# Classical Mechanics and Field Theory
## QFT Seminar, Fall 2001

We start with **Newton's law**:

$$F = ma$$

which we think about as follows: once an initial velocity $x'(t_0)$ and position $x(t_0)$ of a particle of mass $m$ are given, as well as the force $F$ exerted on the particle at all times (as a function of position and/or velocity), then all future positions and velocities are obtained by solving the differential equation:

$$F(x(t), x'(t)) = mx''(t)$$

**Examples:** (a) No force exerted ($F = 0$). Then:

$$x'(t) = x'(t_0) \text{ and } x(t) = x(t_0) + (t - t_0)x'(t_0)$$

i.e. the motion is straight-line motion with constant velocity.

(b) The (simple) harmonic oscillator ($F(x) = -kx$). Then:

$$x(t) = A\cos(\omega t) + B\cos(\omega t)$$

where $\omega = \sqrt{\frac{m}{k}}$ and $A, B$ are determined by the initial conditions.

**The Lagrangian formulation** of mechanics characterizes the "classical" paths $x(t)$, or more precisely their lifts $(x(t), x'(t))$ to paths in the tangent space (or "configuration space") $T\mathbf{R}^n$. Start with a **Lagrangian**:

$$L(q, \dot{q}) : T\mathbf{R}^n \to \mathbf{R}$$

and its **action** integrals:

$$S_{[a,b]}[x] := \int_a^b L(x(t), x'(t))dt$$

and declare that paths $(x(t), x'(t))$ are classical if they satisfy the variational principle:

$$\frac{d}{d\epsilon} S_{[a,b]}[x + \epsilon y]|_{\epsilon=0} = 0$$

for all intervals $[a, b]$ and all paths $y(t)$ satisfying $y(a) = y(b) = 0$.

This principle gives us the **Euler-Lagrange** equations of motion:

$$\frac{\partial L}{\partial q_i}(x(t), x'(t)) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(x(t), x'(t)) \quad \forall i = 1, \cdots n$$

which also characterize the classical paths $(x(t), x'(t))$.

We see this by integrating by parts:

$$0 = \int_a^b \frac{d}{d\epsilon} L(x + \epsilon y, x' + \epsilon y')|_{\epsilon=0} = \int_a^b \frac{\partial L}{\partial q_i} y_i + \frac{\partial L}{\partial \dot{q}_i} y_i' =$$

$$\int_a^b \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) y_i + \left[ \frac{\partial L}{\partial \dot{q}_i} y_i \right]_a^b$$

and the last term is zero because $y_i(a) = y_i(b) = 0$. Of course this holds for **all** the paths $y(t)$ if and only if the Euler-Lagrange equations are satisfied.

**Remark:** The Einstein summation convention is in effect! In this context, that will mean that we sum over the index $i$ without writing the summation symbol.

**Examples:** (a) The Euler-Lagrange equations for $L(q, \dot{q}) = \frac{1}{2} m \dot{q}_i^2$ are:

$$0 = m x''(t)$$

which is Newton's law (with no force).

When we introduce a **potential energy** term: $L(q, \dot{q}) = \frac{1}{2} m \dot{q}_i^2 - V(q)$ we obtain the Euler-Lagrange equations:

$$-\nabla V(x(t)) = m x''(t)$$

which is Newton's law with force $F = -\nabla V(x)$.

(b) When $n = 1$ and $V = \frac{1}{2} k q^2$, we get the simple harmonic oscillator.

(c) We can "couple" simple harmonic oscillators via a Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2} m_i \dot{q}_i^2 + \frac{1}{2} k_i q_i^2 + m_{ij} (q_i - q_j)^2$$

Here our $T\mathbf{R}^n$ is the configuration space of $n$ particles in $\mathbf{R}^1$, and it is interesting to note that one can always "uncouple" such harmonic oscillators via an orthogonal change of coordinates $u = Aq$ giving:

$$L(u, \dot{u}) = \frac{1}{2} m_i \dot{u}_i^2 + \frac{1}{2} K_i u_i^2$$

in the new "normal modes" coordinates.

**Some More Vocabulary:** A term in the Lagrangian which is quadratic in the $\dot{q}_i$ (as in example (a)) is called a **kinetic energy** term. A term which is a function of position only is (as above) called a **potential energy** term, and in addition the quantities:

$$p_i := \frac{\partial L}{\partial \dot{q}_i}$$

are called the **momenta**.

In the **Hamiltonian formulation**, we define:

$$H(q, \dot{q}, p) := \dot{q}_i p_i - L(q, \dot{q})$$

as a function of $T\mathbf{R}^n$ and the momenta $p_i$ (this is the "Legendre transform"). When we compute:

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i$$

we find that $H$ is (locally) a function of the $q_i$ and $p_i$ only. Moreover, when $L(q, \dot{q})$ is of the form "kinetic" $-$ "potential" then the $p_i$ are naturally sections of the cotangent bundle $T^*\mathbf{R}^n$ and they frame the cotangent bundle when the kinetic term is non-degenerate. Thus in this situation, we can think of:

$$H(q, p) : T^*\mathbf{R}^n \to \mathbf{R}$$

as a function on the cotangent bundle (also called the **phase space**).

**Example:** When $L(q, \dot{q}) = \frac{1}{2} m \dot{q}_i^2 - V(q)$, then:

$$H(q, p) = \frac{1}{2} m \dot{q}_i^2 + V(q) = \frac{p_i^2}{2m} + V(q)$$

The Hamiltonian has some exceedingly useful properties:

$$\frac{d}{dt} H(x(t), p(t)) = 0$$

on all classical paths (where $p(t)$ are the $p$-coordinates of the path). That is, the Hamiltonian is a **conserved** quantity along paths. This is an immediate consequence of the Euler-Lagrange equations:

$$\frac{d}{dt} H(x(t), p(t)) = x_i'' \frac{\partial L}{\partial \dot{q}_i} + x_i' \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} x_i' - \frac{\partial L}{\partial \dot{q}_i} x_i'' = 0$$

And from the expression for $dH$, we obtain **Hamilton's equations**:

$$\frac{\partial H}{\partial p_i} = x_i'(t) \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -p_i'(t)$$

of motion along classical paths. Thus the **Hamiltonian vector field**

$$\mathcal{H} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}$$

is precisely the velocity vector field along the classical paths in phase space!

A "quantity" $A : T^*\mathbf{R}^n \to \mathbf{R}$ is conserved (on classical paths) if and only if:

$$\{A, H\} := \mathcal{H}(A) = \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} = 0$$

(and one double-checks that $H$ itself is a conserved quantity)

Of course, the momenta $p_i$ and position $q_i$ are **not** (in general) conserved:

$$\{p_i, H\} = \frac{\partial L}{\partial q_i} =: \dot{p}_i \quad \text{and} \quad \{q_i, H\} = \dot{q}_i$$

The **Poisson bracket** is the pairing $\{,\}$ on "quantities" defined by:

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

which satisfies the following identities (check!):

(1) (Skew symmetry) $\{A, B\} = -\{B, A\}$

(2) (Jacobi identity) $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

(3) (Leibniz rule) $\{AB, C\} = A\{B, C\} + \{A, C\}B$

**Examples:** $\{q_i, q_j\} = 0 = \{p_i, p_j\}$ and $\{q_i, p_j\} = \delta_{ij}$

Let's consider the role of **symmetry** before moving on to field theory:

**Example:** Consider the Lagrangian (in polar coordinates, why not?):

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

(which **is** of the form "kinetic" − "potential"). This has an obvious rotational symmetry:

$$L(r, \theta, \dot{r}, \dot{\theta}) = L(r, \theta + s, \dot{r}, \dot{\theta})$$

When we apply the Euler-Lagrange equations, we therefore get:

$$0 = \frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}mr^2\dot{\theta} = 0$$

on classical paths, so the "angular momentum" $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is a conserved quantity. This generalizes to an easy version of Noether's theorem: any one-parameter group $\Phi_s : \mathbf{R}^n \to \mathbf{R}^n$ of symmetries of the Lagrangian yields a conserved quantity. (Choose a coordinate system so that $\frac{\partial}{\partial q_1}$ is the derivative of $\Phi_s$ at $s = 0$. In that coordinate system, $p_1$ will be conserved.)

There is a time translation symmetry which will also lead to a conserved quantity, though it a little is harder to see (but crucial for field theory). Let $\Phi_s$ be one-parameter group acting on **paths**: $\Phi_s(x(t)) = x(t + s)$. While it is not true that $L(x(t), x'(t))$ is invariant under $\Phi_s$, the **action** satisfies:

$$S_{[a,b]}[\Phi_s(x)] = \int_a^b L(x(t + s), x'(t + s))dt = S_{[a+s,b+s]}[x]$$

We get:

$$\int_a^b L(x(t+s), x'(t+s)) - L(x(t), x'(t)) = \int_b^{b+s} L(x(t), x'(t)) - \int_a^{a+s} L(x(t), x'(t))$$

and when we take $\frac{d}{ds}$ and set $s = 0$, we get:

$$\int_a^b \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) x_i' + \left[ \frac{\partial L}{\partial \dot{q}_i} x_i' \right]_a^b = L(x(b), x'(b)) - L(x(a), x'(a))$$

with $x'$ here playing the role that $y$ played previously, except that $x'$ doesn't vanish at the endpoints. But now assume that $x(t)$ is a path satisfying the Euler-Lagrange equations. Then the **first** term above vanishes, and we get:

$$p_i \dot{q}_i - L(q, \dot{q})$$

as a conserved quantity. This is the Hamiltonian! Thus the Hamiltonian is the conserved quantity corresponding to time translation.

(I think that this is the more serious version of Noether's theorem.)