BASIC GENERAL RELATIVITY

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Contents

"Either the well was very deep, or she fell very slowly, for she had plenty of time as she went down to look about her, and to wonder what was going to happen next"
Lewis Carroll, Alice's Adventures in Wonderland

1. Introduction

Weinberg [2] is a beautiful explanation of general relativity. Hawking & Ellis [1] present the most influential examples of spacetime models, and prove the necessity of gravitational collapse under mild physical hypotheses. Unfortunately, in any approach to this subject we have to use some messy tensor calculations. We will assume at least one course in differential geometry. Ultimately we want to consider quantum field theories using path integrals, so we are forced to set up all of our classical physical theories in terms of Lagrangians and principles of least action.

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2. Special relativity

Following Jim's discussion of classical electrodynamics, we see that electrodynamics leads us to believe that spacetime is Minkowski space $\mathbb{R}^{1+3}$, and we will write it as $\mathbb{R}^{1+n}$ to make generalizations easier. How does a material particle move in this space in the absence of any electromagnetic field?

3. Notation

Let us establish notation for Minkowski space. The speed of light is c. Write coordinates of points as

$$(t, \vec{x}) = \left(x^0, x^1, \dots, x^n\right).$$

Write x^i for components with $i = 1, \dots, n$ and x^μ for components with $\mu = 0, \dots, n$. In these coordinates, our Minkowski metric is

$$(g_{\mu\nu}) = \begin{pmatrix} -c^2 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Write $g^{\mu\nu}$ for the elements of the inverse matrix to $g_{\mu\nu}$. We use Einstein summation notation: an upper and lower index, like $a^\mu b_{\mu\nu}$ implies a sum, in this case over all values of μ.

4. The action of a relativistic particle

The action physicists ascribe to a classical relativistic particle is the "length"

$$S_{\text{massive particle}} = -mc \int d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

where τ is a parameter which is used to parameterize the particles trajectory in Minkowski space. To keep the action real, particle velocities must satisfy

$$-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = c^2 \left(\frac{dx^0}{d\tau}\right)^2 - \sum_i \left(\frac{dx^i}{d\tau}\right)^2 > 0,$$

called *time-like* velocities.

$$\begin{aligned} &\text{time-like} && -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} > 0 \\ &\text{light-like} && -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \\ &\text{space-like} && -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} < 0 \end{aligned}$$

Consider the action of a particle with small velocity in the x^i directions compared to its x^0 velocity ("very time-like"). Parameterize it by $\tau = t = x^0$.

$$\begin{aligned} S_{\text{massive particle}} &= -mc \int dt \sqrt{c^2 - \left(\frac{d\vec{x}}{dt}\right)^2} \\ &\sim -mc \int dt \left(c - \frac{1}{2c}\left(\frac{d\vec{x}}{dt}\right)^2\right) \end{aligned}$$

(using the Taylor expansion for the square root, $\sqrt{a^2+x} = a + x/2a + \dots$)

$$= \int dt \left(\frac{m}{2} \left(\frac{d\vec{x}}{dt} \right)^2 - mc^2 \right).$$

This is just the nonrelativistic action for a free particle, plus a rest energy constant of mc^2. So this Poincaré and reparameterization invariant action looks at low velocities like a nonrelativistic free particle.

Returning to a general particle, the Lagrangian is

$$L_{\text{massive particle}} = -mc\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}},$$

so the momentum is

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial(dx^\mu/d\tau)} \\ &= mc\frac{g_{\mu\sigma}\frac{dx^\sigma}{d\tau}}{\sqrt{-g_{\nu\eta}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}. \end{aligned}$$

Essentially, the momentum is just velocity rescaled. The Euler–Lagrange equations are

$$\frac{d}{d\tau}p_\mu = \frac{\partial L}{\partial x^\mu} = 0.$$

If we reparameterize to get

$$\sqrt{-g_{\nu\eta}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = 1$$

we find that these equations are just

$$\frac{dx^\mu}{d\tau} = 0$$

rectilinear motion, so they behave a lot like free particles.

The Legendre transform is

$$\left(x^\mu, \frac{dx^\mu}{d\tau} \right) \mapsto \left(x^\mu, p_\mu \right).$$

But p_μ satisfies

$$g^{\mu\nu}p_\mu p_\nu + m^2c^2 = 0 \tag{1}$$

(just plug it in and see), which we can write as

$$p_0^2 = p_1^2 + \dots + p_n^2 + m^2c^2.$$

So the image of the Legendre transformation is the variety of (x, p) cut out by this equation.

The Hamiltonian is

$$\begin{aligned} H &= p_\mu\frac{dx^\mu}{d\tau} - L \\ &= mc\frac{g_{\mu\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}}{\sqrt{-g_{\nu\eta}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}} + mc\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \\ &= 0. \end{aligned}$$

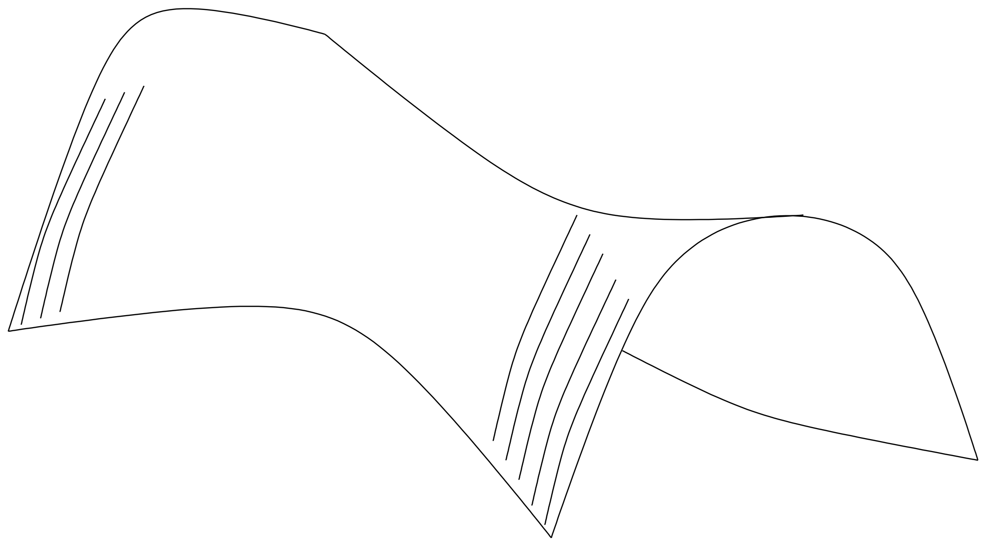

FIGURE 1. In any symplectic manifold, every smooth hypersurface is ruled by bicharacteristics

This is strange: $H = 0$ means no total energy. How do we find the Hamiltonian paths?

Exercise 1. Bicharacteristics

Prove the following: fix a hypersurface Σ inside the phase space, i.e. the space of (x, p) variables (the cotangent bundle). Take any function H of the (x, p) variables (i.e. on the phase space) which vanishes on that hypersurface and use it as our Hamiltonian. On that hypersurface the classical paths of H are given by the bicharacteristics, the curves $(x(t), p(t))$ whose velocity v satisfies (i) v is tangent to the hypersurface and (ii)

$$v \lrcorner \omega|_\Sigma = 0$$

where

$$\omega = dx^\mu \wedge dp_\mu.$$

The Hamiltonian paths change only by reparameterization if we change the choice of H.

Therefore in some sense our Hamiltonian is

$$\begin{aligned} H &= -p_0^2 + p_i p_i + m^2 c^2 \\ &= g^{\mu\nu} p_\mu p_\nu + m^2 c^2 \end{aligned}$$

(up to replacing H by a function of H) but in another sense this is zero.

Summing up, the relativistic action for a particle makes particles move on straight time-like paths in spacetime ("falling freely"), which agrees with experiment, and the action is Poincaré invariant. It is difficult to come up with any other action with the same properties.

5. The metric on the world line

We have a problem for particles with no mass: the action is just $S = 0$. Lets fix this.

The parameter τ is called the *proper time* of the particle, and its path in spacetime is called its *world line*. There is another way to obtain the same action for massive particles (*massive* doesn't mean having large mass—it just means having positive mass). Adjoin to the particle's world line its own internal metric, which we write as $\gamma(\tau)d\tau^2$. This has a volume 1-form attached to it (actually a volume density),

$$\eta(\tau)d\tau = \sqrt{\gamma(\tau)}d\tau.$$

We consider the action

$$S_{\text{particle}} = \frac{1}{2}\int d\tau \left(\eta^{-1}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \eta m^2c^2\right).$$

Now we vary not just the path, but also the 1-form $\eta\, d\tau$ (or equivalently the world line metric). Varying η, we find that the Euler–Lagrange equations are

$$\eta^2 = -\frac{1}{m^2c^2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}.$$

Plugging this into S_{particle} we recover the old $S_{\text{massive particle}}$, so the same equations of motion. But the new Lagrangian makes sense even for massless particles; the Euler–Lagrange equations of a massless particle do not determine η, but determine that the particle moves in straight lines at the speed of light.

6. Wave equations

Light is a phenomenon of electromagnetic radiation, as Jim explained. Moreover it is a wave phenomenon, satisfying a wave equation. We wish to explore wave equations in a diffeomorphism invariant fashion. Let us consider some very general wave equation, of second order, say $P[\phi] = 0$, on some manifold. For simplicity, take ϕ a scalar field. Then we can linearize the equation about a solution ϕ:

$$P[\phi + \varepsilon\psi] = \varepsilon P'[\phi]\psi + O\left(\varepsilon^2\right).$$

A small perturbation of the wave ϕ to a wave $\phi + \varepsilon\psi$ will nearly satisfy this linearization. Suppose that the linearized equation looks like

$$P'[\phi]\psi = \sum_{\mu\nu} g^{\mu\nu}\frac{\partial^2\psi}{\partial x^\mu x^\nu} + \dots$$

(in local coordinates x^μ) where the $\dots$ represent lower derivative terms. The functions $g^{\mu\nu}$ depend on the choice of the unperturbed wave ϕ. Plug in a very high frequency wave

$$P'[\phi]e^{i\lambda\psi} = -\lambda^2 e^{i\lambda\psi}\sum_{\mu\nu} g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu} + \dots$$

where the $\dots$ are lower order in λ, the high frequency. So a small amplitude perturbation of u with very high frequency will nearly satisfy the *eikonal equation*

$$\sum_{\mu\nu} g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}.$$

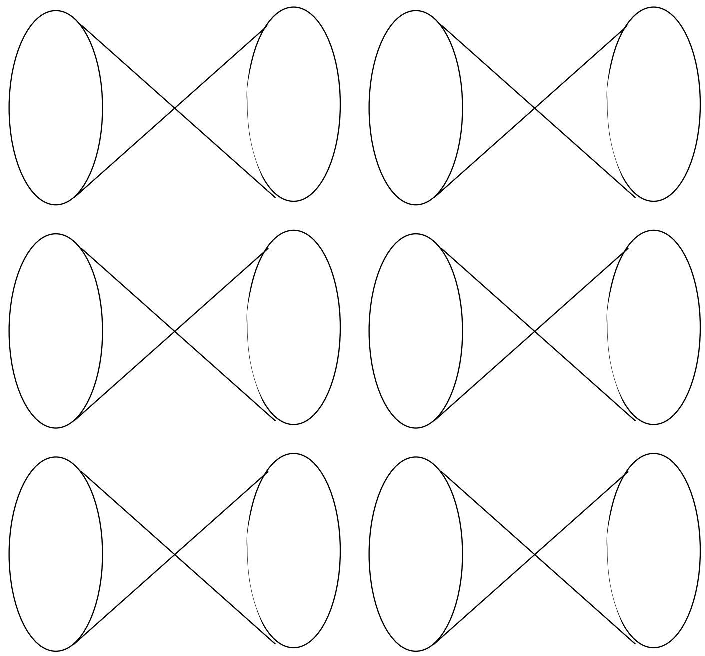

FIGURE 2. Every wave equation has a light cone in each tangent space

The eikonal equation determines a quadric $g^{\mu\nu} p_\mu p_\nu = 0$ in each cotangent space, and a dual quadric $g_{\mu\nu} v^\mu v^\nu = 0$ in each tangent space. Geometrically, this means that a choice of wave ϕ determines a choice of *conformal* Lorentz metric, i.e. a smoothly varying family of quadratic cones in tangent spaces. Consider the hypersurface

$$\sum_{\mu\nu} g^{\mu\nu} p_\mu p_\nu = 0$$

inside the cotangent bundle. The eikonal equation requires precisely that the differential $d\psi$ of our function ψ belong to this hypersurface.

Exercise 2. The eikonal equation

> Calculate the eikonal equation for the usual wave equation (in Minkowski space) using this approach. What happens if we consider not just a phase but also a magnitude like
>
> $$P'[\phi] a e^{i\lambda\psi}$$
>
> with $a > 0$?

Consider the path $x(t)$ defined by

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu} \frac{\partial \psi}{\partial x^\nu}.$$

Differentiate

$$\frac{d}{d\tau}\psi(x) = \frac{\partial \psi}{\partial x^\mu}\frac{dx^\mu}{d\tau} = 0.$$

So wave perturbations are constant along such paths.

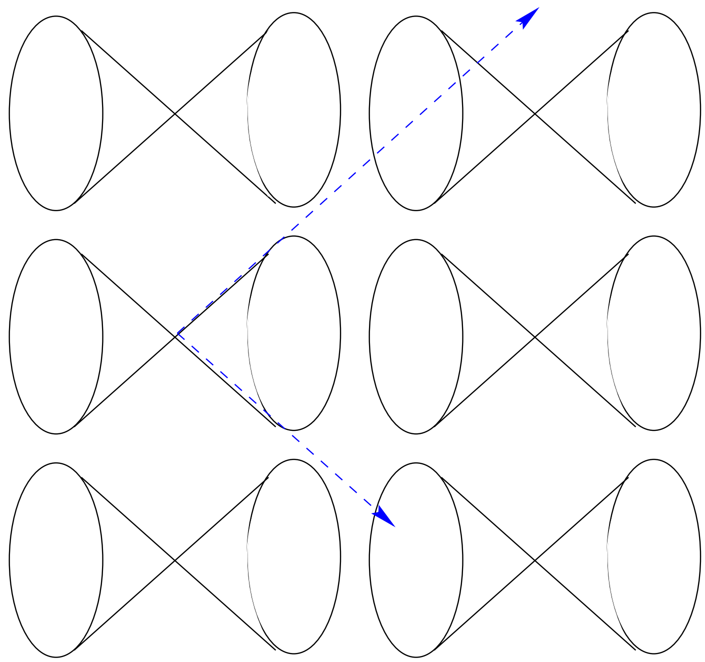

FIGURE 3. Small high frequency perturbations of a wave look like bursts of massless particles

Let us look at the Hamiltonian paths along this hypersurface, with Hamiltonian

$$H = \frac{1}{2} \sum_{\mu\nu} g^{\mu\nu} p_\mu p_\nu .$$

Hamilton's equations are

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\mu$$
$$\frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} = -\frac{1}{2} \frac{\partial g^{\nu\sigma}}{\partial x^\mu} p_\nu p_\sigma .$$

If we plug in the path $x(t)$ we have just looked at, given by

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu} \frac{\partial \psi}{\partial x^\mu}$$

and take

$$p_\mu = \frac{\partial \psi}{\partial x^\mu}$$

we find that this is precisely a bicharacteristic.

Summing up, every high frequency small perturbation of a wave equation consists (approximately) in a sum of constant phase contributions propagating along bicharacteristics, i.e. geodesics of massless particles. For electromagnetic waves, we call these particle-like bursts *photons*.

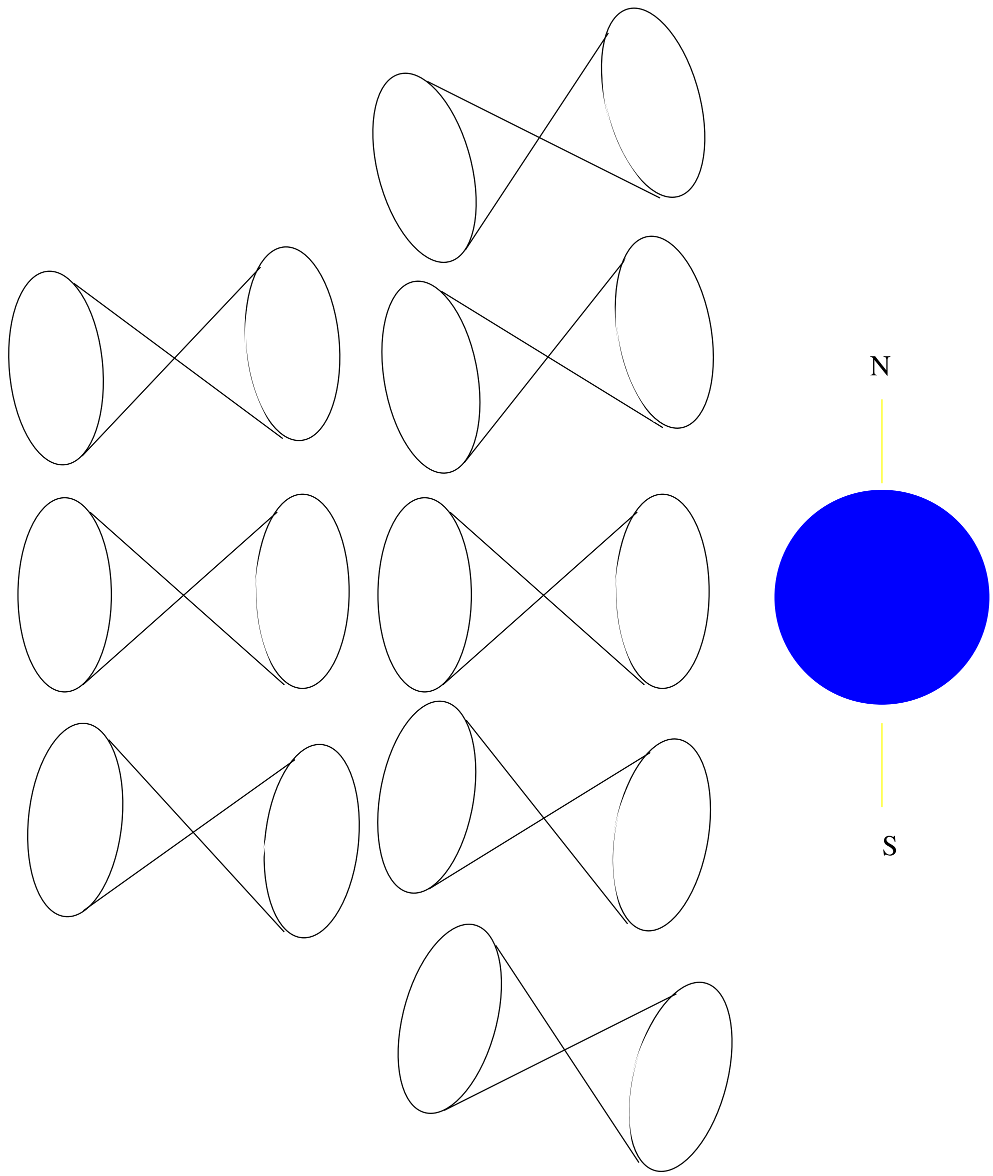

FIGURE 4. Light cones bend due to gravitational influences

7. General relativity

The main problem facing our description of electromagnetism is that electromagnetic waves (light) are deflected by very heavy objects, like the sun. Therefore the relevant wave equation must depend on the presence of matter, and we are not in Minkowski space. On the other hand, matter reacts to electromagnetic stimuli (it is *charged*) so the physics is coupled.

We can measure lengths, and carry rulers around with us. If two people carry rulers around with them, and check that they have the same length, then whenever and wherever they meet again the rulers still have the same length. Therefore the conformal Lorentz metric must at least be upgraded to a Lorentz metric: we can figure out which lengths are unit lengths in spatial directions.

There is now no special choice of coordinate system, so we take all local coordinates on an equal footing. Spacetime is just a manifold M^{1+n}. A *field* (or *matter field*) is a section of a bundle over spacetime; but to ensure that the theory is invariant under diffeomorphisms we ask that the diffeomorphism group of spacetime act as bundle automorphisms. The fields are required to satisfy *field equations*, which are Euler–Lagrange equations of some Lagrangian. The action of this Lagrangian must be invariant under the diffeomorphism group (this is called the *principle of general covariance*), when we carry all of the fields (including the metric) through that diffeomorphism. Indeed this invariance is required even for integrals of the action over regions of the manifold, so it is a local requirement. To form the field equations of a field, we take the variation of its action with regard to variations of that field. In particular, we have one special field, the Lorentz metric, which is required in any such theory. The variation of the action of a relativistic Lagrangian with respect to variations in the metric is called the *energy-momentum tensor* of that Lagrangian. To put the whole theory together, we just add all of the Lagrangians of all of the fields, and demand that the action be stationary for variations of all of the fields.

The *principle of constancy of the speed of light* says that no signal can travel faster than the speed of light. We interpret this in the context of fields as saying that no perturbation of solutions of the field equations concentrated at a point of space (i.e. a point of any space-like hypersurface) can propagate faster than the speed of light, so the perturbation must not be felt outside the light cone (the set of light-like vectors in each tangent space). Moreover all fields must be governed by wave equations as consequences of the Euler–Lagrange equations, and the light cones of all of those wave equations must be the same: the light cones of the metric. This strongly restricts the class of Lagrangians we can use.

The *principle of relativity* in Newtonian mechanics says that it is impossible to tell whether one is moving or not in an inertial frame of reference, i.e. in the absence of external forces. In special relativity and Newtonian mechanics this holds perfectly, because there are no forces at all. Moreover, in Newtonian mechanics, a similar notion tells us that we can not tell if we are falling to the ground, or if the ground is moving up to meet us.

Exercise 3. Newtonian principle of relativity

> Prove that you can't tell if you are falling or the earth is flying up at you, using $F = ma$ and a change of coordinates. Note that the change of coordinates is not a Galilean transformation.

This result is not going to work in a general gravitational field, for a large falling object. For example in falling to the earth, all of the parts of the object are pulled toward the center of the earth, so they feel a force of contraction.

In general relativity, we will ask the same principal to hold for an arbitrary gravitational field, but only in the direction of motion of our particle. So for any particle not interacting with any other field, there is a map of a neighborhood of the particle's world line to Minkowski space, taking the world line of the particle to the world line of a particle in Minkowski space, and matching up the Lorentz metrics along the world lines. The map will not match up the metrics away from the world lines. In fact, we can construct a canonical such map, called the *developing map*. Intuitively, we roll our curved manifold along Minkowski space, getting Minkowski space to touch our manifold only along the world line. Call the existence of this map the *relativistic principle of equivalence*. It is the same as saying that free relativistic particles follow time-like or light-like geodesics of the Lorentz metric.

8. Notation

I will use tensor notation, so that coordinate functions are written x^μ, a tangent vector is write $v^\mu \partial/\partial x^\mu$, a 1-form is $a_\mu dx^\mu$, etc. Further notation is presented, without explanation, in the *Sign conventions* guide on the web page for this lecture series.

Exercise 4.

If Ω is any volume form, and X is a vector field supported in the interior of a region R then

$$\int_R \mathcal{L}_X \Omega = 0.$$

9. Examples of fields and field equations

Consider the action for a particle

$$S_{\text{particle}} = \frac{1}{2} \int d\tau \left(\eta^{-1} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \eta m^2 c^2 \right).$$

Here we are using the Lorentz metric $g_{\mu\nu}$ instead of the flat Minkowski metric. But it is still invariant under world line reparameterization. It is also invariant under the obvious action of the diffeomorphism group of M^{1+n}. Let us consider how this action varies under variation of the world line volume form $\eta\, d\tau$, the curve $x^\mu(t)$, and the spacetime metric $g_{\mu\nu}$.

Exercise 5. Euler–Lagrange equations of the free particle

Let $S = S_{\text{massive particle}}$.

(a) Show that

$$\frac{\delta S}{\delta x^\nu(\tau)} = \frac{dp_\mu}{d\tau}$$

where the momentum p_μ is

$$p_\mu = \frac{g_{\mu\nu} \frac{dx^\nu}{d\tau}}{\sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}}.$$

(b) Show that this implies that

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

where the *Christoffel symbols* $\Gamma^\mu_{\nu\sigma}$ are determined by

$$g_{\varepsilon\mu} \Gamma^\mu_{\nu\sigma} = \frac{1}{2} \left(\frac{\partial g_{\varepsilon\nu}}{\partial x^\sigma} + \frac{\partial g_{\varepsilon\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\varepsilon} \right).$$

(c) For any vector v^μ define the *covariant derivative operator* on tensors by

$$\nabla_v f = \mathcal{L}_v f$$

for f a function, and

$$\nabla_v X^\nu = v^\mu \frac{\partial X^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma} X^\nu X^\sigma .$$

Show that

$$\nabla_{\frac{dx}{d\tau}} \frac{dx}{d\tau} = 0$$

is precisely the equation of free particle motion. This says that the covariant derivative replaces Newton's ordinary spatial derivatives.

(d) Show that the energy-momentum tensor is

$$T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}(x)} = \frac{mc}{2} \frac{\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}{\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}} .$$

Exercise 6. More Euler–Lagrange equations of the free particle

Let $S = S_{\text{particle}}$. Show that

(a)

$$\frac{\delta S}{\delta x^\nu(\tau)} = \eta^{-1} g_{\mu\nu} \left(\frac{d^2 x^\mu}{d\tau^2} + g^{\mu\varepsilon} \frac{\partial g_{\varepsilon\beta}}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \eta^{-1} \frac{d\eta}{d\tau} \frac{dx^\mu}{d\tau} \right) .$$

(b) The energy-momentum tensor is

$$T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}(x)} = \eta^{-1} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} .$$

(c) Calculate the variation in the world line metric:

$$\frac{\delta S}{\delta \eta(\tau)} = -\left(\eta^{-2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + m^2 c^2 \right) .$$

Exercise 7. Volume

Calculate the energy-momentum tensor of the volume of a region of our spacetime manifold. Recall that volume is the integral

$$V = \int dV$$

where dV is the volume form associated to the Lorentz metric, i.e.

$$dV = \sqrt{-\det(g)}\, dx^0 \wedge \cdots \wedge dx^n .$$

You should get

$$T^{\mu\nu} = g^{\mu\nu} .$$

Exercise 8. Electromagnetic field equations

Take a 1-form $A = A_\mu dx^\mu$, and its exterior derivative

$$\begin{aligned} F &= dA \\ &= F_{\mu\nu} dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) dx^\mu \wedge dx^\nu . \end{aligned}$$

Consider the Maxwell action

$$S(A) = -\frac{1}{16\pi} \int g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'}\, dV$$

Show that

$$\begin{aligned}\frac{\delta S}{\delta g_{\mu\nu}} &= T^{\mu\nu} \\ &= -\frac{1}{8\pi}\left(-g^{\mu\beta} F_{\mu\alpha} g^{\nu\gamma} g^{\alpha\delta} F_{\gamma\delta} + \frac{1}{2} g^{\mu\nu} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha\beta} F_{\alpha'\beta'}\right)\end{aligned}$$

(Be careful: the second term comes from varying dV.)

10. Parallel transport and covariant derivatives

As the Newtonian or special relativistic particle falls, it carries a frame of reference with it, by affine translation.

For us, a *frame* means a choice of orthonormal basis of cotangent vectors

$$\omega^0, \dots, \omega^n$$

which, in local coordinates, have components $\omega^\mu = a^\mu_\nu dx^\nu$. Orthonormal here means that the metric is

$$g_{\mu\nu} dx^\mu dx^\nu = -\left(\omega^0\right)^2 + \sum \left(\omega^\mu\right)^2.$$

Write

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

so that

$$g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \omega^\mu \omega^\nu.$$

Suppose that we want a law that says that a frame will be carried along with a falling particle, so that the frame stays orthonormal. This law, by analogy with special relativity, should give affine translations in the limit as the metric $g_{\mu\nu}$ becomes nearly the Minkowski metric. So it should look like

$$\frac{d}{d\tau} \omega^\mu = 0$$

for Minkowski space, with a translation invariant frame. We won't be able to pick translation invariant frames in general spacetimes, so let us first try to describe how a frame Ω^μ is carried along by a particle, in terms of an arbitrary smoothly varying frame ω^μ in some open set.[1]

We want to say that the observer who picks a frame Ω^μ to measure things in, say

$$\Omega^\mu = a^\mu_\nu \omega^\nu$$

will carry this frame with him by an equation which becomes $da^\mu_\nu = 0$ when the ω^μ framing is translation invariant.

[1] When we have a smoothly varying choice of frame on an open subset of our manifold, we will call that a *framing*; physicists call it a *vierbein* if $1+n=4$, and a *veilbein* if $1+n \neq 4$; sometimes an *einbein* if $1+n=1$, etc.

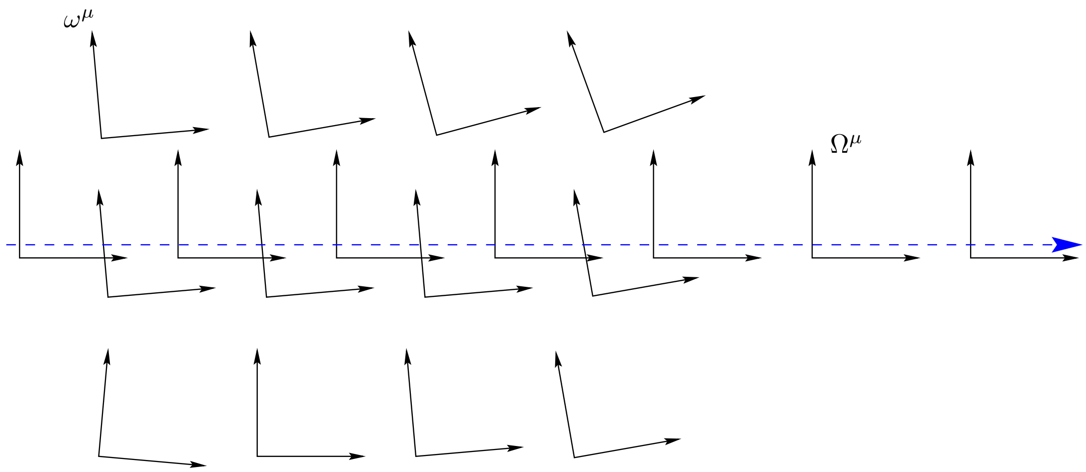

FIGURE 5. In terms of any moving frame ω^μ we can use parallel transport to carry our fixed frame Ω^μ with us

In Minkowski space coordinates x^μ, suppose $\omega^\mu = g^\mu_\nu dx^\nu$, with $g^\mu_\nu(x)$ a smoothly varying element of the Lorentz group. Write this $\omega = g\,dx$. Then calculate

$$\begin{aligned} d\omega &= dg \wedge dx \\ &= -\gamma \wedge \omega \end{aligned}$$

where

$$\gamma = -dg\, g^{-1}.$$

This $\gamma = \gamma^\mu_\nu$ is a 1-form valued in the Lie algebra of the Lorentz group. Translation of $\Omega = a\omega = ag\,dx$ is expressed as

$$0 = d(ag) = da\, g + a\, dg.$$

We can express this as

$$\begin{aligned} 0 &= da + a\, dg\, g^{-1} \\ &= da - a\gamma. \end{aligned}$$

So in terms of the arbitrarily chosen framing ω, a frame $\Omega = a\omega$ is carried by translation precisely when

$$da - a\gamma = 0.$$

Note that this is expressed purely in terms of the arbitrary ω framing. Write this as a differential operator: given a velocity vector v, write

$$\nabla_v \Omega = (v \lrcorner (da - a\gamma))\, \omega$$

or

$$\nabla_v \Omega^\mu = (v \lrcorner (da^\mu_\nu - a^\mu_\sigma \gamma^\sigma_\nu))\, \omega^\nu.$$

Exercise 9. Structure equations of Minkowski space

> Prove the existence and uniqueness of 1-forms γ^μ_ν on Minkowski space, for each choice of framing ω^μ, which satisfy $d\omega = -\gamma \wedge \omega$ with γ a 1-form valued in the Lie algebra of the Lorentz group.

Exercise 10. The futility of $\mathfrak{L}$

Why can't we use the Lie derivative to do this? Hint: show that the Lie derivative $\mathcal{L}_X Y$ depends on how the vector field X varies in all directions, for general vector field Y. Therefore there is no concept of differentiating along a curve with the Lie derivative. Use simple coordinate changes to show that there is no coordinate invariant concept of differentiating a vector field along a curve on a smooth manifold with no "fields" on it.

So our differential operator ∇ must come from the metric, not just the manifold.

Exercise 11. Part of the fundamental lemma of (pseudo)Riemannian geometry

On a manifold M with Lorentz metric, and an orthonormal moving framing ω:

(a) prove the existence and uniqueness of a 1-form
$$\gamma = (\gamma^\mu_\nu)$$
valued in the Lie algebra of the Lorentz group, so that
$$d\omega = -\gamma \wedge \omega.$$
(This requires a lot of algebra).

(b) If we change ω to another orthonormal moving framing, it will become $\omega' = g\omega$ for g a function valued in the Lorentz group. Show that γ changes by
$$\gamma' = -dg\, g^{-1} + g\gamma g^{-1}.$$

Now define the *covariant derivative* to be the operator on tensors so that

(1)
$$\nabla_X \omega^\mu = -\left(X \lrcorner \gamma^\mu_\nu\right) \omega^\nu$$
(2)
$$\nabla_X f = \mathcal{L}_X f$$
(3) ∇ is a derivation on tensors for the tensor product and for contractions.

Exercise 12.

Show that the covariant derivative is well defined, independent of the choice of framing ω^μ. Hint: any other framing looks like $\omega' = g\omega$ for some Lorentz-group-valued function g.

Exercise 13.

Writing $X^\mu = X \lrcorner \omega^\mu$, show that
$$\left(\nabla_X Y\right)^\mu = \mathcal{L}_X \left(Y^\mu\right) + \gamma^\mu_\nu(X) Y^\nu.$$

Exercise 14.

Show that
$$\nabla_{f} X = f \nabla_X$$
for any function f.

Exercise 15. Invariance of the metric

Formally, write
$$d\tau^2 = -\left(\omega^0\right)^2 + \sum_i \left(\omega^i\right)^2.$$

Show that for any vector field X,

(a)

$$\nabla_X d\tau^2 = 0.$$

(b)

$$\nabla_X \omega^0 \wedge \cdots \wedge \omega^n = 0.$$

(c)

$$\nabla_X \left(d\tau^2 \left(Y, Z\right)\right) = d\tau^2 \left(\nabla_X Y, Z\right) + d\tau^2 \left(Y, \nabla_X Z\right).$$

(d) If we write X_μ for the basis of vector fields dual to ω^μ, so that

$$X_\mu \lrcorner \omega^\nu = \delta^\mu_\nu$$

then for any vector field $Y = Y^\mu X_\mu$,

$$\nabla_X Y = \left(\mathcal{L}_X Y^\mu + X \lrcorner \gamma^\mu_\nu Y^\nu\right) X_\mu.$$

(e)

$$\left(\mathcal{L}_X d\tau^2\right)(Y, Z) = \nabla_Y d\tau^2(X, Z) + \nabla_Z d\tau^2(X, Y).$$

(f)

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(g)

$$\mathcal{L}_X \left(d\tau^2\right)(Y, Z) = d\tau^2 \left(\nabla_Y X, Z\right) + d\tau^2 \left(\nabla_Z X, Y\right).$$

Hint: start with Y, Z orthonormal, and look at $\nabla_X \left(d\tau^2(Y, Z)\right)$.

Exercise 16. Coordinate freedom

Use the results from the previous problem to calculate the Euler–Lagrange equations of a particle, without using coordinates. You should find

$$\nabla_{dx/d\tau} dx/d\tau = 0.$$

Exercise 17. Holonomy

Define the parallel transport operator along a path $x(\tau)$ to be the linear operator taking a vector v along by the equation

$$\nabla_{dx/d\tau} v(\tau) = 0.$$

Show that this is a Lorentz transformation of the tangent space at one end of of the path to the tangent space at the other end.

Exercise 18. Gauß's theorem

We assume Stokes' theorem:

$$\int_R d\eta = \int_{\partial R} \eta.$$

Now prove *Gauß's theorem*:

$$\int_{\partial R} Y \lrcorner dV = \int_R \nabla_\nu Y^\nu \, dV$$

where ∇_ν means taking the covariant derivative in the direction X_μ dual to ω^μ, and $Y^\nu = Y \lrcorner \omega^\nu$. Hint: use the Cartan formula

$$\mathcal{L}_X dV = X \lrcorner d(dV) + d(X \lrcorner dV),$$

and define some functions $\Gamma^\mu_{\nu\sigma}$ by

$$\gamma^\mu_\nu = \Gamma^\mu_{\nu\sigma} \omega^\sigma.$$

Also use the equation

$$dV = \omega^0 \wedge \cdots \wedge \omega^n.$$

Exercise 19. The principle of relativity

Take any curve C in a Lorentz manifold M and carry a frame Ω^μ along C by parallel transport. Take the velocity vector of the curve C, say v, and calculate its components in the frame Ω^μ, say $v^\mu = v \lrcorner \Omega^\mu$. This determines a curve in Minkowski space, C', by requiring that C' start at the origin at time $\tau = 0$, and have velocity $v^\mu \frac{\partial}{\partial x^\mu}$ at each time τ. This C' is the developing curve of C. Show that C' is a geodesic precisely when C is, so that a falling particle has trouble telling if he is falling on a curve in M or on its developing curve in Minkowski space.

Exercise 20. Hodge star

In an orthonormal frame ω^μ any differential form α can be written as $\alpha = \alpha_M \omega^M$ with a multiindex $M = (\mu_1, \dots, \mu_p)$. Recall that our frame being orthonormal means that

$$\eta_{\mu\nu}\omega^\mu\omega^\nu = -\left(\omega^0\right)^2 + \sum_i \left(\omega^i\right)^2.$$

If

$$\begin{aligned} M &= (\mu_1, \dots, \mu_p) \\ N &= (\nu_1, \dots, \nu_p) \end{aligned}$$

then let

$$\eta_{MN} = \eta_{\mu_1\nu_1} \dots \eta_{\mu_p\nu_p}.$$

Define

$$\langle \alpha, \beta \rangle = \alpha_M \beta_N \eta^{MN}.$$

It is clear that this is independent of the choice of orthonormal frame. Write $\alpha^\sharp$ for $\langle \alpha, \rangle$. Prove existence and uniqueness of a linear operator $*$ so that

$$\langle \alpha, \beta \rangle \, dV = \alpha \wedge *\beta.$$

Show that

$$\begin{aligned} *\omega^0 &= -\omega^1 \wedge \cdots \wedge \omega^n \\ *\omega^i &= (-1)^i \omega^0 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^n \end{aligned}$$

or in other words

$$*\omega^\mu = \eta^{\mu\mu}(-1)^\mu \omega^0 \wedge \cdots \wedge \widehat{\omega^\mu} \wedge \cdots \wedge \omega^n.$$

Also show that

$$*\left(\omega^\nu \wedge \omega^\mu\right) = (-1)^{\nu+\mu+1} \eta^{\nu\nu}\eta^{\mu\mu} \omega^0 \wedge \cdots \wedge \widehat{\omega^\nu} \wedge \cdots \wedge \widehat{\omega^\mu} \wedge \cdots \wedge \omega^n.$$

Exercise 21. Maxwell's equations

Use the last problem to show that the electromagnetic field action can be written as

$$S(A) = -\frac{1}{16\pi} \int dA \wedge *dA$$

and that the functional derivative is

$$\frac{\delta S}{\delta A(x)} = \frac{1}{8\pi} \left(*^{-1} d * dA \right)^{\sharp}$$

so that the Euler–Lagrange equations are

$$d * dA = 0.$$

This compact notation does not make clear that this is a wave equation.

Exercise 22. Back to coordinates

Show that in coordinates, this operator is

$$\nabla_X Y^\mu = X^\nu \frac{\partial Y^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma} X^\nu Y^\sigma$$

where the $\Gamma^\mu_{\nu\sigma}$ are the Christoffel symbols.

11. Conservation of the energy-momentum tensor

We wish to make clear the significance of the energy-momentum tensor.

Exercise 23. Invariance of the metric

Prove the following:

(a)

$$\nabla_v g_{\mu\nu} = 0.$$

(Hint: you did this already.) How do you read this? It is not

$$\nabla_v \left(g_{\mu\nu} \right)$$

but rather

$$\left(\nabla_v g \right)_{\mu\nu} .$$

(b)

$$\nabla_v \left(g_{\mu\nu} X^\mu Y^\nu \right) = g_{\mu\nu} \left(X^\mu \nabla_v Y^\nu + Y^\nu \nabla_v X^\mu \right).$$

(c)

$$\nabla dV = 0.$$

Essentially we are repeating what Aaron did to show that the Hamiltonian is the conservation law of time translation. Suppose we write the action for some field ϕ as an integral

$$S = \int \Lambda$$

where Λ is a volume form whose coefficients depend on the field and its derivatives, and also on the metric and its derivatives:

$$\Lambda = L(\phi, \partial\phi, \dots, g, \partial g, \dots)\, dV.$$

The advantage to this notation is that we don't have to keep in mind that variations in the metric vary the volume form dV, so they hit something like $L\,dV$ in a complicated way.

The Euler–Lagrange equations are described by the vanishing of the functional derivative

$$dS[\phi, g](\delta\phi, \delta g) = \left.\frac{d}{d\varepsilon} S[\phi + \varepsilon\delta\phi, g + \varepsilon\delta g]\right|_{\varepsilon=0}.$$

Think of the $\delta\phi$ and δg as tangent vectors to the space of all fields ϕ and g. The magic is that we can calculate this derivative as an integral:

$$dS[\phi, g](\delta\phi, \delta g) = \int \frac{\partial\Lambda}{\partial\phi}\delta\phi + \frac{\partial\Lambda}{\partial g}\delta g.$$

If not convinced, try some examples. You will always use integration by parts some number of times. We can of course write this in our functional derivative notation as

$$\frac{\delta S}{\delta\phi(x)} = \frac{1}{dV}\frac{\partial\Lambda}{\partial\phi}.$$

This $\frac{\partial\Lambda}{\partial\phi}$ eats a variation $\delta\phi$ and spits out a volume form. Our energy-momentum tensor is

$$T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{dV}\frac{\partial\Lambda}{\partial g}.$$

Theorem 11.1. *Suppose ϕ satisfies the Euler–Lagrange equations of a diffeomorphism invariant action function $S[\phi]$. Then its energy-momentum tensor satisfies*

$$\nabla_\nu T^{\mu\nu} = 0.$$

Proof. Consider taking the action of our field ϕ just over a region R (a compact subset with smooth boundary). Under a family of diffeomorphisms Φ^s, we can write move the fields ϕ and g and then take the action:

$$S[\Phi^s\phi, \Phi^s g] = \int_{\Phi^s R} \Lambda[\Phi^s\phi, \Phi^s g].$$

By invariance of the action

$$\begin{aligned}
0 &= S[\Phi^s\phi, \Phi^s g] - S[\phi, g] \\
&= \int_{\Phi^s R} \Lambda[\Phi^s\phi, \Phi^s g]. - \int_R \Lambda[\phi, g] \\
&= \int_{\Phi^s R \cap R} (\Lambda[\Phi^s\phi, \Phi^s g] - \Lambda[\phi, g]) + \int_{\Phi^s R \setminus R} \Lambda[\Phi^s\phi, \Phi^s g] - \int_{R \setminus \Phi^s R} \Lambda[\phi, g]
\end{aligned}$$

Now as $s \to 0$, we get

$$0 = \left.\frac{d}{ds}\right|_{s=0} S[\Phi^s\phi, \Phi^s g] = \int_R \mathfrak{L}_X \Lambda[\phi, g] + \int_{\partial R} X \lrcorner \Lambda[\phi, g].$$

The first term becomes

$$\int_R \mathfrak{L}_X \Lambda[\phi, g] = \int_R \frac{\partial\Lambda}{\partial\phi}\mathfrak{L}_X \phi + \int_R \frac{\partial\Lambda}{\partial g}\mathfrak{L}_X g.$$

By the Euler–Lagrange equations, the first term here vanishes, and we have

$$0 = \int_R \frac{\partial\Lambda}{\partial g}\mathfrak{L}_X g + \int_{\partial R} X \lrcorner \Lambda[\phi, g].$$

Plugging in the definition of energy-momentum tensor (and returning to physicists' notation)

$$0 = \int_R T^{\mu\nu}\mathfrak{L}_X g_{\mu\nu}\, dV + \int_{\partial R} X \lrcorner \Lambda[\phi, g].$$

In particular, if our vector field X vanishes on the boundary of R, then

$$0 = \int_R T^{\mu\nu} \mathcal{L}_X g_{\mu\nu} \, dV.$$

Exercise 24.

Prove that

$$\mathcal{L}_X g_{\mu\nu} = g_{\mu\sigma} \nabla_\nu X^\sigma + g_{\nu\sigma} \nabla_\mu X^\sigma$$

where we are working in some local coordinates x^μ and

$$\nabla_\nu = \nabla_{\partial/\partial x^\nu}.$$

Using this identity:

$$\begin{aligned} 0 &= 2 \int_R T^{\mu\nu} g_{\mu\sigma} \nabla_\nu X^\sigma \, dV \\ &= 2 \int_R dV \left[\nabla_\nu \left(T^{\mu\nu} g_{\mu\sigma} X^\sigma \right) - \nabla_\nu T^{\mu\nu} g_{\mu\sigma} X^\sigma \right]. \end{aligned}$$

Thinking of $T^{\mu\nu} g_{\mu\sigma} X^\sigma \frac{\partial}{\partial x^\nu}$ as a vector field, Gauß's theorem applies to the first term, so it vanishes, giving

$$0 = -2 \int_R \nabla_\nu T^{\mu\nu} g_{\mu\sigma} X^\sigma \, dV.$$

Since this vanishes for every vector field X which vanishes on ∂R, we must have

$$\nabla_\nu T^{\mu\nu} = 0.$$

□

Theorem 11.2. *Conversely, if S is an action functional given by integrating a Lagrangian for some field ϕ, then the energy-momentum tensor of S satisfies*

$$\nabla_\nu T^{\mu\nu} = 0$$

precisely if S is invariant under the identity component of the diffeomorphism pseudogroup.

The proof is immensely more difficult, and I only know how to do it in the real analytic case. It requires proving that there are "a lot" of solutions of the field equations, hard PDE.

Theorem 11.3. *Suppose that K is a vector field on spacetime so that flow along K preserves the spacetime metric. Let*

$$P^\mu = T^{\mu\nu} g_{\nu\sigma} K^\sigma.$$

If P decays at infinity sufficiently rapidly then

$$\int_{t=t_0} P \lrcorner \, dV = \int_{t=t_1} P \lrcorner \, dV,$$

i.e. the integral $\int P \lrcorner \, dV$ over space is conserved over time. Here we are splitting into space and time by taking t any function with no critical points, whose level sets are spacelike.

Proof.

$$\begin{aligned}\nabla_\mu P^\mu &= \left(\nabla_\mu T^{\mu\nu}\right) g_{\nu\sigma} K^\sigma + T^{\mu\nu} g_{\nu\sigma} \nabla_\mu K^\sigma \\ &= T^{\mu\nu} g_{\nu\sigma} \nabla_\mu K^\sigma \\ &= \frac{1}{2} T^{\mu\nu} g_{\nu\sigma} \nabla_\mu K^\sigma + \frac{1}{2} T^{\mu\nu} g_{\mu\sigma} \nabla_\nu K^\sigma \\ &= T^{\mu\nu} \mathcal{L}_K g_{\mu\nu} \\ &= 0.\end{aligned}$$

Start by integrating over a large box R with one of its sides at $t = t_0$ and another at $t = t_1$.

$$\int_{\partial R} P \lrcorner\, dV = \int_R \nabla_\mu P^\mu \, dV$$

by Gauß's theorem

$$= 0.$$

□

In Minkowski space, we have 10 dimensions of these infinitesimal symmetries: the 10 dimensions of the Poincaré group. But in a general curved manifold, we can only hope to find that in small regions of the manifold, where the metric is nearly flat, there are vector fields whose flows are nearly symmetries. We interpret this as saying that there are 10 nearly conserved quantities.

The time translation generator of the Poincaré group is $\partial/\partial x^0$, and it gives a kind of "Hamiltonian":

$$\mathcal{H}\left(t_0\right) = \int_{t=t_0} T^{\mu\nu} g_{\nu 0} \, dV.$$

Recall from Aaron's talk that the Hamiltonian is the generator of time translation. Similarly, the space translations are generated in classical mechanics by momenta, and the space translations of the Poincaré group give nearly conserved quantities

$$\mathcal{P}^i\left(t_0\right) = \int_{t=t_0} T^{\mu\nu} g_{\nu i} \, dV$$

which we think of as momentum in the x^i direction.

According to Noether, there are actually conserved quantities given by diffeomorphism invariance, but this approach doesn't seem to find them.

12. Curvature

Consider a family of geodesics, all coming out of the same point of spacetime. Suppose we parameterize each of them by τ, and parameterize the members of the family by s. So we have a map taking an (s, τ) square into our manifold M^{1+n}, so that the τ direction is mapped into geodesics. Also, one side of the square, $\tau = 0$, is mapped into a single point. Along the image of this square, we have two vector fields tangent to the square, ∂_s and ∂_τ. These vector fields commute, because they are images of commuting vector fields from the square. Moreover the geodesic condition is

$$\nabla_t \partial_t = 0.$$

Exercise 25. Framing definition of curvature

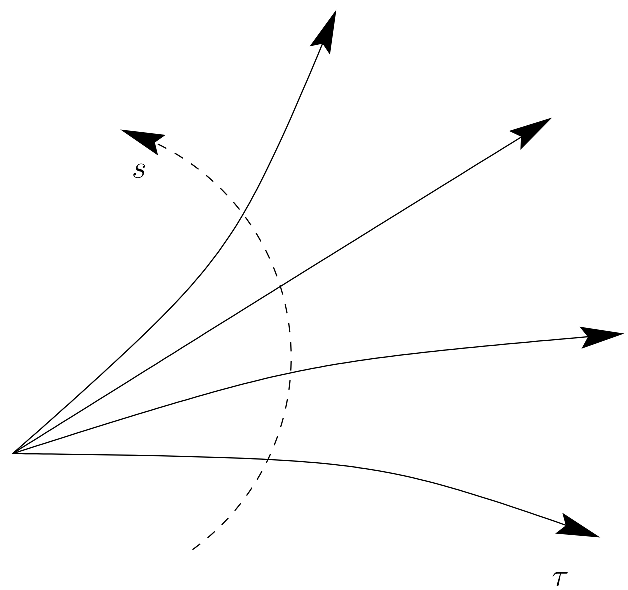

FIGURE 6. Geodesic spreading is controlled by curvature

Show that for any framing ω^μ there is a unique set of functions $R^\mu_{\nu\alpha\beta}$ so that

(1)
$$d\gamma^\mu_\nu = -\gamma^\mu_\sigma \wedge \gamma^\sigma_\nu + \frac{1}{2} R^\mu_{\nu\alpha\beta}\omega^\alpha \wedge \omega^\beta,$$

(2) $\frac{1}{2}R\omega \wedge \omega$ is a 2-form valued in the Lie algebra of the Lorentz group and

(3)
$$\sum \eta_{\alpha\beta} R^\beta_{\gamma\delta\epsilon} = 0$$
where the sum is a cyclic sum on α, γ, δ (the first Bianchi identity).

Exercise 26. Differential operator definition of curvature

Using a framing ω^μ, show that
$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = R(X,Y)Z$$
is a tensor. It is called the *Riemann curvature tensor*. Show that in terms of a framing it matches up with the framing definition of curvature.

Applying this to our square,
$$\begin{aligned} \nabla_s \nabla_t \partial_t &= \nabla_t \nabla_s \partial_t + R(\partial_s, \partial_t)\partial_t \\ &= \nabla_t \nabla_t \partial_s + R(\partial_s, \partial_t)\partial_t. \end{aligned}$$

So finally we have
$$\nabla_t^2 \partial_s + R(\partial_s, \partial_t)\partial_t$$

our Sturm–Liouville equation, which says that curvature R measures how fast geodesics spread apart, since the field ∂_s describes the spreading out of our family of geodesics.

We can split the curvature up into pieces, for example

$$R = R^{\mu}_{\nu\mu\sigma} g^{\nu\sigma}$$

is the scalar curvature. The curvature tensor belongs to a representation of the Lorentz group which splits into 4 irreducibles, called the scalar curvature, the Ricci curvature (which is $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$), and the Weyl tensor.

Exercise 27. Flatness

> Show that a Lorentz manifold is locally isometric to Minkowski space precisely when the curvature vanishes. Hint: parallel transport around a frame. This is independent of path (why?). Then show that each 1-form in the frame is closed, so locally the differential of a function. Here are your coordinates.

13. The field equations of Einstein

The simplest Lagrangians are the best. Which ones can we find? Starting with a metric, we can form its volume form dV. This can be integrated, and we can try $\int dV$ as action. We have seen that the energy-momentum tensor is essentially just the metric itself, so it can't vanish. But the Euler–Lagrange equations for a theory with just a metric express precisely the vanishing of the energy-momentum tensor.

There are no first order invariants of a metric, which is a difficult thing to prove. There are second order invariants: the curvature tensor is one. Also difficult to prove is that there aren't any other second order differential invariants than those contained in the curvature. However, there are lots of these. For example, horrible things like

$$g_{\alpha\alpha'} g^{\beta\beta'} g^{\gamma\gamma'} g^{\delta\delta'} W^{\alpha}_{\beta\gamma\delta} W^{\alpha'}_{\beta'\gamma'\delta'}$$

where W is the dreaded Weyl tensor.

However, the curvature tensor is linear in the second derivatives of the metric. So naturally we will look for Lagrangians which (1) depend on as few derivatives as possible and (2) depend on them as polynomials of the lowest possible order. This essentially determines the action: the *Einstein–Hilbert action*

$$S[g] = \int A\,(R - \Lambda)\; dV$$

where A and Λ are constants. The Euler–Lagrange equations, i.e. the energy-momentum tensor is

$$\frac{\delta S}{\delta g_{\mu\nu}} = T^{\mu\nu} = R^{\mu\nu} + \left(\Lambda - \frac{1}{2}\right) R g^{\mu\nu} = 0.$$

(The derivation of this result is quite long—a triumph of tensor calculus).

If we add in additional matter fields, we have to add their actions to this one, so if $T^{\mu\nu}_{\text{matter}}$ is the sum of the energy-momentum tensors of all of the other fields, then our equations for the whole theory become

$$R^{\mu\nu} + \left(\Lambda - \frac{1}{2}\right) R g^{\mu\nu} + T^{\mu\nu}_{\text{matter}} = 0.$$

But is this the right choice of action? Suppose that our particle moves slowly in a weak time-independent gravitational field. Slow means $dx^i/d\tau$ is small compared to $dx^0/d\tau$, so we can write out our geodesic equation

$$\frac{dx^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

as approximately

$$\frac{dx^\mu}{d\tau} + \Gamma^\mu_{00} \left(\frac{dx^0}{d\tau} \right)^2 = 0.$$

Weak means that $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric, and $\varepsilon h_{\mu\nu}$ is small. Time independent means

$$\frac{\partial h_{\mu\nu}}{\partial x^0} = 0,$$

so that

$$\Gamma^\mu_{00} = -\varepsilon \frac{1}{2} \eta^{\mu\nu} \frac{\partial h_{00}}{\partial x^\nu} + O\left(\varepsilon^2\right).$$

The equations of a geodesic are then

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \left(\frac{dx^0}{d\tau} \right)^2 \frac{\partial h_{00}}{\partial x^i}$$
$$\frac{d^2 x^0}{d\tau^2} = 0.$$

So $t = x^0/c$ is a linear function of τ, which we take to be $t = a\tau$. Dividing by a^2, we get

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i}.$$

Compare this to Newton's law

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial V}{\partial x^i}$$

where V is the gravitational potential energy. We have

$$h_{00} = -2V + \text{constant}.$$

Assuming that the perturbation $h_{\mu\nu}$ to the gravitational field dies off at great distances, as does the Newtonian gravitational potential, we have

$$h_{00} = -2V$$

and so

$$g_{00} = \eta_{00} + h_{00} = -(1 + 2V).$$

Newton's law of gravitation says that a point of mass m creates a gravitational potential

$$-\frac{Gm}{r}$$

which is a harmonic function away from the point itself. So if we suppose that our gravitational field is away from sources, we should find that it is harmonic. In the presence of matter, we should get

$$\Delta g_{00} = 2G\rho$$

where ρ is the density of matter.

The fact that this is a second order partial differential equation leads us to believe that any gravitational field $g_{\mu\nu}$ will satisfy second order field equations.

Here is a big surprise (proven in Weinberg [2] pp. 155–157):

Theorem 13.1. *There is no diffeomorphism invariant second order action for a Lorentz metric which has second order Euler–Lagrange equations other than*

$$S = \int A \left(R - \Lambda \right) \, dV.$$

So we surely have the correct action. The constant Λ is called the *cosmological constant*, and controls the expansion of the universe.

References

[1] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1. 1

[2] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, Wiley, New York, 1972. 1, 24

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