

**MATH 3160: APPLIED COMPLEX VARIABLES**  
**TEST #1 (VERSION B)**

1. What are all of the values of  $i^{2i}$  in the form  $x + iy$ ?

**Solution:**

$$e^{-2\pi(1+2n)}$$

where  $n$  is any integer.

2. Calculate

$$\frac{2+i}{3-i}$$

in the form  $x + iy$ .

**Solution:**

$$\frac{1}{2} + \frac{1}{2}i$$

3. Show that under the map

$$w = f(z) = z^2$$

vertical lines in the  $z$  plane become parabolas in the  $w$  plane, as in figure 1 on the following page.

**Solution:** A vertical line is a line with constant  $x$  value, say  $x = x_0$ , and varying  $y$  value. Then

$$u + iv = w = x_0^2 - y^2 + 2x_0yi$$

so

$$u = x_0^2 - y^2$$

$$v = 2x_0y.$$

Solving for  $y$ :

$$y = \frac{v}{2x_0}$$

then

$$\begin{aligned} u &= x_0^2 - \left(\frac{v}{2x_0}\right)^2 \\ &= x_0^2 - \frac{v^2}{4x_0^2} \end{aligned}$$

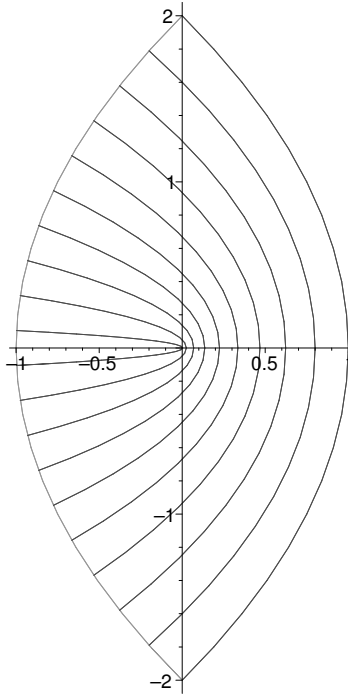


FIGURE 1. The images of vertical lines under the transformation  $w = z^2$

which is the equation of a parabola. There is one exceptional case,  $x_0 = 0$ , which becomes a half line:

$$x = 0 \rightarrow v = 0, u = -y^2 \leq 0.$$

4. At which points  $z$  does the function

$$f(z) = |z|^2$$

satisfy the Cauchy–Riemann equations? Explain your answer.

**Solution:**

$$u = x^2 + y^2$$

$$v = 0$$

so that

$$u_x = 2x$$

$$u_y = 2y$$

$$v_x = 0$$

$$v_y = 0.$$

The Cauchy–Riemann equations are satisfied precisely at

$$x = y = 0.$$

5. (a) What are all of the values of

$$\log i^3?$$

(b) What are all of the values of

$$3 \log i?$$

**Solution:**

(a)

$$\frac{3\pi}{2}i + 2\pi ni$$

with  $n$  any integer.

(b)

$$\frac{3\pi}{2} + 6\pi ni$$

with  $n$  any integer.

6. At what points  $z$  does the function

$$f(x) = \sin \bar{z}$$

satisfy the Cauchy–Riemann equations?

**Solution:**

$$u = \sin x \cosh y$$

$$v = -\cos x \sinh y$$

so the derivatives are

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

$$v_x = \sin x \sinh y$$

$$v_y = -\cos x \cosh y$$

so that the Cauchy–Riemann equations are satisfied exactly when

$$\cos x \cosh y = 0$$

and

$$\sin x \sinh y = 0.$$

Since  $\cosh y$  never vanishes, being a sum of two positive numbers:

$$\cosh y = \frac{e^y + e^{-y}}{2},$$

we see that  $\cos x = 0$  and therefore  $\sin x \neq 0$ , so  $\sinh y = 0$ , giving  $y = 0$ , and  $x = \pi \left(n + \frac{1}{2}\right)$  so that

$$z = \pi \left(n + \frac{1}{2}\right).$$