MANIFOLDS WITH G_2 HOLONOMY

Introduction

These are notes from Robert Bryant's 1998 lectures at Duke University about Riemannian manifolds with G_2 holonomy.

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Table 1. Nonclassical representations with open orbits

1. Representation theory of G_2

1.1. **G_2 as a classical group.** In Robert's view (but not Weyl's) G_2 can be seen as a classical group. The classical groups all act as stabilizers of objects belonging to open orbits of the general linear group. For example:

$\mathbf{SO(p,q)}$

$GL(n,\mathbb{R})$ acts on $S^2\mathbb{R}^n$ (quadratic forms on $\mathbb{R}^n$) with dense open orbits - quadratic forms with nonzero eigenvalues constitute these orbits. The stabilizer of a quadratic form in one of these open orbits is isomorphic to $SO(p,q)$ for some p,q with $p+q=n$.

$\mathbf{Sp(2n,\mathbb{R})}$

$GL(2n,\mathbb{R})$ acts on $\Lambda^2\mathbb{R}^{2n}$ with a single dense open orbit - nondegenerate skew symmetric quadratic forms constitute this orbit. The stabilizer of a nondegenerate skew symmetric quadratic form is isomorphic to $Sp(2n,\mathbb{R})$. (We could have used $\Lambda^{2n-2}\mathbb{R}^{2n}$ instead of $\Lambda^2\mathbb{R}^{2n}$ because they are dual and have equivalent $GL(2n,\mathbb{R})$ representations.)

$\mathbf{SL(n,\mathbb{R})}$

$GL(n,\mathbb{R})$ acts on $\Lambda^n\mathbb{R}^n$ (which is a one dimensional vector space) with open orbit consisting of nonzero elements, i.e. volume forms. The stabilizer of a volume form is isomorphic to $SL(n,\mathbb{R})$.

When does this pattern work? By highest weight theory for $GL(n,\mathbb{R})$ we can prove that these are all of the open orbits of $GL(n,\mathbb{R})$ acting on any of its representations, except those listed in table 1.

In the first row of table 1, the stabilizer of a generic homogeneous polynomial of degree 3 in 2 variables is a finite group. Our attention focuses on the second row. Robert leaves the study of the third row as an exercise.

1.2. **G_2 defined.** Let V be a 7 dimensional vector space over a field $\mathbb{F}$, which we will assume is either $\mathbb{R}$ or $\mathbb{C}$. Take $\phi\in\Lambda^3V^*$. Define

$$B_\phi : V\times V\to\Lambda^7V^* \qquad B_\phi(x,y)=\frac{1}{6}(x\lrcorner\phi)\wedge(y\lrcorner\phi)\wedge\phi$$

Claim: for ϕ generic, this is nondegenerate. Indeed for generic ϕ Robert claims that

$$0\neq\det B_\phi\in S^9\Lambda^7V^*$$

To define this determinant, consider how we define the determinant of a symmetric quadratic form. First, the determinant of a linear map $\psi: U\to W$ between vector spaces is zero unless $\dim U=\dim W$, in which case it is

$$\operatorname{Det}\psi:\operatorname{Det}U=\Lambda^{\dim U}U\to\operatorname{Det}W=\Lambda^{\dim W}W$$
$$\operatorname{Det}\psi(u_1\wedge\cdots\wedge u_{\dim U})=\psi(u_1)\wedge\cdots\wedge\psi(u_{\dim U})$$

If $U = W$, then we can compare $\operatorname{Det}\psi$ to $\operatorname{Det} Id$ and use this to define the number $\det\psi = \operatorname{Det}\psi / \operatorname{Det} Id$. Now for a quadratic form, suppose $Q \in S^2V^*$. We can treat Q as a symmetric map $Q : V \to V^*$ defined by

$$Q(v) = Q(v, \cdot)$$

Taking Det we have

$$\operatorname{Det} Q : \operatorname{Det} V \to \operatorname{Det} V^*$$

as a symmetric map, and define the quadratic form

$$\det Q(\Omega) := \langle \operatorname{Det} Q(\Omega), \Omega\rangle$$

We have as determinant

$$\det Q \in S^2 \operatorname{Det} V^*$$

Now for our object B_ϕ, we can pick some nonzero volume element $\Omega \in \Lambda^7 V$, and then look at the quadratic form

$$Q_\Omega(x, y) = \langle B_\phi(x, y), \Omega\rangle$$

which is an element of S^2V^*. Note

$$Q_{\lambda\Omega} = \lambda Q_\Omega$$

We can define as above the determinant of this quadratic form. We can then plug in Ω again to this determinant

$$\det B_\phi(\Omega) = \det Q_\Omega(\Omega)$$

This gives us our determinant object, and it scales like

$$\begin{aligned} \det B_\phi(\lambda\Omega) &= \det Q_{\lambda\Omega}(\lambda\Omega) \\ &= \lambda^2 \det(\lambda Q_\Omega)(\Omega) \\ &= \lambda^2\lambda^7 \det Q_\Omega(\Omega) \\ &= \lambda^9 \det Q_\Omega(\Omega) \end{aligned}$$

Therefore

$$\det B_\phi \in S^9\Lambda^7V^*$$

Moreover

$$\phi \mapsto \det B_\phi$$

is degree 21. We need to show that it is not everywhere 0. This requires computing an example. But if we have some ϕ for which $\det B_\phi \neq 0$, then the equation

$$\det B_\phi(\Omega) = 1$$

has as many solutions $\Omega \in \Lambda^7 V$ as the field $\mathbb{F}$ has ninth roots of unity. Over $\mathbb{R}$, this determines a unique volume form. In terms of this volume form, the quadratic form

$$\langle B_\phi(x, y), \Omega\rangle$$

will give a nondegenerate inner product on V, so the symmetry group of ϕ will preserve a quadratic form and a volume form, and hence will be a subgroup of $SO(V)$.

Let

$$e_1, e_2, \ldots, e_7$$

be a basis of V, and let

$$e^1, e^2, \ldots, e^7$$

be the dual basis. Set

$$e^{ijk} := e^i \wedge e^j \wedge e^k$$

and let

$$\phi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

Definition 1. G_2 is the stabilizer subgroup of ϕ_0 in $GL(7, \mathbb{R})$.

Lemma 1. *G_2 acts transitively on the sphere of rays through the origin in $\mathbb{R}^7$.*

Proof. The 3 form ϕ_0 can be written

$$\phi_0 = e^1 \wedge \left(e^{23} + e^{45} + e^{67}\right) + \operatorname{Re}\left(\left(e^2 + ie^3\right) \wedge \left(e^4 + ie^5\right) \wedge \left(e^6 + ie^7\right)\right)$$

The linear transformations fixing ϕ_0 and e_1 must also fix the Kähler form

$$e^{23} + e^{45} + e^{67}$$

on $\mathbb{R}^7/\text{span } e_1$, and the volume form

$$\operatorname{Re}\left(\left(e^2 + ie^3\right) \wedge \left(e^4 + ie^5\right) \wedge \left(e^6 + ie^7\right)\right)$$

Therefore it acts on $\mathbb{R}^7/\text{span } e_1 = \mathbb{C}^3$ as $SU(3)$, and acts transitively on $S^5 \subset \mathbb{C}^3$, so acts transitively on the equatorial $S^5 \subset S^6$. We could also write

$$\phi_0 = e^2 \wedge (e^{31} + e^{46} + e^{75}) + \operatorname{Re}(\dots \text{exercise} \dots)$$

and see that our stabilizer group would act transitively on another $S^5 \subset S^6$. It is fairly easy to get the result from here. $\square$

We could just as well have defined G_2 as the stabilizer of

$$\psi = e^{4567} + e^{2367} + e^{2345} + e^{1357} + e^{1346} + e^{1256} + e^{1247}$$

Proposition 1.

$$\det B_{\phi_0} \neq 0$$

Proof. By the last lemma, it suffices to calculate

$$B_{\phi_0}(e_1, e_1) = (e_1 \lrcorner \phi_0)^2 \wedge \phi_0$$

But this is not zero because

$$\begin{aligned} e_1 \lrcorner B_{\phi_0}(e_1, e_1) &= (e_1 \lrcorner \phi_0)^3 \\ &= \left(e^{23} + e^{45} + e^{67}\right)^3 \\ &= \text{cube of Kähler form on } \mathbb{C}^3 \\ &\neq 0 \end{aligned}$$

$\square$

1.3. **Some properties of G_2.** We can calculate that the quadratic form preserved by G_2 is the usual

$$\left(e^1\right)^2 + \dots + \left(e^7\right)^2$$

and so the stabilizer subgroup in G_2 of e_1 is $SU(3)$. This gives the principal bundle

$$\begin{array}{ccc} SU(3) & \longrightarrow & G_2 \quad \subset SO(7) \\ & & \downarrow \\ & & S^6 \end{array}$$

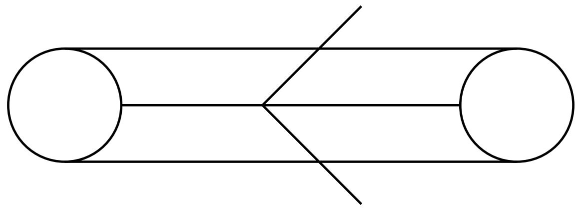


FIGURE 1. The root system of G_2

This determines essentially all of the topological properties of G_2. For instance its dimension must be

$$\dim G_2 = \dim SU(3) + \dim S^6 = 8 + 6 = 14$$

Also its homotopy groups can be determined, for instance:

$$\pi_1(G_2) = 1$$
$$\pi_2(G_2) = 1$$
$$\pi_3(G_2) = \pi_2(SU(3)) = \mathbb{Z}$$

The dimension count tells us that

$$\dim G_2 = 14 = \binom{7}{2} = \dim \Lambda^3\mathbb{R}^7$$

which shows that the orbit of $GL(7,\mathbb{R})$ on $\Lambda^3\mathbb{R}^7$ through ϕ_0 must be an open orbit.

Note that there are 2 open orbits of $GL(7,\mathbb{R})$ on $\Lambda^3\mathbb{R}^7$, one of which gives G_2 as stabilizer as above. This is the compact form. The noncompact split real form is the other stabilizer, say the stabilizer of

$$\phi_1 := -e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

We will not need to study the split form any further.

The principal bundle structure shows us that S^6 has a remarkable almost complex structure with trivial canonical bundle. It is not integrable, and no complex structure is known on S^6. The holomorphic curves in S^6 with this almost complex structure were studied in [4].

It appears as if G_2 were some kind of special unitary group since it contains $SU(3)$ as a maximal subgroup. However G_2 is not a subgroup of $SU(4)$.

The homotopy groups given above show that G_2 is a simple Lie group, since any compact Lie group K will satisfy

$$\text{rank } H_1(K) = \text{ number of } S^1 \text{ factors in } K$$
$$\text{rank } \pi_3(K) = \text{ number of simple factors in } K$$

The center of $SU(3)$ is $\mathbb{Z}/3\mathbb{Z}$. It turns out that the center of G_2 is trivial. Therefore every representation of G_2 is faithful or trivial. G_2 is a linear algebraic group by its definition.

1.4. **Representations of G_2.** The maximal tori of $SU(3)$ are maximal tori for G_2. The root system of G_2 is drawn in figure 1 on the preceding page. Choosing a Weyl chamber, we can calculate the irreducible representations, $V^{m,n}$. The first few are

$$\begin{aligned}
V^{1,0} &= V = \mathbb{R}^7 \\
V^{0,1} &= Ad = \mathfrak{g}_2 = \mathbb{R}^{14} \\
V^{2,0} &= S_0^2 V = \mathbb{R}^{27} \\
V^{1,1} &= \mathbb{R}^{64} \\
V^{0,2} &= \mathbb{R}^{77} \\
V^{3,0} &= \mathbb{R}^{77} \\
V^{n,0} &= S_0^n V
\end{aligned}$$

Using an expression found in [9], we can find the decomposition into irreducibles of tensor products of the irreducible representations. For example

$$V^{1,0} \otimes V^{0,1} = V^{1,1} \oplus V^{1,0} \oplus V^{2,0}$$

which can be determined from the above table by the "rule of 3".

1.5. **Decomposition of the exterior algebra $\Lambda^*\mathbb{R}^7$ under G_2.**

$$\begin{aligned}
\Lambda^1 V^* &= \text{irreducible} \\
\Lambda^2 V^* &= \mathfrak{g}_2 \oplus V = \Lambda^2_{14} V^* \oplus \Lambda^2_7 V^* \\
\Lambda^3 V^* &= \mathfrak{gl}(7,\mathbb{R})/\mathfrak{g}_2 \\
&= (V \otimes V)/\mathfrak{g}_2 \\
&= (S^2 V \oplus \Lambda^2 V)/\mathfrak{g}_2 \\
&= (\text{trivial} \oplus S_0^2 V \oplus \mathfrak{g}_2 \oplus V)/\mathfrak{g}_2 \\
&= \mathbb{R} \oplus V \oplus S_0^2 V \\
\Lambda^4 V^* &= \Lambda^3 V^* \\
\text{etc. } \ldots
\end{aligned}$$

2. G_2 MANIFOLDS AND G_2 GEOMETRIES

2.1. **Positive 3-forms.** We have seen that the orbit of

$$\phi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

in $\Lambda^3\mathbb{R}^7$ under the action of $GL(7,\mathbb{R})$ is an open orbit, call it

$$\Lambda^3_+\mathbb{R}^7$$

Call these positive 3 forms. Note that if ϕ is positive then so is $-\phi$. We have seen how to define equivariant maps

$$\begin{array}{ccc}
\Lambda^3_+ V^* & \longrightarrow & \Lambda^7 V^* \\
& \searrow & \\
& & S^2_+ V^*
\end{array}$$

(the volume element and metric associated to the choice of ϕ). For $\phi \in \Lambda^3_+ V^*$, we write $*_\phi$ for the Hodge star operator, and g_ϕ for the metric.

2.2. *Spin*(7) **and** G_2**.** We need to recognize the placement of G_2 inside $Spin(7)$. Recall that $Spin(7)$ acts transitively on $S^7 \subset \mathbb{R}^8$. The stabilizer subgroup of a nonzero vector in $\mathbb{R}^8$ turns out to be G_2. To see this, suppose we call the stabilizer subgroup H. We get a principal bundle

$$\begin{array}{ccc} H & \longrightarrow & Spin(7) \\ & & \downarrow \\ & & S^7 \end{array}$$

Notice that $\dim Spin(7) = 21$. So $\dim H = 14$, and H is a compact Lie group, indeed a subgroup of $SO(7)$. Also we see that

$$\pi_1(H) = 1$$
$$\pi_3(H) = \pi_3\left(Spin\left(7\right)\right) = \mathbb{Z}$$

Therefore H is a 14 dimensional simple Lie group, and by the classification of such groups, it is G_2.

2.3. **Cayley's Octonions.** Associated to any $\phi \in \Lambda^3_+ V^*$, we can define a cross product on $\mathbb{R}^7$ by

$$\langle x \times y, z \rangle = \phi(x, y, z)$$

The group G_2 acts transitively on 2 planes, so the cross product can easily be seen to satisfy

$$||x \times y||^2 = \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2$$

We then define a multiplication on $\mathbb{R} \oplus V$ by

$$(a, x) \cdot (b, y) = \left(ab - \langle x, y \rangle, ay + bx + x \times y\right)$$

This gives us the octonions. We can check that they are a non-associative division algebra, with G_2 acting as automorphisms. It turns out that G_2 is the entire automorphism group.

2.4. G_2 **structures on 7 dimensional manifolds.** A G_2 structure on a 7 dimensional manifold is a section of $\Lambda^3_+ T^*$.

$$\begin{array}{ccc} GL(7, \mathbb{R})/G_2 & \longrightarrow & \Lambda^3_+ T^* M \\ & & \downarrow \\ & & M \end{array}$$

Proposition 2. *A 7 manifold admits a G_2 structure precisely when it is orientable and spinable.*

Proof. A manifold with G_2 structure has an induced volume element, so is orientable. Also, G_2 is simply connected, and a subgroup of $SO(7)$, so we get a reduction of structure group to a simply connected group, hence a spin structure.

Conversely, suppose that M is orientable and spinable. Let

$$\begin{array}{ccc} \mathbb{R}^8 & \longrightarrow & \mathbb{S} \\ & & \downarrow \\ & & M \end{array}$$

be a spin structure. An 8 dimensional vector bundle over a 7 manifold always admits a nowhere vanishing section. We consider the subgroup of $Spin(7)$ fixing our nowhere zero section. This provides a restriction of the structure group to

$$Spin(7) \cap SO(7) = G_2 \subset SO(8)$$

□

The spin bundle in this story emerges from the $\mathbb{R}^7$ part that G_2 acts on together with a trivial $\mathbb{R}$ factor. The algebra Clifford(7) acts on $\mathbb{R}^8$, and gives rise to the action of $Spin(7)$ on $\mathbb{R}^8$.

2.5. **First order flatness of G_2 structures.**

Definition 2. A G_2 structure is 1–flat if it agrees with the constant G_2 structure on $\mathbb{R}^7$ up to first order.

Suppose that $\phi \in \Omega^3_+ M$ is the 3–form giving the G_2 structure, and in local coordinates $x^1, \dots, x^7$

$$\phi = a_{ijk} dx^i \wedge dx^j \wedge dx^k$$

Then 1–flatness means that

$$\frac{\partial a_{ijk}}{\partial x^l} = 0$$

Clearly 1–flatness implies

$$d\phi = 0 \text{ and } d *_\phi \phi = 0$$

This appears to be 56 equations on 35 functions of 7 variables. It turns out the be only 49, because of some overlap.

$$\begin{array}{ccc} \Lambda^3_+ T^*M & \longleftarrow & 35 \text{ functions} \\ \downarrow & & \\ M & \longleftarrow & 7 \text{ variables} \end{array}$$

Proposition 3. *A G_2 structure ϕ is 1–flat precisely if*

$$d\phi = 0 \textit{ and } d *_\phi \phi = 0$$

which happens precisely if

$$\nabla^\phi \phi = 0$$

where ∇^ϕ is the Levi Civita connection of the metric g_ϕ.

Proof. Clearly 1–flatness implies both of the above conditions. If

$$\nabla^\phi \phi = 0$$

then in geodesic normal coordinates, we find that ϕ is 1–flat. A coframing

$$\omega_1, \dots, \omega_7$$

is a ϕ coframing if

$$\phi = \omega_{123} + \omega_{145} + \omega_{167} + \omega_{246} - \omega_{257} - \omega_{347} - \omega_{356}$$

For such a coframing

$$g_\phi = (\omega_1)^2 + \cdots + (\omega_7)^2$$

We can define the connection 1 forms θ_{ij} by

$$d\omega_i = -\theta_{ij} \wedge \omega_j \qquad \theta_{ij} + \theta_{ji} = 0$$

We define ϵ_{ijk} by

$$\phi = \frac{1}{6}\epsilon_{ijk}\omega_{ijk}$$

We let

$$\tau_k = \epsilon_{ijk}\theta_{ij}$$

Noting that the Lie algebra of G_2 is

$$\mathfrak{g}_2 = \{a_{ij} = -a_{ji} \,|\, \epsilon_{ijk}a_{ij} = 0 \text{ for all } k\}$$

we see that $\tau_k = 0$ precisely if the connection matrix has values in the Lie algebra $\mathfrak{g}_2$. This occurs exactly when

$$\nabla^\phi \phi = 0$$

Write

$$\tau_k = t_{kj}\omega_j$$

and we see exactly 49 scalar functions t_{kj} which vanish precisely when the G_2 structure is 1–flat.

The equation $d\phi = 0$ gives 35 independent 1st order equations, since d takes a section of $\Lambda^3_+ T^*$ to a section of $\Lambda^4 T^*$. The equation $d *_\phi \phi = 0$ gives 27 equations for the same reason.

$$\Lambda^4\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^7 \oplus \mathbb{R}^{27}$$

with the $\mathbb{R}^{27}$ factor being identified as a representation with $S^2_0\mathbb{R}^7$.

$$\Lambda^5\mathbb{R}^7 = so(7) = \mathbb{R}^7 \oplus \mathfrak{g}_2$$

The terms t_{kj} give

$$\begin{aligned}(t_{kj}) \in \mathbb{R}^7 \otimes \mathbb{R}^7 &= S^2\mathbb{R}^7 \oplus \Lambda^2\mathbb{R}^7 \\ &= \mathbb{R} \oplus S^2_0\mathbb{R}^7 \oplus \mathbb{R}^7 \oplus \mathfrak{g}_2\end{aligned}$$

Therefore the equations $d\phi = d *_\phi \phi = 0$ account for all 49 conditions. □

Proposition 4. *The metric g_ϕ associated to a 1–flat G_2 structure ϕ has holonomy contained in G_2. Moreover a Riemannian metric with holonomy G_2 leaves parallel the 3–form ϕ of a unique 1–flat G_2 structure*

Proof. This says just that ϕ is parallel with respect to ∇^ϕ. □

Proposition 5. *A Riemannian spin manifold of dimension 7 has holonomy contained in G_2 iff it has a parallel nowhere vanishing spinor.*

Proof. We know that a G_2 structure is spinable, and that the spin bundle splits into $TM \oplus \mathbb{R}$, one trivial factor, from our study of $G_2 \subset \text{Spin}(7)$. It remains to check that this is parallel. Conversely, we also have a G_2 structure from the isotropy of the spinor, and have to see that it is torsion free. □

2.6. **Curvature, analyticity and fundamental groups of G_2 manifolds.**

Proposition 6. *Every Riemannian manifold with G_2 holonomy is Ricci flat.*

Proof. In terms of the ϕ coframings used above, our curvature forms are

$$\begin{aligned}\Omega_{ij} &= d\theta_{ij} + \theta_{ik} \wedge \theta_{kj} \\ &= \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l\end{aligned}$$

and

$$\epsilon_{ijm} R_{ijkl} = \epsilon_{ijm} R_{klij} = 0$$

Therefore

$$(R_{ijkl}) \in S^2 \mathfrak{g}_2 = S^2 V^{0,1}$$

and the curvature takes values in

$$\begin{aligned}S^2 V^{0,1} &= S^2 \mathbb{R}^{14} \\ &= \mathbb{R}^{105} \\ &= V^{0,2} \oplus \mathbb{R} \oplus S^2_0 \mathbb{R}^7\end{aligned}$$

The equation

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

turns out to force the $\mathbb{R} \oplus S^2_0 \mathbb{R}^7$ part to vanish. The curvature is taking values in $V^{0,2}$ which is 77 dimensional. There are 105-77=28 equations satisfied by the curvature. These include the Ricci curvature. □

There are $7^2(7^2-1)/12 = 196$ components of the curvature tensor of a 7 dimensional Riemannian manifold: 28 components of Ricci tensor, and 168 of Weyl. Holonomy in G_2 forces $168 - 77 = 91$ equations on the Weyl tensor, so the angle geometry is quite special as well. Ricci flatness forces the metric (and the 3–form) to be analytic in harmonic local coordinates, so without lose of generality, G_2 manifolds are real analytic. Modulo diffeomorphism, the 1–flatness condition is an elliptic equation.

Proposition 7. *A compact manifold M with 1–flat G_2 structure is (after perhaps taking a finite cover) a product*

$$M = T^d \times N^{7-d}$$

where $d = b_1(M)$, and the Riemannian metric is the product metric of a flat metric on the torus T^d, with a Riemannian metric on N that has no parallel 1–forms. The holonomy group of the Riemannian metric is G_2 precisely if $b_1(M) = 0$.

Proof. By Bochner's identity,

$$\Delta \alpha = \nabla^* \nabla \alpha + \mathrm{Ric} \cdot \alpha$$

on any Ricci flat manifold, every harmonic 1–form is parallel, since

$$\int \langle \alpha, \Delta \alpha \rangle = \int |\nabla \alpha|^2$$

By the Cheeger-Gromoll theorem, after taking a finite cover, we get the indicated splitting. If the holonomy is G_2, then $b_1(M) = 0$. By Berger's classification of holonomy groups, we can check that every subgroup of G_2 which can occur as the holonomy group of a Riemannian metric leaves a 1–form parallel. If $b_1(M) = 0$, then the holonomy group is G_2. □

The manifolds we are interested in have finite fundamental group, and we may restrict attention to the compact, simply connected, spinable real analytic manifolds, and look for 1–flat G_2 structures on them. We will see more obstructions to existence of 1–flat G_2 structures shortly.

2.7. **Comparison to $SO(3)$ structures.** Consider the irreducible representations

$$SO(3) \hookrightarrow SO(2n+1)$$

If $n = 1$ then a $SO(3)$ structure on a $2n+1 = 3$ manifold is just a Riemannian metric, even if the structure is only 0 flat. If $n = 2$ on the other hand then an $SO(3)$ structure on a $2n+1 = 5$ manifold is 1–flat only if the manifold is a flat Riemannian 5 manifold, or a locally symmetric space locally of the form

$$SL(3,\mathbb{R})/SO(3) \text{ or } SU(3)/SO(3)$$

In all higher dimensions, an $SO(3)$ structure on a $2n+1$ dimensional manifold is 1–flat precisely if it is a flat Riemannian metric.

2.8. **Cohomology of G_2 manifolds and obstructions.** There are locally no more identities in the curvature. In fact in [2], Robert proved that all of the elements of $V^{0,2}$ occur as curvature tensors of Riemannian metrics with G_2 holonomy. We won't look at the curvature tensor for more information.

The equation

$$\nabla^{\phi}\phi = 0$$

implies that ϕ is harmonic, so $H^3(M) \neq 0$, and also $*_{\phi}\phi$ is harmonic so $H^4(M) \neq 0$.

Proposition 8. *If a compact manifold M^7 has a 1–flat G_2 structure ϕ, then*

$$[\phi] \cup [p_1(M)] \neq 0$$

unless the G_2 structure is flat.

Proof. Recall

$$p_1(M) = \operatorname{tr}\left(\Omega^2\right)$$

where Ω is the curvature 2–form.

$$\phi \wedge p_1(M) = q(R) *_{\phi} 1$$

where $q(R)$ is some quadratic form in the curvature tensor R. By irreducibility of the representation of G_2 on curvature, q must be definite or zero. We only need to see that $q \neq 0$.

$$\int \phi \wedge p_1(M) = \int q(R) *_{\phi} 1 = c||R||^2_{L^2}$$

for some universal constant c. We actually only need that $c \neq 0$. It suffices to check one example. Not only does G_2 contain $I_1 \times SU(3)$, but it contains $I_3 \times SU(2)$ as well. Thus, an example is $M = T^3 \times X$ where X is a K3 surface. However the 7-form $\phi \wedge p_1(M)$ is clearly just $\phi \wedge p_1(X)$ in this case, and the integral of this latter form over M is

$$\int_M \phi \wedge p_1(X) = 3 \cdot \operatorname{vol}(T^3) \cdot \operatorname{sign}(X)$$

which is non-zero. Finally, this proves that a Calabi-Yau 3-fold always has c_2 non-zero. In fact, though, a straightforward calculation shows that for a CY 3-fold Y with Kahler form ω,

$$\int_Y \omega \wedge c_2(Y) = (\text{a non-zero universal constant}) \cdot ||R||^2_{L^2(Y)}$$

so this can be seen directly. □

Proposition 9. *If a compact Riemannian manifold has holonomy G_2 then it is not diffeomorphic to a product.*

Proof. After taking a finite cover we can assume that our manifold M is compact, simply connected and spinable, with $H^3(M) \neq 0$, and $H^4(M) \neq 0$ and

$$H^3(M) \cup p_1(M) \neq 0$$

If M was a product, we know none of the factors could be 1 dimensional, since it is simply connected. If M had a factor which was a surface then, being simply connected, the factor would be a sphere

$$M^7 = S^2 \times N^5$$

with N simply connected, so $0 = H^1(N) = H^4(N)$ (Poincaré duality). I will write $\mathcal{H}$ for harmonic forms: we know that

$$\mathcal{H}^2(M, \mathbb{R}) = \mathcal{H}^2_{14}(M, \mathbb{R}) = \{\beta \,|\, \Delta\beta = 0 \text{ and } \beta \wedge \phi = - * \beta\}$$

If $\beta \in \mathcal{H}^2(M, \mathbb{R})$, then

$$\beta^2 \wedge \phi = -|\beta|^2 * 1$$

which implies that

$$\int \beta^2 \wedge \phi \neq 0 \text{ unless } \beta = 0$$

But the cohomology class $\beta = [S^2]$ dual to the S^2 factor satisfies

$$\beta = [S^2] \neq 0 \text{ although } \beta^2 = [S^2]^2 = 0$$

which is a contradiction.

This leaves the case of $M = X^3 \times Y^4$. We find

$$H^*(X, \mathbb{R}) = \mathbb{R} \cdot 1 \oplus 0 \oplus 0 \oplus \mathbb{R} \cdot [X]$$
$$H^*(Y, \mathbb{R}) = \mathbb{R} \cdot 1 \oplus 0 \oplus \operatorname{span}\{\beta_1, \ldots, \beta_k\} \oplus 0 \oplus \mathbb{R} \cdot [Y]$$

where the β_i are some basis for the harmonic 2-forms on Y. We find

$$\left(\sum_i \lambda_i \beta_i\right)^2 \wedge \phi \neq 0$$

for any numbers $\lambda_1, \ldots, \lambda_k$ not all zero. Clearly the cohomology class $[\phi]$ is a multiple of $[X] \in H^*(M, \mathbb{R})$, and $[*\phi]$ is a multiple of $[Y] \in H^*(M, \mathbb{R})$ so

$$\left(\sum_i \lambda_i \beta_i\right)^2 = Q(\lambda_1, \ldots, \lambda_k)[Y]$$

for some definite quadratic form Q. Therefore Y is a simply connected 4 dimensional manifold with definite intersection form. By Donaldson's theorem, such a manifold can not be spin.

We know that M is spin, and by the Wu formula,

$$w_1(X) = w_2(X) = 0$$

so

$$w_2(M) = w_2(Y)$$

which means that Y is spin, a contradiction. □

$H^i_j(M,\mathbb{R})$	$j=1$	$j=7$	$j=14$	$j=27$
$i=0$	1	0	0	0
$i=1$	0	$\star$	0	0
$i=2$	0	$\star$	$\star$	0
$i=3$	ϕ	$\star$	0	$\star$
$i=4$	$*_\phi\phi$	$\star$	0	$\star$
$i=5$	0	$\star$	$\star$	0
$i=6$	0	$\star$	0	0
$i=7$	$*_\phi 1$	0	0	0

TABLE 2. $H^i_j(M)$, $\star$ means could be nonzero

It is not known if a G_2 manifold can be a fiber bundle.

3. Deformation theory and the explicit examples

3.1. Hodge-like decomposition of cohomology.

Theorem 1. *If M^n is a compact Riemannian manifold with holonomy $H \subset O(n)$ and*

$$\pi : \Lambda^* T^* \to \Lambda^* T^*$$

is any parallel map then π commutes with H and with Δ.

For $G_2 \subset O(7)$,

$$\begin{aligned}
\Lambda^1 T^* &= \Lambda^1_7 T^* \\
\Lambda^2 T^* &= \Lambda^2_7 T^* \oplus \Lambda^2_{14} T^* \\
\Lambda^3 T^* &= \Lambda^3_1 T^* \oplus \Lambda^3_7 T^* \oplus \Lambda^3_{27} T^* \\
\Lambda^4 T^* &= \Lambda^4_1 T^* \oplus \Lambda^4_7 T^* \oplus \Lambda^4_{27} T^*
\end{aligned}$$

$$\begin{aligned}
\Lambda^2_7 T^* &= \{ v \lrcorner \phi \,|\, v \in T \} \\
\Lambda^3_1 T^* &= \mathbb{R}\phi \\
\Lambda^3_7 T^* &= \{ v \lrcorner *_\phi \phi \,|\, v \in T \} \\
\Lambda^4_1 T^* &= \mathbb{R} *_\phi \phi
\end{aligned}$$

This decomposes the cohomology groups (in a fashion analogous to Hodge theory) into $H^*_*(M)$. We indicate in the table 2 which of these groups might be nonzero with a $\star$ star, or with an explicit generator if it is one dimensional.

On a compact Riemannian manifold with G_2 holonomy, we can use Bochner identities to force $H^1(M) = 0$, as we saw before. Using $*_\phi$ and $\wedge\phi$ we can thereby kill all $\star$ entries in the 7 dimensional column. There are only two independent "Betti numbers" left unknown: b^2_{14} and b^3_{27}, as in table 3 on the following page. There are no known bounds on these numbers. See [12] for a table of Betti numbers known to occur among G_2 manifolds.

$H^i_j(M,\mathbb{R})$	$j=1$	$j=7$	$j=14$	$j=27$
$i=0$	1	0	0	0
$i=1$	0	0	0	0
$i=2$	0	0	b^2_{14}	0
$i=3$	ϕ	0	0	b^3_{27}
$i=4$	$*_\phi\phi$	0	0	b^3_{27}
$i=5$	0	0	b^2_{14}	0
$i=6$	0	0	0	0
$i=7$	$*_\phi\ 1$	0	0	0

TABLE 3. $H^i_j(M)$ for M compact

3.2. **The G_2 cone.** If ψ is a harmonic 2-form then

$$\psi \wedge *_\phi\psi = -\psi \wedge \phi \wedge \psi = *_\phi|\psi|^2$$

We have a pairing

$$H^2(M) \times H^2(M) \to H^7(M) \cong \mathbb{R} \qquad [\psi_1]\,,[\psi_2] \mapsto -\int \psi_1 \wedge \psi_2 \wedge \phi$$

which is negative definite. Consequently to have a 1–flat G_2 structure,

$$H^2(M) \to H^4(M) \qquad [\psi] \mapsto [\psi \wedge \psi]$$

must have image contained on the opposite side of $*_\phi\phi$. So $*_\phi\phi$ must lie on a cone across from the squares of 2-forms. This might sound unlikely, since ϕ is a positive 3-form iff $-\phi$ is, but note that ϕ and $-\phi$ determine opposite orientations.

This last observation rules out products $X^3 \times Y^4$ because we would need a definite intersection form and spin, which does not occur by Donaldson's theorem.

3.3. **Deformation theory of G_2 manifolds.** Consider the map

$$(M,\phi)\ \text{1–flat} \xmapsto{\tau} [\phi] \in H^3(M,\mathbb{R})$$

If ϕ is a 1–flat G_2 structure, then

$$\phi \wedge *_\phi\phi \neq 0$$

so $\tau(\phi) \neq 0$. Also τ is invariant under $\mathrm{Diff}_0(M)$, the group of diffeomorphisms of M which are path connected to the identity element. Consequently,

$$\frac{\text{1–flat } G_2 \text{ structures}}{\mathrm{Diff}_0(M)} \xrightarrow{\bar\tau} H^3(M,\mathbb{R})$$

is well defined.

Proposition 10. *$\bar\tau$ is a local diffeomorphism.*

Proof. (Sketch) Suppose that ϕ_t is a 1 parameter family of closed and coclosed G_2 structures.

$$d\phi_t = d *_{\phi_t} \phi_t = 0$$

Our equations on ϕ_t are diffeomorphism invariant and nonlinear. We will linearize them first. Take a Taylor expansion

$$\phi_t = \phi_0 + t\phi_1 + \frac{1}{2}t^2\phi_2 + \dots$$

Write

$$\phi_1 = f_0 \phi_0 + X \lrcorner *_0 \phi_0 + \langle h, \phi_0 \rangle$$

where h is a traceless quadratic form, and $\langle h, \phi_0 \rangle$ is the natural G_2 invariant pairing

$$h \in S^2_0 T^* \mapsto \langle h, \phi_0 \rangle \in \Lambda^3_{27} T^*$$

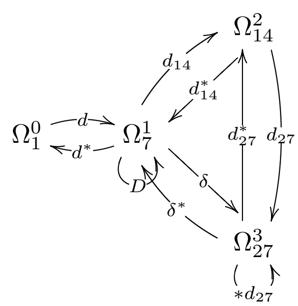
Laborious calculation reveals

$$g_t = g_0 + t \left(\frac{2}{3} f_0 g_0 + 2h \right) + O\left(t^2\right)$$
$$*_t \phi_t = *_0 \phi_0 + t \left(\frac{4}{3} f_0 *_0 \phi_0 + X^* \wedge \phi_0 - * \langle h, \phi_0 \rangle \right) + O\left(t^2\right)$$

and from our assumptions of 1–flatness,

$$d\left(f_0 \phi_0 + X \lrcorner *_0 \phi_0 + \langle h, \phi_0 \rangle\right) = 0$$
$$d\left(\frac{4}{3} f_0 * \phi_0 + X^* \wedge \phi_0 - * \langle h, \phi_0 \rangle \right) = 0$$

The invariantly defined first order operators on differential forms are



where

$$D\alpha = * \left(d\alpha \wedge *\phi\right)$$
$$\delta\alpha = *d * \left(\alpha \wedge \phi\right)$$

All invariantly defined first order operators are combinations of these operators. (In working with G_2 manifolds, one needs to know the linear dependencies among the operators produced by running around between chosen vertices but on different paths through this diagram. These linear relations have been computed. For instance, the Laplace operators that arise from travelling along different two-edge loops are linearly related.) We split all of our equations above into their irreducible components expressed in terms of these operators. We use the "Kähler identities" involving these operators and the associated Laplace operators. We end up splitting our objects $f_0, X, \dots,$ into a vector field and a harmonic 3-form. This is where the splitting into an infinitesimal diffeomorphism and a solution of a linearized elliptic equation comes in. This splits the problem into an elliptic pde modulo diffeomorphism. □

3.4. **Explicit examples.** There are no proper homogeneous examples because homogeneous Ricci flat manifolds are flat. Compact G_2 manifolds have at most a discrete isometry group, because a vector field X whose flow preserves the 3-form ϕ must be dual to a harmonic 1-form, and we know that there are no harmonic 1–forms on compact G_2 manifolds. We try for cohomogeneity 1. There are two complete G_2 manifolds with cohomogeneity 1 symmetry groups:

$$\begin{array}{ccc} \Lambda^2_+\left(\mathbb{CP}^2\right) & & \mathbb{S} \\ \downarrow & & \downarrow \\ \mathbb{CP}^2 & & S^3 \end{array}$$

where $\mathbb{S}$ indicates the spin bundle.

First, the first example. Recall that for the flat example

$$\begin{aligned} \phi_0 &= \omega^{123} + \omega^{145} + \omega^{167} + \omega^{246} - \omega^{257} - \omega^{356} - \omega^{247} \\ &= \omega^{123} + \omega^1 \wedge \left(\omega^{45} + \omega^{67}\right) + \omega^2 \wedge \left(\omega^{46} + \omega^{75}\right) + \omega^3 \wedge \left(\omega^{47} + \omega^{56}\right) \end{aligned}$$

so that the last three terms are $\omega^j \wedge \eta^j$ where each η^j is a self dual 1–form on $\mathbb{R}^4$. We split

$$\mathbb{R}^7 = \underset{123}{\mathbb{R}^3} + \underset{4567}{\mathbb{R}^4}$$

and we are thinking of the diagram

$$\begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \Lambda^2_+\left(\mathbb{CP}^2\right) \\ & & \downarrow \\ & & \mathbb{CP}^2 \end{array}$$

Let M be an oriented Riemannian 4 manifold, and let ψ be the canonical 2–form on $\Lambda^2_+(M)$. Let α be the canonical volume form on the fibers. Extend α to be 0 on the orthogonal spaces to the fibers. Let

$$r : \Lambda^2_+(M) \to \mathbb{R}$$

be the norm squared. We want to try

$$\phi = -f(r)\alpha + g(r)d\psi$$

with some $f, g > 0$. In a small open set on M, we can pick a basis of self-dual 2-forms

$$\Omega_1, \Omega_2, \Omega_3$$

and treat any self-dual 2-form Ω as

$$\Omega = a_i \Omega_i$$

giving function $a_i : \Lambda^2_+(M) \to \mathbb{R}$. We can write the canonical 2-forms α_i on $\Lambda^2_+(M)$ as

$$\alpha_i = da_i + a_j \beta_{ji}$$

where these β are the connection 1-forms. Writing a section ψ of $\Lambda^2_+(M)$ as

$$\psi = a_i \Omega_i$$

we have

$$d\psi = \alpha_i \wedge \Omega_i$$

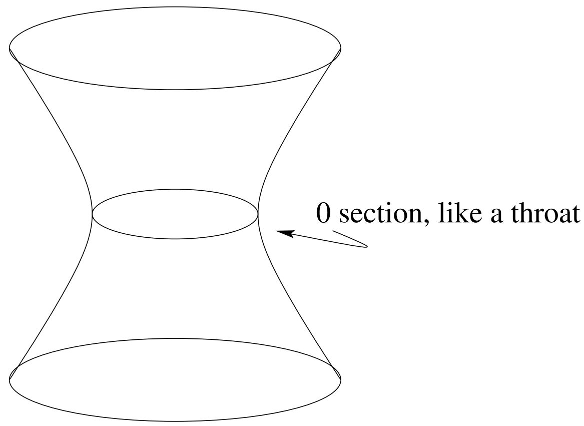


FIGURE 2. There is a complete G_2 metric on $\Lambda^2_+\left(\mathbb{CP}^2\right)$

and our volume form on the fibers is

$$\alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$$

We can calculate that the 3-form ϕ is positive

$$\phi \in \Omega^3_+\left(\Lambda^2_+ M\right)$$

Indeed comparing to the usual ϕ_0 in flat $\mathbb{R}^7$, we see

$$\begin{aligned} \phi_0 &= \omega^{123} + \omega^1 \wedge \left(\omega^{45} + \omega^{67}\right) + \dots \\ &= \alpha^{123} + \alpha_1 \wedge \Omega_1 + \dots \end{aligned}$$

We wonder what are the conditions on the functions f, g and the metric on M under which

$$d\phi = d *_\phi \phi = 0$$

The answer: These equations are incompatible unless M is self-dual Einstein, from straightforward (probably long) local calculation. So S^4 and $\mathbb{CP}^2$ are the only possibilities. The Einstein constant is necessarily positive. We get some ordinary differential equations on f, g as well. Only one solution of these equations gives a complete metric on $\Lambda^2_+(M)$, and only for $M = \mathbb{CP}^2$.

Is it a G_2 manifold? If the holonomy were smaller, we would need to have a parallel 1-form. Cross products would generate more parallel 1-forms. There turn out to be either 0 or 1 or 3 parallel 1-forms. But there can't be any parallel 1-forms because of the behaviour of the isometry group of $\mathbb{CP}^2$. Therefore the holonomy is G_2. The zero section is a totally geodesic minimal submanifold, looking like a throat, as in figure 2. Asymptotically, the 7 manifold $\Lambda^2_+\left(\mathbb{CP}^2\right)$ at infinity is

$$\Lambda^+_2\left(\mathbb{CP}^2\right) \sim \frac{SU(3)}{\text{maximal torus}} = \frac{SU(3)}{U(1) \times U(1)}$$

The same ideas work for the spin bundle

$$\begin{array}{c} \mathbb{S} \\ \downarrow \\ S^3 \end{array}$$

so that $\mathbb{S}$ becomes $S^3 \times S^3$ at infinity. Again the zero section is a minimal throat. We will revisit these "throats" in studying calibrated submanifolds.

4. Calibrated submanifolds in G_2 manifolds

4.1. What is a calibration?

Definition 3. If M is a Riemannian manifold, we say that a closed p-form ϕ is a *calibration* if

$$||\phi|| \leq 1$$

i.e. $\phi|_E \leq \operatorname{vol}(E)$, for any oriented p-plane E. A p-plane E is *calibrated* by ϕ if $\phi|_E = \operatorname{vol}(E)$. A p dimensional submanifold $\Sigma \subset M$ is calibrated by ϕ if all of its tangent spaces are:

$$\phi|_{T_x\Sigma} = \operatorname{vol}(T_x\Sigma)$$

For example, if M is a Kähler manifold, with Kähler form ω, then

$$\phi = \frac{\omega^p}{p!}$$

is a calibration. By Wirtinger's inequality, the calibrated p planes are the complex p planes, and the calibrated submanifolds are the complex submanifolds.

Theorem 2 (The Fundamental Theorem of Calibrations)**.** *A calibrated submanifold has the least volume among all of its compactly supported variations.*

Proof. We remark that this is the only known method for proving absolute minimality of a submanifold. Suppose that Σ is a p dimensional submanifold calibrated by a p–form ϕ. Then if Σ' is a compactly supported deformation of Σ then since $d\phi = 0$:

$$\operatorname{vol}(\Sigma) = \int_\Sigma \phi = \int_{\Sigma'} \phi \leq \operatorname{vol}(\Sigma')$$

□

Corollary 1. *Every minimal submanifold homologous to a calibrated submanifold is calibrated.*

4.2. Examples of calibrations.

(1) On a Riemannian manifold, we can pick a moving frame of orthonormal 1-forms $\omega^1, \ldots, \omega^n$ and let

$$\phi = \omega^1 \wedge \cdots \wedge \omega^p$$

This calibrates a single p plane in each tangent space, and generically this p plane field won't be holonomic, so there won't be any calibrated submanifolds.

(2) On a Kähler manifold, the Kähler form calibrates.

(3) On a Riemannian manifold, pick a closed 2–form ϕ. In an orthonormal moving frame of 1-forms $\omega^1, \ldots, \omega^n$,

$$\phi = a_{ij}\omega^i \wedge \omega^j$$

where $a_{ij} = -a_{ji}$. If we can get the $A = (a_{ij})$ matrix to satisfy $A^2 = -I$, then ϕ determines an almost Kähler structure, and calibrates pseudo-holomorphic curves.

(4) Let M be a Calabi Yau manifold (a Ricci-flat Kähler manifold) with holomorphic volume form ζ of norm 1. Then $\operatorname{Re}(\zeta)$ is a calibration and it calibrates Lagrangian manifolds of "unit phase". This phase means the following: take N a Lagrangian manifold. Then

$$\zeta|_N = e^{i\theta} \cdot \operatorname{vol}_N$$

for some function $\theta : N \to \mathbb{R}/2\pi\mathbb{Z}$. The Lagrangian manifolds with $\theta = 0$ are calibrated by $\mathrm{Re}(\zeta)$. The form

$$\phi = \mathrm{Re}\left(e^{-i\theta_0}\zeta\right)$$

calibrates Lagrangian manifolds N with phase

$$\zeta|_N = e^{i\theta_0} \cdot \mathrm{vol}_N$$

For example, in the Calabi Yau manifold $\mathbb{C}^2$ with holomorphic coordinates z_1, z_2

$$\zeta = dz_1 \wedge dz_2$$

and $\mathbb{R}^2 \subset \mathbb{C}^2$ is calibrated by $\mathrm{Re}(\zeta)$. Or consider a $K3$ surface $S \subset \mathbb{P}^3$, given by a real quartic equation, so that $S \cap \mathbb{RP}^3$ is a real surface. Then there exists a Calabi Yau structure on S which is invariant under complex conjugation so that this real slice is calibrated by $\mathrm{Re}(\zeta)$.

4.3. **Calibrations as differential equations.** Given a calibration ϕ on a manifold M we can let $G(\phi)$ be the set of all oriented p planes in tangent spaces of M, calibrated by ϕ. We can view $G(\phi)$ as a constraining equation for p dimensional manifolds, and study it as a first order system of partial differential equations for p dimensional submanifolds. We can ask when there are solutions, and how general they are.

4.4. **Example: G_2 calibrations.** If $\phi \in \Lambda^3_+ V^*$ is fixed, and G_2 is its stabilizer, then

$$\phi|_E \le \mathrm{vol}\,(E)$$

for any 3 plane $E \subset V$. How do we prove this? Pick an orthonormal basis $e_1, \dots, e_7$ so that ϕ is identified with the standard G_2 structure

$$\phi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

We know that G_2 acts transitively on unit vectors S^6, so we can arrange that $e_1 \in E$. The stabilizer of e_1 is $SU(3) \subset G_2$, and $SU(3)$ acts transitively on S^5, so we can arrange that $e_2 \in E$. The stabilizer of e_1, e_2 in $SU(3)$ is $SU(2)$, acting on $\mathbb{C}^2$ as usual. We can compare E to the 3-plane spanned by

$$e_1, e_2, e_3 = Je_2$$

where J is the complex structure on $\mathbb{C}^2$. We will find that an oriented orthonormal basis for E looks like

$$e_1, e_2, e_3' = \cos\theta \cdot e_3 + \sin\theta \cdot f$$

where f is unit length in $\mathbb{C}^2$. We can calculate

$$\phi|_E = \cos\theta \cdot \mathrm{vol}\,(E)$$

Consequently E is calibrated by ϕ precisely if

$$E = e_1 \wedge e_2 \wedge e_1 \times e_2$$

for some orthonormal pair e_1, e_2. We map

$$\begin{aligned} G_2/SU(2) &\to G(\phi) \\ e_1, e_2 &\mapsto e_1 \wedge e_2 \wedge e_1 \times e_2 \end{aligned}$$

to describe the principal bundle

$$\begin{array}{cccc} SO(3) \longrightarrow & G_2/SU(2) & & \\ & \downarrow & & \\ & G(\phi) & & \subset \tilde{G}_3\mathbb{R}^7 \\ & & & \\ & 8 \text{ dim'l} & & 12 \text{ dim'l} \end{array}$$

There are 4 first order partial differential equations expressing the condition that a 3 dimensional submanifold is calibrated by a G_2 structure. Take a 3 dimensional submanifold in a 7 dimensional G_2 manifold, thought of as a product of our 3 manifold with a 4 manifold, and deformations of the 3 manifold being graphs of 4 functions over those 3 variables. But we have 4 first order partial differential equations to solve, so it is reasonable to believe that these equations are a determined system.

The form $*\phi \in \Lambda^4 V^*$ gives a calibration as well. It calibrates the 4 planes E which are perpendicular to those calibrated by ϕ:

$$G(*\phi) = \{E^\perp \mid E \in G(\phi)\}$$

So $G(*\phi)$ has dimension 8, and is a submanifold of $\tilde{G}_4\mathbb{R}^7$ which has dimension 12. So again the codimension is 4. But by the same argument as before, this says that the equations for a calibrated 4 dimensional manifold in a G_2 manifold are overdetermined.

4.5. **Deformations through calibrated submanifolds.** If we have a calibrated submanifold, and we want to deform through a family of calibrated submanifolds, we need to start with the infinitesimal construction: the linearization of the calibrated submanifold equations about the given submanifold. Let N be our calibrated submanifold of M. Then at each point $x \in N$ we can look at $E = T_x N$, a calibrated plane:

$$E \in G_p(T_x M)$$

Then we consider the tangent space to our differential equations:

$$T_E G(\phi_x) \subset T_E \tilde{G}_p(T_x M) \cong E^\perp \otimes E$$

We write A_E for

$$A_E = T_E G(\phi_x) \subset E^\perp \otimes E$$

the *tableau*; $A_E \subset E^\perp \otimes E$ has codimension r equal to the number of equations imposed by the calibration condition on a submanifold. Write

$$E^\perp \otimes E = A_E \oplus C_E$$

as the orthogonal decomposition. We define the linearization

$$D : J^1(\nu) \to C_N$$

taking sections of the normal bundle ν of N to sections of C_N, by

$$D\sigma = \pi_C \cdot \nabla\sigma$$

where ∇ is the covariant derivative, and π_C is the orthogonal projection

$$E^\perp \otimes E \xrightarrow{\pi_C} C_E$$

Proposition 11. *If σ_t is a family of sections of the normal bundle ν_N with $\sigma(0) = 0$, and so that*

$$exp\,(\sigma_t) \subset M$$

is smooth and ϕ calibrated, for all t, then

$$D\sigma'(0) = 0$$

For example, in Kähler geometry,

$$D = \bar{\partial}$$

It is important in Kähler–Einstein geometry that E and $E^{\perp}$ are canonically isomorphic, by

$$E^{\perp} = J \cdot E$$

with J the complex structure automorphism of the tangent bundle. The normal bundle is then isomorphic to the tangent bundle

$$\nu_N = TN$$

We find that

$$A_E \subset E \otimes E^{\perp}$$

is just

$$S^2_0 E \subset E \otimes E$$

the traceless symmetric bilinear forms, which leaves

$$C_E = \Lambda^2 T \oplus \Lambda^0 T$$

and we find

$$D\sigma = (d\sigma, d^*\sigma)$$

So the kernel of D consists of the harmonic 1–forms. Therefore, the kernel has dimension given by the DeRham theorem:

$$\dim \ker D = \dim H^1(N, \mathbb{R})$$

and this fact was employed by Robert McLean in [14] to prove smoothness of the moduli spaces of special Lagrangian manifolds.

Christopher Michael, in [15], considered the 4–form

$$p \in \Omega^4\left(G_4\left(\mathbb{R}^n\right)\right)$$

which is harmonic and parallel and not vanishing. There is only one up to scaling, and the right multiple of it is a calibration. In that case, A_E is not an involutive tableau, and one has to work harder to determine if there are any calibrated submanifolds. Prolonging like mad, eventually you get some inhomogeneous finite dimensional family of calibrated objects, with not very bad singularities.

If G is a compact, simply connected Lie group, and

$$\kappa \in \Omega^3(G)$$

is the Cartan–Killing form:

$$\kappa(x, y, z) = \langle [x, y], z \rangle$$

then $\frac{1}{4}\kappa$ is a calibration, and it calibrates 3–planes spanning an $su(2)$ subalgebra. The only calibrated submanifolds are $SU(2)$ subgroups and their translates. Gluck, Morgan, Mackenzie and Ziller have studied this quite a bit, see [5] and [6].

Take a $K3$ surface S in $\mathbb{CP}^3$ defined over $\mathbb{R}$. Then the real points $S \cap \mathbb{RP}^3$ form a 2–torus, and this 2–torus lies in a 2 dimensional family of 2–tori, foliating the $K3$

surface in a neighborhood of the real points. As we deform the complex structure, this family must survive.

In the G_2 case, let us look at the three dimensional calibrated submanifolds. Given a tangent space for such a submanifold, think of it as

$$E \subset \tilde{G}_3\left(\mathbb{R}^7\right)$$

we have the splitting

$$\mathbb{R}^7 = E \oplus E^\perp$$

preserved by $SO(4)$, represented as

$$\begin{pmatrix} \rho(A) & 0 \\ 0 & A \end{pmatrix}$$

where

$$SO(4) \xrightarrow{\rho} SO(3)$$

is one of the two nontrivial homomorphisms. Our normal plane $E^\perp$ is acted on by $SU(2)$ stabilizing E. So for a calibrated submanifold

$$N^3 \subset M^4$$

the normal bundle is like a spin bundle. It is actually a twisted spin bundle, say $\check{\mathbb{S}}$. Our linear operator

$$D : J^1(\check{\mathbb{S}}) \to \check{\mathbb{S}}$$

is the twisted Dirac operator. The index vanishes by the Atiyah–Singer Index Theorem, since the manifold N has dimension 3, which is odd.

Now consider the four dimensional calibrated submanifolds in a G_2 holonomy manifold. Our vector bundles are

$$\nu_N = \Lambda^2_+(T^*N)$$
$$C_N = \Lambda^3(T^*N)$$

while our linearized operator

$$J^1\left(\nu_N\right) \xrightarrow{D} C_N$$

is just

$$D\sigma = d\sigma$$

so the kernel of D consists of the self–dual closed 2–forms. The tangent space to the moduli space is

$$H^2_+(N)$$

We have the exact sheaf sequence

$$0 \longrightarrow \Omega^2_+(N) \xrightarrow{D} \Omega^3(N) \xrightarrow{d} \Omega^4(N)$$

so harmonic 3–forms look like an obstruction to construction of a smooth moduli space. Surprisingly, there is never a nonzero obstruction, and the moduli space is smooth of dimension

$$b^2_+(N)$$

We could also consider 2–forms on G_2 manifolds for calibrations. These split into

$$\Omega^2_7 \oplus \Omega^2_{14}$$

but the first part vanishes on compact G_2 manifolds. We recall that as G_2 representations,

$$\Omega^2_{14} \cong \mathfrak{g}_2$$

the Lie algebra, under adjoint representation. The stabilizer of a generic element of Ω^2_{14} is therefore a maximal torus in G_2. It turns out that this prevents it calibrating anything. There are 2–forms with stabilizer $SO(4)$, and about them nothing is known.

In our examples of G_2 manifolds constructed in the last lecture, the manifolds are vector bundles, and the zero sections are associative or coassociative.

Consider the flat torus T^7 as a manifold with holonomy (trivial) contained in G_2. If $N^3 \subset T^7$ is calibrated, then translations of T^7 give families of calibrated submanifolds. This forces

$$b^2_+(N) \geq 3$$

If $b^2_+(N) = 3$, then we must have four dimensions of trivial action on it, so it must be

$$T^3 \subset T^7$$

We know that

$$S^1 \times CY, T^3 \times K3, \text{ or } T^7$$

are the only compact manifolds with holonomy contained in G_2 which have circle factors. If we have a circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & M^7 \\ & & \downarrow \\ & & CY \end{array}$$

over a Calabi–Yau 3–fold, then M has a natural G_2 holonomy metric, and any rational curve in CY has, as preimage in M, a coassociative submanifold.

Also, complex hypersurfaces in Calabi–Yau 3–folds and special Lagrangian 3-folds in Calabi-Yau 3–folds give coassociative manifolds in M.

5. Further comments

These are a few things I have come across since Robert gave his lectures.

Merkulov has a construction of exceptional and exotic holonomy groups on moduli spaces of Legendre submanifolds of holomorphic contact manifolds. In particular, every complex G_2 manifold (i.e. with holonomy in the complexification $G_2^{\mathbb{C}}$) comes about locally from a moduli space of 5-quadrics with fixed normal bundle in an 11 dimensional holomorphic contact manifold. The objects that live in the $G_2^{\mathbb{C}}$ manifold, like associative and coassociative complex submanifolds, must have some interpretation on the other side, in the holomorphic contact geometry. But Merkulov does not know the conditions on the holomorphic contact manifold under which his moduli space will have G_2 holonomy. These conditions must be global, and I can only imagine that they have to do with the canonical bundle and the contact line bundle (the bundle of (1,0)-forms vanishing on the contact planes). For example, positivity or nonpositivity of some tensor products of these bundles, or of their restriction to one of the quadrics.

The construction seems to be taking the manifold of null geodesics (if it is a manifold) as our contact manifold. The manifold of pointed null geodesics is then mapped to points and to null geodesics, forming the sort of double fibration which

Cartan was interested in. A coassociative submanifold can be represented in this picture by the null geodesics perpendicular to it, which should form a $\mathbb{P}^1$ bundle over the coassociative submanifold, given by the null elements of the projectivized Λ^2_+ bundle. These null geodesics will also form a Legendre immersion, I believe.

Every hypersurface in a G_2 manifold is equipped with a canonical $SU(3)$ structure. We might try to use the magnitude of the torsion tensor for that $SU(3)$ structure as a Lagrangian, and look at the heat flow associated to such a hypersurface. The singular Calabi-Yau-like objects which would arise as the critical objects for such a flow might be fundamental. The idea is that perhaps many G_2 manifolds look like circle bundles over Calabi-Yau-like manifolds.

Paul Aspinwall told me a little about the string theory story. To compactify M-theory, preserving exactly one supersymmetry in the effective theory on a four manifold, we need to compactify seven dimensions, and those seven dimensions must have G_2 holonomy. Moreover, the resulting theory will have chiral symmetry if the compactified dimensions form a smooth manifold. Therefore, to have a chiral theory (i.e. to break chiral symmetry) we need a G_2 orbifold, and to use Witten's notion of twisting (or so it would appear).

In studying the coassociative and associative submanifolds, we would like to think of them as D-branes. This suggests looking at conformal minimal immersions of Riemann surfaces with boundary into the G_2 manifold, with boundaries in the calibrated submanifolds, or looking at the same maps, but assuming that the entire image is contained in the calibrated submanifold.

Suppose that we have a coassociative torus fibration. This is the analogue of the Strominger– Yau–Zaslow picture for Calabi–Yau manifolds. The Calabi–Yau story gives the base of a special Lagrangian torus fibration a lot of structure. We obtain a lattice in the cotangent bundle of the base, using standard ideas from symplectic geometry (forgetting that the fibers are *special* Lagrangian, and remembering only that they are Lagrangian). The analogue for G_2 manifolds is that the spin bundle of the base 3-manifold must have a lattice in it. We may pick any spin structure on our coassociative tori, and any spin structure on the base. Since

$$\operatorname{Spin}(4) = \operatorname{Spin}(3)_- \times \operatorname{Spin}(3)_+$$

is canonical, we should be able to identify the $\operatorname{Spin}(3)_-$ with the spin group of the base manifold. This should get the lattice going, using identifications of elements of the spin bundle of the base with vector fields tangent to the fibers, and forming the lattice of spinors which form unit time periodic vector fields.

To identify the G_2 manifold with the spin bundle of the base, we need to pick a section. I expect that this section must be associative to get nice things to happen. In the symplectic geometry situation, the section is Lagrangian, but there we only want symplectic structure.

All of this should repeat with Cayley torus fibrations of $\operatorname{Spin}(7)$ holonomy manifolds, again using the spin bundle of the base. Perhaps it also holds for quaternionic Kähler manifolds, with complex torus fibrations. I don't know what the appropriate vector bundle is on the base. This may lead out of mirror symmetry and into some deeper nonperturbative quantum field theoretic geometry.

Gromov, in [7], has suggested that one might think of convexity in terms of calibrated submanifolds in exceptional geometries. For G_2 manifolds this leads to associative and coassociative convexity of hypersurfaces. The condition should be essentially that any tangent associative (resp. coassociative) submanifold lies on the exterior side of the hypersurface, at least near the point of tangency. I expect that this can be expressed in terms of the induced $SU(3)$ structure on the hypersurface as the condition that the exterior derivative of the $(1,1)$-form (the "Kähler" form) is positive on special Lagrangian planes (resp. that the exterior derivative of the $(3,0)$-form is positive on complex 2-planes).

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