geq C_2(A^*).$$

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Naturally one might ask if (1) holds for all Riesz capacities. This problem was stated by P. Mattila in [8], where the author finds lower bounds for the Hausdorff measure of the projection of $A$ on an $m$-dimensional subspace in terms of $C_m(A)$. In this paper we present a very short proof of the following result:

**Theorem 1.** Let $\alpha \in (0,2)$, and $A$ be a compact set of $\mathbb{R}^n$ such that $|A| > 0$. Then,

$$C_{\alpha}(A) \geq C_{\alpha}(A^*),$$

where $A^*$ is the ball, centered at the origin, such that $|A| = |A^*|$.

Theorem 1 was previously proved by D. Betsakos. His proof relies on some polarization inequalities for transition densities of killed symmetric stable processes and a well-known relationship between Green’s functions and Riesz capacities.

Luttinger obtained (1) from a probabilistic representation of the Newtonian capacity, due to F. Spitzer [14], and the following rearrangement inequality, proved by R. Friedberg and J. M. Luttinger [7].

**Theorem 2.** Let $F_0, \ldots, F_m : \mathbb{R}^n \to [0,1]$, and let $H_0, \ldots, H_m$ be nonnegative nonincreasingly symmetric functions in $\mathbb{R}^n$. Then,

$$\int \ldots \int \left[ 1 - \prod_{j=0}^m (1 - F_j(z_j)) \right] \prod_{j=0}^m H_i(z_j - z_{j+1}) \, dz_0 \cdots dz_m \geq$$

$$\int \ldots \int \left[ 1 - \prod_{j=0}^m (1 - F_{i,j}^*(z_j)) \right] \prod_{j=0}^m H_i(z_j - z_{j+1}) \, dz_0 \cdots dz_m,$$

where $F_{i,j}^*$ is the symmetric decreasing rearrangement of $F_i$, and $z_{m+1} = z_0$.

Actually Friedberg and Luttinger proved Theorem 2 for the Steiner symmetrization of functions in $\mathbb{R}^n$. However, it was proved in [4] that the symmetric decreasing rearrangement of a function in $\mathbb{R}^n$ can be obtained as the limit of a sequence of Steiner symmetrizations with respect to different planes. It is known that such rearrangement inequalities, combined with a probabilistic representation of the heat kernel, imply the classical Raleigh–Faber–Krahn inequality and many other generalized isoperimetric inequalities for heat kernels and Green’s functions of the Laplacian and fractional Laplacian; see [1], [9], [10].

The other key result, in Luttinger’s proof of (1), is Spitzer’s study of the expected volume of the Brownian sausage generated by $A$. The
Markov processes associated to Riesz kernels are the symmetric α-stable processes. Let $X_t$ be an $n$-dimensional symmetric α-stable process of order $\alpha \in (0, 2)$, and let $T_A^\alpha = \inf\{ t > 0 : X_t \in A \}$ be the first time $X_t$ hits $A$. Define

$$E_A^\alpha(t) = \int P^\alpha( T_A^\alpha \leq t ) \, dx,$$

this quantity can be interpreted as the expected Lebesgue measure of $\cup_{s \leq t} [X_s + A]$.

In the case that $\alpha = 2$,

$$E_A^2(t) - |A| = \int_{A^c} P^2( T_A^\alpha \leq t ) \, dx$$

can be interpreted as the total heat flow, up to time $t$, from $A$ to the surroundings. This was the original motivation of Spitzer to study the behavior of $E_A^\alpha(t)$. R. K. Getoor [6] proved that

$$\lim_{t \to \infty} \frac{E_A^\alpha(t)}{t} = C_\alpha(A),$$

extending Spitzer’s result to all symmetric stable processes.

Not only do Theorem 2 implies isoperimetric inequalities for the expected area of the stable sausage, but this method applies, without change, to any Lévy processes whose transition probability densities are radially symmetric and nonincreasing. This class of processes includes the relativistic α-stable processes and any processes of the form $B_{s\alpha}$, where $B_t$ is a Brownian motion and $A_t$ is a subordinator. We will prove Theorem 1 in §2, and we will discuss extensions of Theorem 1 to other processes in §3.

2. Proof of Theorem 1

Recall that the process $X_t$ has right continuous sample paths and stationary independent increments. Its infinitesimal generator is

$$(-\Delta)^{\alpha/2},$$

where $0 < \alpha \leq 2$ and $\Delta$ is the Laplacian in $\mathbb{R}^n$. When $\alpha = 2$ the process $X_t$ is just an $n$-dimensional Brownian motion $B_t$ running at twice the speed. If $0 < \alpha < 2$, then $X_t = B_{2\alpha t}$, where $\sigma_t$ is a stable subordinator of index $\alpha/2$ that is independent of $B_t$; see [2]. Thus,

$$p_\alpha(t, x, y) = \int_0^\infty \frac{1}{(4\pi tu)^{n/2}} \exp \left[ -\frac{|x - y|^2}{4tu} \right] g_{\alpha/2}(t, u) \, du,$$

where $g_{\alpha/2}(t, u)$ is the transition density of $\sigma_t$. Hence, for every positive $t$, $p_\alpha(t, x, y) = f_t^\alpha(|x - y|)$ and the function $f_t^\alpha(r)$ is decreasing.
Let $A_k$ be a decreasing sequence of compact sets such that the interior of $A_k$ contains $A$ for all $k$, and $\cap_{k=1}^{\infty} A_k = A$. By the right-continuity of the sample paths and the Markov property of stable processes, we have

$$\int P^{z_0} \{ T_1^\alpha \leq t \} d\mathbb{P}_0 = \int \left[ 1 - P^{z_0} \{ T_1^\alpha > t \} \right] d\mathbb{P}_0 =$$

$$\int \left[ 1 - P^{z_0} \{ X_s \in A^c, 0 \leq s \leq t \} \right] d\mathbb{P}_0 =$$

$$\lim_{k \to \infty} \lim_{m \to \infty} \int \left[ 1 - P^{z_0} \{ X_s \in A_k^c, j = 1, \ldots, m \} \right] d\mathbb{P}_0 =$$

$$\lim_{k \to \infty} \lim_{m \to \infty} \int \ldots \int \left[ 1 - \prod_{j=1}^{m} I_{A_k^c}(z_j) \right] \prod_{j=1}^{m} p_{A}(t/m, z_j - z_{j-1}) d\mathbb{P}_0 \cdots d\mathbb{P}_m =$$

$$\lim_{k \to \infty} \lim_{m \to \infty} \int \ldots \int \left[ 1 - \prod_{j=1}^{m} [1 - I_{A_k^c}(z_j)] \right] \prod_{j=1}^{m} p_{A}(t/m, z_j - z_{j-1}) d\mathbb{P}_0 \cdots d\mathbb{P}_m,$$

where $I_{A_k}$ is the indicator function of $A_k$. Since $f_t^\alpha(x)$ is nonincreasing and radially symmetric, we can take $H_m = 1$ and $F_0 = 0$ in Theorem 2 to obtain

$$\lim_{k \to \infty} \lim_{m \to \infty} \int \ldots \int \left[ 1 - \prod_{j=1}^{m} [1 - I_{A_k^c}(z_j)] \right] \prod_{j=1}^{m} p_{A}(t/m, z_j - z_{j-1}) d\mathbb{P}_0 \cdots d\mathbb{P}_m \geq$$

$$\lim_{k \to \infty} \lim_{m \to \infty} \int \ldots \int \left[ 1 - \prod_{j=1}^{m} [1 - I_{A_k^c}(z_j)] \right] \prod_{j=1}^{m} p_{A}(t/m, z_j - z_{j-1}) d\mathbb{P}_0 \cdots d\mathbb{P}_m.$$

Hence,

$$E_1^\alpha(t) = \int P^{z_0} \{ T_1^\alpha \leq t \} d\mathbb{P}_0 \geq \int P^{z_0} \{ T_1^\alpha \leq t \} d\mathbb{P}_0 = E_{1^*}^\alpha(t).$$

Finally we conclude from (3) that

$$C_\alpha(A) = \lim_{t \to \infty} \frac{E_1^\alpha(t)}{t} \geq \lim_{t \to \infty} \frac{E_{1^*}^\alpha(t)}{t} = C_\alpha(A^*).$$
3. Isoperimetric Inequalities for Radial Lévy Processes

Let $X_t$ be a Lévy process whose transition density is radially symmetric and decreasing. This class includes any process of the form $B_{A_t}$, where $B_t$ is a Brownian motion and $A_t$ is a subordinator. An important example of such processes is the relativistic $\alpha$-stable processes. The infinitesimal generator of the relativistic $\alpha$-stable process is

$$[m - \Delta]^{\alpha/2} - m,$$

where $m > 0$ is the mass of a relativistic particle. These processes arise in the study of relativistic Hamiltonian systems in physics; see [2], [5], [13] and the references there.

Let $T_A = \inf\{t > 0 : X_t \in A\}$ be the first time $X_t$ hits $A$, and consider

$$E_A(t) = \int P^t(T_A \leq t)dx.$$

This quantity can be interpreted as the expected Lebesgue measure of $\cup_{s \leq t}[X_s + A]$, the $X_t$-sausage generated by $A$. The following result generalizes the results of the previous section:

**Theorem 3.** Let $X_t$ be a transient Lévy processes whose transition density is radially symmetric and decreasing. Let $A$ be a compact set of $\mathbb{R}^n$ with positive measure. If $A^c$ is the ball centered at the origin with $|A| = |A^c|$, then for all $t \geq 0$

$$E_A(t) \geq E_{A^c}(t),$$

and

$$C(A) \geq C(A^c),$$

where $C(A)$ is the capacity associated to $X_t$.

An examination of the proof of Theorem 1 shows that (4) follows from Theorem 2 and the fact that the transition densities of $X_t$ are radially symmetric and decreasing. On the other hand S. C. Port and C. J. Stone [12] proved that

$$\lim_{t \to \infty} \frac{E_A(t)}{t} = C(A),$$

thus (5) follows from (4). We include a proof of (6) for the convenience of the reader. The capacity, associated to $X_t$, of a compact set $A$ is defined as

$$C(A) = \lim_{\lambda \to 0} \lambda \int E^\lambda e^{-\lambda T_A} dx = \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} E_A(dt).$$
On the other hand the Markov property and the symmetry of $X_t$ imply

$$E_A(t) - E_A(t-h) = \int P^x(t-h < T_A \leq t) dx$$

$$= \int \int P^x(T_A > t-h, X_{t-h} \in dy) P^y(T_A \leq y) dy dx$$

$$= \int P^y(T_A > t-h) P^y(T_A \leq y) dy.$$  

Then

$$\lim_{t \to \infty} \left( E_A(t) - E_A(t-h) \right) = \int P^y(T_A = \infty) P^y(T_A \leq y) dy.$$  

This is an additive function of $h$ which is bounded and measurable. Hence it is a linear function of $h$ and there exists a constant $\gamma(A)$ such that

$$\lim_{t \to \infty} \left( E_A(t) - E_A(t-h) \right) = h \gamma(A).$$

It follows that

$$\lim_{t \to \infty} \frac{1}{t} E_A(t) = \gamma(A),$$

and integration by parts implies that

$$C(A) = \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} E_A(dt) = \gamma(A),$$

which proves (6).

References


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