MATH 3160: APPLIED COMPLEX VARIABLES
TEST #2 (VERSION A)

1. **(a)** Calculate the Laurent expansion about $z = 0$ of

$$\frac{\cos z}{\sin z}$$

up to z^2 terms.

(b) Calculate

$$\operatorname*{Res}_{z=0} \frac{\cos z}{\sin z}$$

Solution:

(a)

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + O\left(z^5\right)$$

You only need the first two terms.

(b)

$$\operatorname*{Res}_{z=0} \frac{\cos z}{\sin z} = 1$$

2. Calculate the integral

$$\int_{|z|=5} \frac{1 - \cos z}{\sin z} dz \ .$$

(where the circle $|z| = 5$ is positively oriented).

Solution: The only singular points inside the circle are at

$$z_0 = -\pi, \ z_1 = 0, \ z_2 = \pi \ .$$

The residues there are

$$\operatorname*{Res}_{z=z_0} \frac{1 - \cos z}{\sin z} = -2$$

$$\operatorname*{Res}_{z=z_1} \frac{1 - \cos z}{\sin z} = 0$$

$$\operatorname*{Res}_{z=z_2} \frac{1 - \cos z}{\sin z} = -2$$

Therefore the value of the integral is

$$\begin{aligned} \int_{|z|=5} \frac{1 - \cos z}{\sin z} dz &= 2\pi i \left(\operatorname*{Res}_{z=z_0} + \operatorname*{Res}_{z=z_1} + \operatorname*{Res}_{z=z_2} \right) \frac{1 - \cos z}{\sin z} \\ &= 2\pi i(-2 + 0 + -2) \\ &= -8\pi i \end{aligned}$$

Date: April 4, 2001.

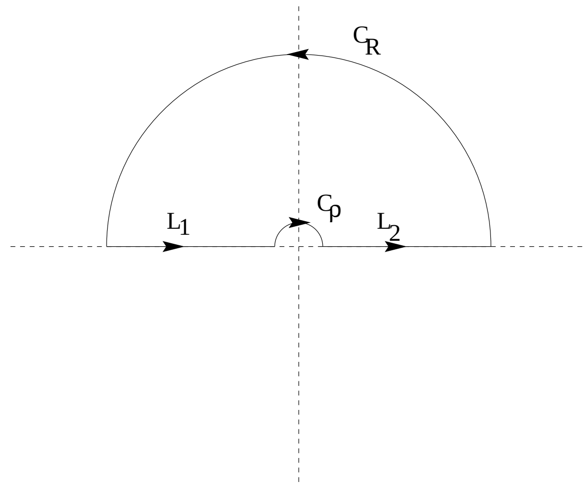

FIGURE 1. A contour, with ρ (the radius of C_ρ) small, and R (the radius of C_R) large.

3. Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

by integration of the function

$$f(z) = \frac{e^{iz}}{z}$$

along a contour like that shown in figure 1.
Solution: See page 220 of Churchill & Brown.

4. Calculate

$$\int_{|z|=100} \frac{dz}{\sin(1/z)}$$

where the circle $|z| = 100$ is positively oriented.
Solution: The singular points are the simple poles

$$z = \frac{1}{\pi N}$$

for N any nonzero integer. They accumulate at $z = 0$, and are all contained inside the circle, so we will calculate residue at infinity. There is a triple pole at $z = \infty$:

$$\begin{aligned}\int_{|z|=100} \frac{dz}{\sin(1/z)} &= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{w^2}\frac{1}{\sin w} \\ &= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{w^2}\frac{1}{w\left(1-\frac{w^2}{3!}+\cdots\right)} \\ &= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{w^3}\frac{1}{\left(1-\frac{w^2}{3!}+\cdots\right)} \\ &= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{w^3}\left(1+\frac{w^2}{3!}+\cdots\right) \\ &= 2\pi i \operatorname*{Res}_{w=0} \frac{1}{w^3}+\frac{1}{6w}+\cdots \\ &= 2\pi i \frac{1}{6} \\ &= \frac{\pi}{3} i\end{aligned}$$

Another way to try to solve the problem: we could try to add the infinitely many residues inside the circle. This doesn't actually work, surprisingly, since the singular points accumulate to $z = 0$. Let us try it anyway. We can expand (but you don't need to):

$$\frac{1}{\sin(1/z)} = \frac{(-1)^{N+1}}{\pi^2 N^2}\frac{1}{z-\frac{1}{\pi N}} + \frac{(-1)^{N+1}}{\pi N} + O(z) \ .$$

There is also an essential singularity at $z = 0$ which looks like

$$\frac{1}{\sin(1/z)} = z + \frac{1}{6z} + O\left(z^{-2}\right)$$

(i.e. up to higher order terms in $1/z$).

This appears to give

$$\int_{|z|=100} \frac{dz}{\sin(1/z)} = 2\pi i\left(\frac{1}{6} + \sum_{N\neq 0}\frac{(-1)^{N+1}}{\pi^2 N^2}\right) \ .$$

The fact that the series

$$\sum_{N\neq 0}\frac{(-1)^{N+1}}{\pi^2 N^2} = \frac{1}{6}$$

is not obvious. It suggests a value of $2\pi i/3$ for the integral, which is incorrect. Either approach gets full points, although the residue at infinity gives the right answer and is much easier.

The reason why the second approach doesn't work is that the expansion at $z = 0$ is not actually valid in any region around $z = 0$, since there are poles in every such region. It is really only an *asymptotic series*, not a convergent Laurent series.

5. Bonus: Calculate

$$\int_0^\infty \frac{dx}{x^{1/4}(x+4)} \ .$$

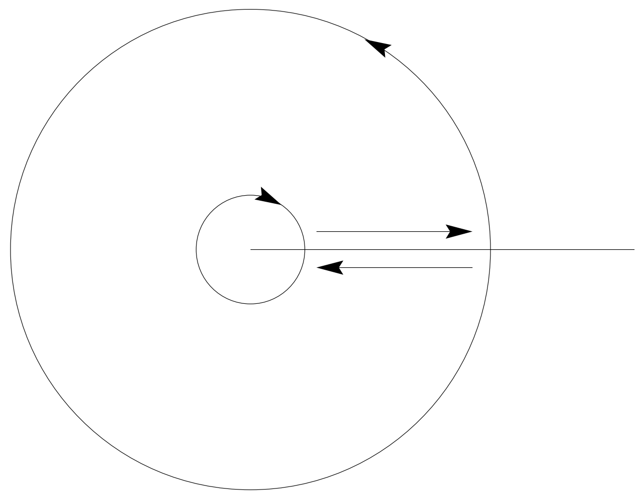

FIGURE 2. Integrating along the branch, then around a large circle, then backwards along the branch, and then around a small circle.

Solution:

$$\int_0^\infty \frac{dx}{x^{1/4}(x+4)} = \pi \ .$$

I arrived at this by integration of the function

$$f(z) = \frac{1}{z^{1/4}(z+4)}$$

along the branch cut, setting

$$z^{1/4} = r^{1/4} e^{i\theta/4}$$

whenever

$$0 < \theta < 2\pi$$

with

$$z = re^{i\theta} \ .$$

We integrate along a contour like that shown in figure 2. The only singularity off the branch is at $z = -4$, where the residue is

$$\begin{aligned} \operatorname*{Res}_{z=-4} \frac{1}{z^{1/4}(z+4)} &= \frac{1}{(-4)^{1/4}} \\ &= \frac{1}{\sqrt{2}e^{\pi i/4}} \\ &= \frac{1}{\sqrt{2}\left(1/\sqrt{2} + i/\sqrt{2}\right)} \\ &= \frac{1}{1+i} \\ &= \frac{1-i}{2} \ . \end{aligned}$$

The integral of the function $f(z)$ around a circle of radius R is of order

$$\frac{R}{R^{1/4}\,(R+4)}$$

which for large R is like

$$\frac{R}{R^{5/4}} = R^{-1/4} \to 0$$

while for small R is of order

$$\frac{R}{R^{1/4}} = R^{3/4} \to 0\ .$$

So we can ignore these circular arcs, and find that the limit of the contour integral (as the large circle gets very large and the small one very small) is

$$\int_0^\infty \frac{dx}{x^{1/4}\,(x+4)} + \int_\infty^0 \frac{dx}{x^{1/4}e^{i\pi/2}\,(x+4)} = (1+i)\int_0^\infty \frac{dx}{x^{1/4}\,(x+4)}\ .$$

But this must equal

$$2\pi i \operatorname*{Res}_{z=-4} f(z) = 2\pi i \frac{1-i}{2} = \pi(1+i)$$

and therefore we have the answer given above.