

# Complex Differentials and the Stokes, Goursat and Cauchy Theorems

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## 1 Stokes theorem

**Theorem 1 (Stokes)**

$$\int_{\partial U} f(x, y) dx + g(x, y) dy = \int_U \left( \frac{\partial g}{\partial y} - \frac{\partial f}{\partial x} \right) dx dy$$

where  $U$  is a region of the plane,  $\partial U$  is the boundary of that region, and  $f(x, y), g(x, y)$  are functions (smooth enough—we won't worry about that).

The main problem is to orient things correctly. It works if we orient the plane according to figure 1 on the following page and orient the curve  $\partial U$  following the left hand rule: as you travel along the curve  $\partial U$ , the region  $U$  should always be on your left, as in figure 2 on the next page.

**Exercise 1.1** Draw the arrows to orient the boundary of the region drawn in figure 3 on page 3.

**Example 1.1** Try  $f(x, y) = y$  and  $g(x, y) = 0$ . Let  $U$  be the square

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

We calculate

$$\begin{aligned} \int_{\partial U} f(x, y) dx &= \int_{\text{bottom}} f(x, y) dx + \int_{\text{right}} f(x, y) dx + \int_{\text{top}} f(x, y) dx + \int_{\text{left}} f(x, y) dx \\ &= \int_{x=0}^{x=1} f(x, 0) dx + \int_{y=0}^{y=1} f(1, y) dx + \int_{x=1}^{x=0} f(x, 1) dx + \int_{y=1}^{y=0} f(0, y) dx \end{aligned}$$

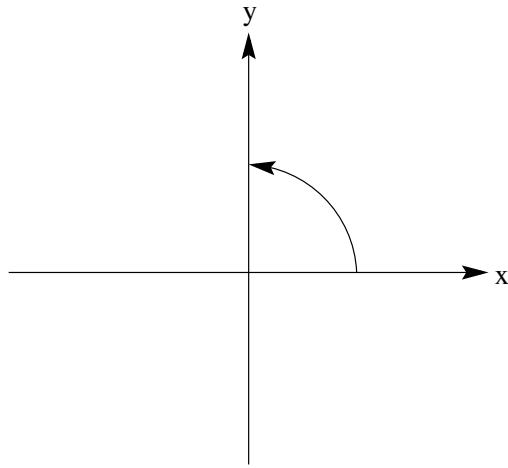


Figure 1: The standard orientation of the plane

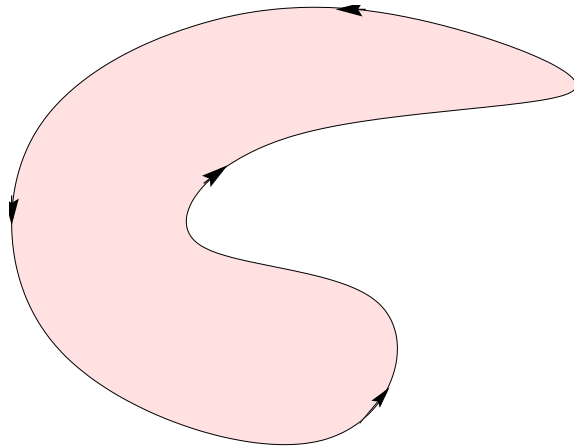


Figure 2: The standard orientation of the boundary of a region

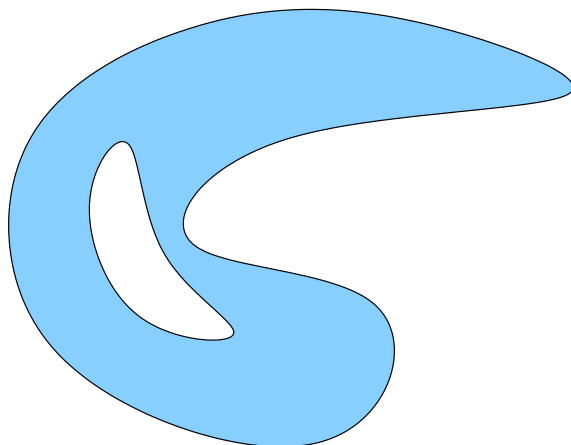


Figure 3: A region with a hole

Along the right and left,  $x$  is constant so  $dx = 0$ . In other words, if we parameterize these sides any way we like, say moving along them with time  $t$ , we get  $dx/dt = 0$ . So these drop out:

$$\int_{\partial U} f(x, y) dx = \int_{x=0}^{x=1} f(x, 0) dx + \int_{x=1}^{x=0} f(x, 1) dx.$$

Now plug in  $f(x, y) = y$  to get the bottom integral (along  $y = 0$ ) to drop out. We are left with

$$\int_{\partial U} f(x, y) dx = \int_{x=1}^{x=0} dx = -1.$$

Now let's check the other side of Stokes theorem:

$$\int_U \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} (-1) dx dy = -1.$$

**Example 1.2** Try  $f(x, y) = y$  and  $g(x, y) = 0$ . Let  $U$  be the unit circle  $x^2 + y^2 \leq 1$ . Then use polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta$$

so that

$$\begin{aligned} dx &= dr \cos \theta - r \sin \theta d\theta \\ &= dr \cos \theta - r \sin \theta d\theta. \end{aligned}$$

Similarly

$$dy = dr \sin \theta + r \cos \theta d\theta.$$

Let's calculate:

$$\int_{\partial U} f dx.$$

The  $dx$  must be

$$dx = -\sin \theta d\theta$$

because  $r = 1$  is constant.

$$\begin{aligned} \int_{\partial U} f dx &= \int_{\partial U} (r \sin \theta) (-\sin \theta) d\theta \\ &= - \int_{\partial U} \sin^2 \theta d\theta \\ &= -\pi. \end{aligned}$$

What is  $dx dy$  in terms of  $dr, d\theta$ ? This is tricky. But the right hand side of Stokes theorem is

$$- \int_U dx dy = -\text{area}(U) = -\pi.$$

So it works.

The orientation matters, since reversing the direction of the curve changes the sign. So we have to watch orientations if we swap  $x$  and  $y$  variables, as in figure 4 on the next page. This leads to the rule

$$dx dy = -dy dx$$

(anticommuting!) Strange, but essential. Generalizing to problems with many variables, this rule must hold for any two of the variables, since we could integrate over a plane parameterized by them. In particular

$$dx dx = -dx dx$$

so that

$$dx dx = 0.$$

Now we can change coordinates properly:

$$\begin{aligned} dx &= dr \cos \theta - r \sin \theta d\theta \\ dy &= dr \sin \theta + r \cos \theta d\theta \end{aligned}$$

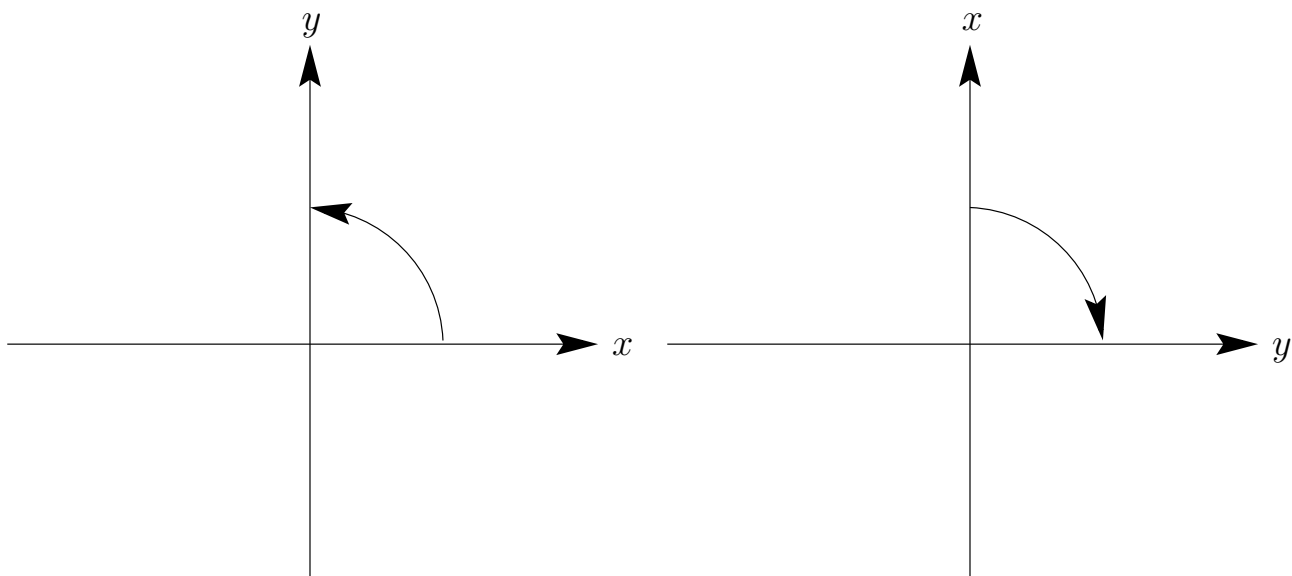


Figure 4: Swapping the orientation of the plane

Multiplying out, and using anticommuting,

$$dx dy = r dr d\theta.$$

**Exercise 1.2** Show that

$$r dr = x dx + y dy.$$

**Solution:** Differentiate

$$r^2 = x^2 + y^2.$$

□

**Exercise 1.3** Show that

$$\frac{x dy - y dx}{2} = \frac{r^2}{2} d\theta.$$

**Exercise 1.4** Use the previous exercise to show that the area of a disk of radius  $r$  is  $\pi r^2$ .

**Solution:**

$$\int_{\partial U} \frac{x dy - y dx}{2} = \int_U dx dy = \text{area}(U).$$

But in polar coordinates

$$\int_{\partial U} \frac{x dy - y dx}{2} = \int_{\partial U} \frac{r^2}{2} d\theta.$$

So if  $r$  is constant on  $\partial U$ , i.e. it is a circle, then this integral becomes

$$\frac{r^2}{2} \int_{\partial U} d\theta = \frac{r^2}{2} 2\pi.$$

□

## 2 Proof of Stokes theorem in a box

**Exercise 2.1** Take  $U$  a box, say

$$a_0 \leq x \leq b_0, \quad a_1 \leq y \leq b_1.$$

Show that

$$\int_{\partial U} g(x, y) dy = \int_U \frac{\partial g}{\partial x} dx dy.$$

Hint: this is really just the fundamental theorem of calculus in  $x$  direction, integrated in the  $y$  variable.

**Exercise 2.2** Show that the last exercise, when you swap the  $x$  and  $y$  variables as in figure 4 on the preceding page, gives

$$\int_{\partial U} g(y, x) dx = \int_U \left( -\frac{\partial g}{\partial y} \right) dx dy.$$

**Exercise 2.3** Put the last two exercises together to get Stokes theorem in a box.

**Exercise 2.4** Why does Stokes theorem hold for the region shown in figure 5 on the next page?

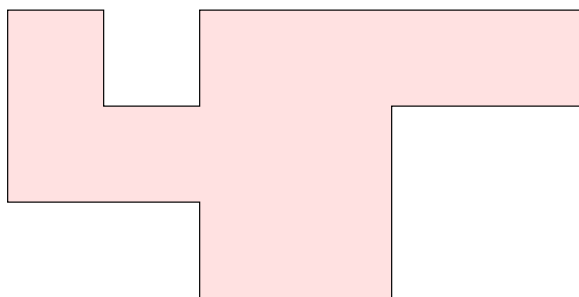


Figure 5: Boxes glued together

### 3 Coordinate changes

Define the differential of any function  $f(x, y)$  by the rule

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Now if we change coordinates, we have to say what happens to the differential. The rule for derivatives in new coordinates  $X, Y$  is the chain rule:

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial X}.$$

**Example 3.1** In polar coordinates

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}.$$

No matter what  $f$  is, we have

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos \theta \\ \frac{\partial y}{\partial r} &= \sin \theta. \end{aligned}$$

Therefore

$$\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}.$$

So if  $f(x, y) = x^2$ , then

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta \\ &= 2x \cos \theta. \end{aligned}$$

Plugging in  $x = r \cos \theta$  gives

$$\frac{\partial f}{\partial r} = 2r \cos^2 \theta.$$

So if we move out radially, fixing our angle  $\theta$ , then this is how fast  $f$  changes.

**Exercise 3.1** Let  $f(x, y) = x^2$ . Calculate  $\frac{\partial f}{\partial \theta}$ .

**Solution:**

$$\frac{\partial f}{\partial \theta} = -2r^2 \cos \theta \sin \theta.$$

□

**Exercise 3.2** Let  $f(x, y) = x^2$ . Calculate  $f$  in terms of  $r$  and  $\theta$ . Now check your answers to the last exercise.

**Solution:**

$$f = r^2 \cos^2 \theta.$$

□

**Exercise 3.3** Show that under any change of coordinates, for any function  $f(x, y)$ , the differential transforms as  $df$  goes to  $df$ , i.e.

$$\frac{\partial f}{\partial X} dX + \frac{\partial f}{\partial Y} dY = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

This is the most confusing part of the story: the differential is automatically matched up by the coordinate change.

**Solution:**

$$\begin{aligned} df_{X,Y \text{ coordinates}} &= \frac{\partial f}{\partial X} dX + \frac{\partial f}{\partial Y} dY \\ &= \frac{\partial f}{\partial X} \left( \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial y} dy \right) + \frac{\partial f}{\partial Y} \left( \frac{\partial Y}{\partial x} dx + \frac{\partial Y}{\partial y} dy \right) \\ &= \left( \frac{\partial f}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial x} \right) dx + \left( \frac{\partial f}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y} \frac{\partial Y}{\partial y} \right) dy \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\ &= df_{x,y \text{ coordinates}} \end{aligned}$$

Stare carefully at these lines.  $\square$

We call an expression like

$$f(x, y) dx + g(x, y) dy$$

a line element, or 1-form. Now define the derivative of this guy to be

$$d(f(x, y) dx + g(x, y) dy) = \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy.$$

**Exercise 3.4** Show that

$$d(F dx + G dy) = dF dx + dG dy.$$

Hint: remember that

$$dx dy = -dy dx$$

and that

$$dx dx = dy dy = 0.$$

**Theorem 2 (Stokes)**

$$\int_{\partial U} \vartheta = \int_U d\vartheta$$

where  $\vartheta$  is any line element.

The same works for functions, integrating over any curve  $C$ :

$$F(b) - F(a) = \int_C dF$$

where  $a$  is the start of  $C$  and  $b$  the end of  $C$ . We will write this as

**Theorem 3 (Fundamental Theorem of Calculus)**

$$\int_{\partial C} F = \int_C dF.$$

Here  $\partial C$  means the point  $b$  with positive orientation (because the curve  $C$  goes from  $a$  to  $b$ ), hence the  $F(b)$ , and the point  $a$  with negative orientation, hence the  $-F(a)$ . So the same Stokes theorem handles both cases:

$$\int_{\partial U} \vartheta = \int_U d\vartheta$$

where  $\vartheta$  can be a function or a line element. This will keep working in any number of dimensions.

On the plane, we only have  $dx$  and  $dy$  to build elements out of, so since

$$dx dx = dy dy = 0$$

we can't multiply three differentials together without getting zero. So the  $d$  of an area element

$$H(x, y) dx dy$$

must be

$$d(H(x, y) dx dy) = 0$$

since there is nothing else for it to be. In higher dimensions we have volume elements like

$$dx dy dz$$

and so on.

**Exercise 3.5** With these definitions of  $d$ , show that

$$dd = 0$$

on a function, line element or area element. (We will write this as  $d^2 = 0$ .)

**Exercise 3.6** Show that

$$d(f\vartheta) = df\vartheta + f d\vartheta$$

for any function  $f$  and line element  $\vartheta$ .

**Exercise 3.7** Show that if  $\delta$  is any other operation satisfying  $\delta^2 = 0$  and agreeing with  $d$  on functions:

$$df = \delta f$$

and satisfying

$$\delta(f\vartheta) = df\vartheta + f(\delta\vartheta)$$

then

$$\delta = d.$$

**Solution:** We start with the observation that

$$\delta^2 f(x, y) = 0$$

for any function. Write out how  $\delta$  behaves in coordinates:

$$\vartheta = F dx + G dy$$

so that

$$\begin{aligned}\delta\vartheta &= dF dx + F\delta(dx) + dG dy + G\delta(dy) \\ &= dF dx + F\delta^2 x + dG dy + G\delta^2 y \\ &= dF dx + dG dy \\ &= d\vartheta.\end{aligned}$$

So  $\delta = d$  on functions and on line elements. The same result on area elements is quite easy.  $\square$

**Exercise 3.8** Use the last result to show that if you take  $d$  and then change coordinates, you get the same result as changing coordinates and then taking  $d$ . In other words,  $d$  is independent of coordinates.

**Solution:** We know that the result is true for functions:

$$d_{X,Y \text{ coordinates}} F = d_{x,y \text{ coordinates}} F.$$

Let  $\delta$  be

$$\delta = d_{X,Y \text{ coordinates}}$$

i.e. the operation given by changing the coordinates, and then taking  $d$  in the new coordinates, and then changing coordinates back again. We see immediately that this satisfies the requirements of the last exercise.  $\square$

**Example 3.2** We will take  $d$  of

$$\vartheta = x^2 dy$$

in both rectangular and polar coordinates. First, in rectangular

$$d\vartheta = 2x dx dy.$$

Now in polar

$$\begin{aligned}\vartheta &= r^2 \cos^2 \theta (dr \sin \theta + r \cos \theta d\theta) \\ &= r^2 \cos^2 \theta \sin \theta dr + r^3 \cos^3 \theta d\theta.\end{aligned}$$

Taking  $d$  (and recalling that  $dr dr = 0$ , etc.)

$$d\vartheta = 2r^2 \cos \theta dr d\theta.$$

But recalling that

$$dx dy = r dr d\theta$$

we see that this is the same result in polar coordinates as we found in rectangular coordinates.

The operation  $d$  is the only coordinate invariant differential operator.

## 4 Proof of Stokes theorem in more generality

### 4.1 Regions with smooth boundary

Given functions  $f(x, y)$  and  $g(x, y)$ , we take our 1-form

$$\vartheta = f(x, y) dx + g(x, y) dy$$

and a region  $U$  with smooth boundary.

Suppose that the line element  $\vartheta$  vanishes everywhere except somewhere deep inside  $U$ ; in particular it vanishes near the boundary  $\partial U$ . Then certainly

$$0 = \int_{\partial U} \vartheta.$$

Put a big box around  $U$  and apply Stokes theorem to the box to conclude that

$$0 = \int_{\partial \text{BOX}} \vartheta = \int_{\text{BOX}} d\vartheta = \int_U d\vartheta$$

so Stokes theorem holds here too. We only need the result near the boundary.

Now suppose that  $\vartheta$  vanishes everywhere except near a little piece of the boundary. All we have to do is to change coordinates to get that piece of the boundary to straighten out to a piece of a straight line. Then we make a box with that piece of straight line appearing along its bottom side.

How do we do this? Suppose that (by rotating the plane if needed) the piece of boundary that we want to handle is the graph of a function  $y = f(x)$ . Then use the new coordinates  $X, Y$  and map  $x = X, y = Y + f(X)$ .

**Exercise 4.1** Where does this change of coordinates send the piece of curve  $y = f(x)$ ?

If we have to prove Stokes theorem for any line element on any region with smooth boundary, we just write the line element as a sum

$$\vartheta = \sum_i \vartheta_i$$

where each  $\vartheta_i$  represents the contribution to  $\vartheta$  coming from near some point; in other words each  $\vartheta_i$  is zero except in some little portion of  $U$  which is either entirely inside  $U$  or entirely outside  $U$  (so irrelevant) or sitting on a little smooth piece of the boundary.

This gives Stokes theorem on any region with smooth boundary.

## 4.2 Corners

If the boundary is not so smooth, say it has a corner, just use the same argument but change coordinates near the corner to get it to become the corner of a box. This requires changing the angle of the corner to a right angle, and then straightening out the sides.

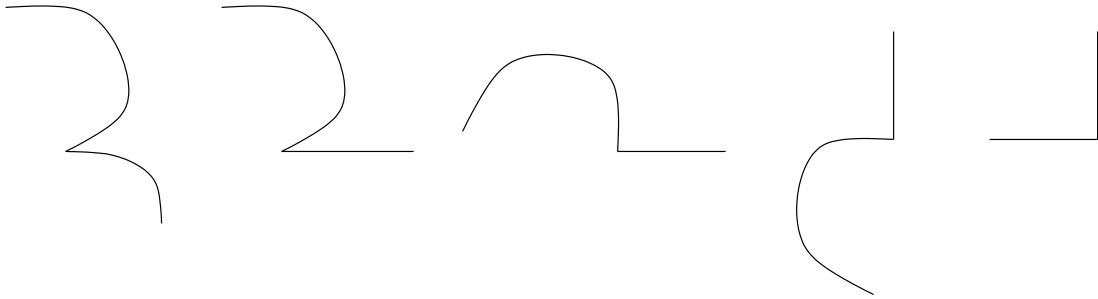


Figure 6: Turning a sharp corner into a square corner

How do we do it? The sequence of operations is shown in figure 6. (1) First, use the trick we worked out above for smooth sides to straighten out one side. (2) Then use a linear change of coordinates to fix the angle to be a right angle. (3) Rotate so that the straight side is the  $y$  axis. Then the bent side looks like the graph of a function  $y = f(x)$ , near the corner. (4) Now straighten out that bent side using the same trick as in (1).

**Exercise 4.2** Why doesn't the last step, straightening the bent side, bend the already-straightened side?

**Exercise 4.3** Calculate the coordinate changes this gives you for the region  $y \leq \sqrt{x}$ ,  $y \geq 0$ ,  $x \geq 0$ .

**Solution:**

$$X = -y, Y = x - y^2.$$

□

### 4.3 Cusps

**Exercise 4.4** Why is the above reasoning not good enough to prove Stokes theorem on the region

$$-1 \leq x \leq 1, y^3 \geq x^2, y \leq 1$$

which is drawn in figure 7 on the next page? Hint: consider the slope of the tangent line to the boundary near the origin. Show that this is not a corner.

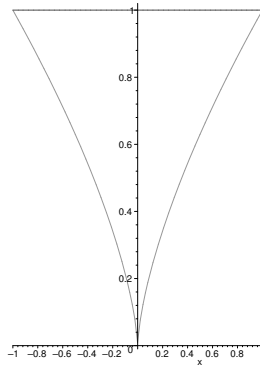


Figure 7: A region whose boundary has a cusp

**Exercise 4.5** What happens to the cusp from the previous exercise under the change of coordinates

$$x = X^3, y = Y?$$

What happens to the region? Calculate the integral

$$\int x \, dy$$

around the boundary of that region, in both  $x, y$  and  $X, Y$  coordinates.

**Solution:**

$$\int x \, dy = \frac{4}{5}.$$

□

**Exercise 4.6** Does this change of coordinates in the previous exercise prove Stokes theorem for the region in figure 7?

If the boundary was as nasty as in figure 8 on the next page, like the graph of

$$y = \text{round}(1/x) - 1/x,$$

then Stokes theorem would not hold, because the integral of a line element on this curve doesn't always make sense. It is too oscillatory.

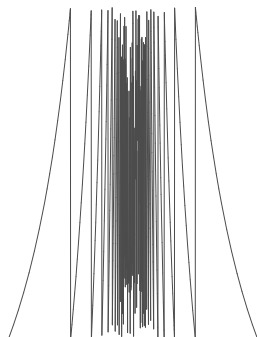


Figure 8: Too nasty for Stokes theorem

## 5 Complex notation

Nothing was said so far about the functions  $f$  and  $g$  appearing in a line element

$$f(x, y) dx + g(x, y) dy$$

so there is no problem letting them be complex valued instead of real valued. Or vector valued. Or matrix valued. Consider them complex valued for now. Writing

$$z = x + iy, \quad \bar{z} = x - iy$$

we are naturally led to define

$$dz = dx + i dy, \quad d\bar{z} = dx - i dy.$$

This tells us what

$$\int_C f(z) dz$$

means: it means

$$\int_C f(x + iy) dx + i \int_C f(x + iy) dy.$$

Following this pattern, by analogy with

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

we want to write

$$dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z}.$$

**Exercise 5.1** Show that to have  $dF$  be the same line element in both cases forces

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{1}{2} \left( \frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} \right) \\ \frac{\partial F}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).\end{aligned}$$

**Exercise 5.2** Show that if  $F = u + iv$  splits  $F$  into real and imaginary parts, then

$$\frac{\partial F}{\partial \bar{z}} = 0$$

precisely when  $u$  and  $v$  satisfy the Cauchy–Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned}$$

We will call  $F(z)$  *holomorphic* or *analytic* if it satisfies these.

**Exercise 5.3** Show that

$$\begin{aligned}\frac{\partial z}{\partial z} &= 1 \\ \frac{\partial z}{\partial \bar{z}} &= 0 \\ \frac{\partial \bar{z}}{\partial z} &= 0 \\ \frac{\partial \bar{z}}{\partial \bar{z}} &= 1.\end{aligned}$$

**Exercise 5.4** Show that

$$\frac{\partial}{\partial \bar{z}} FG = \frac{\partial F}{\partial \bar{z}} G + F \frac{\partial}{\partial \bar{z}}$$

and the same for  $\frac{\partial}{\partial z}$ .

**Exercise 5.5** Put the last two exercises together to show that

$$\frac{\partial}{\partial z} z^p \bar{z}^q = pz^{p-1} \bar{z}^q$$

(differentiation “in  $z$ ” leaving  $\bar{z}$  alone), and a similar identity for  $\frac{\partial}{\partial \bar{z}}$ .

**Exercise 5.6** Show that the sum, product, difference and quotient of holomorphic functions is holomorphic (except, for the quotient, when the denominator vanishes—just ignore that possibility for now).

**Exercise 5.7** Show that

$$dz d\bar{z} = -2i dx dy.$$

**Exercise 5.8** Show that

$$dz dz = 0.$$

**Theorem 4 (Complex Stokes)**

$$\int_{\partial U} F dz + G d\bar{z} = \int_U \left( \frac{\partial G}{\partial z} - \frac{\partial F}{\partial \bar{z}} \right) dz d\bar{z}.$$

**Exercise 5.9** Prove this by rewriting both sides in terms of  $dx$  and  $dy$ .

## 6 Goursat's theorem

This is a little weaker than Goursat's actual theorem, because he didn't require as much smoothness as we will. But ignoring that:

**Theorem 5 (Goursat)** If  $F$  is a holomorphic function on a region  $U$  then

$$\int_{\partial U} F(z) dz = 0.$$

**Proof** Stokes theorem.  $\square$

## 7 The Cauchy Integral Theorem

**Theorem 6** If  $f(z)$  is a holomorphic function on a region  $U$  then at any point  $z_0$  in  $U$

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z) dz}{z - z_0}.$$

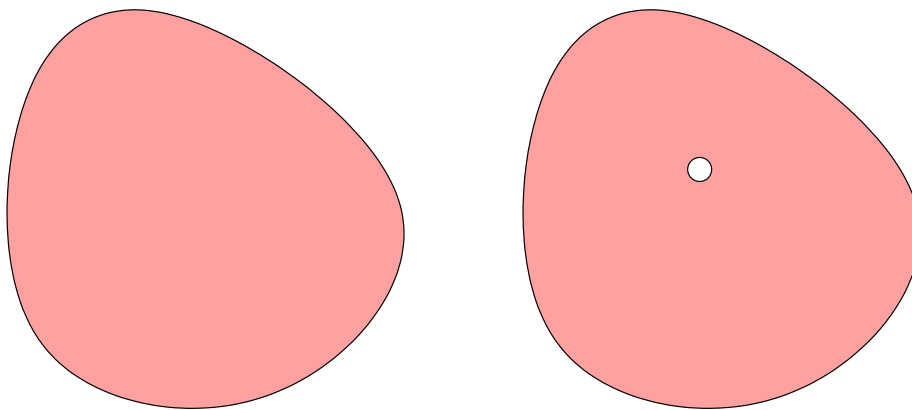


Figure 9: Cut a hole out of the region  $U$  to make the region  $U_\epsilon$

**Proof** Let

$$\vartheta = \frac{1}{2\pi i} \frac{f(z) dz}{z - z_0}$$

be the line element we are trying to integrate. It looks nasty at  $z = z_0$ , because the denominator vanishes there. But away from there

$$\begin{aligned} d\vartheta &= \frac{1}{2\pi i} d\left(\frac{f(z)}{z - z_0}\right) dz \\ &= 0 \end{aligned}$$

because  $f(z)/(z - z_0)$  is a quotient of holomorphic functions. We can't apply Stokes theorem here, because the line element  $\vartheta$  is nasty at  $z = z_0$ . As in figure 9, we cut out a small disk, of radius  $\epsilon$ , from around  $z_0$ ; call that disk  $D_\epsilon$ . Let  $U_\epsilon$  be the whole set  $U$  with  $D_\epsilon$  cut out of it. Then (if the disk is very small)

$$\partial U_\epsilon = \partial U - \partial D_\epsilon.$$

The minus sign here means that we orient this circle backwards. We now get

$$\int_{\partial U_\epsilon} \vartheta = \int_{U_\epsilon} d\vartheta = 0.$$

or

$$\int_{\partial U} \vartheta - \int_{\partial D_\epsilon} \vartheta = 0.$$

Writing this out explicitly,

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f(z) dz}{z - z_0} = \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(z) dz}{z - z_0}.$$

We want to take the limit as  $\varepsilon \rightarrow 0$ . Write out the right hand side in polar coordinates,

$$z = z_0 + \varepsilon e^{i\theta}.$$

You get

$$\begin{aligned} \int_{\partial D_\varepsilon} \vartheta &= \frac{1}{2\pi i} \int_{\theta=0}^{\theta=2\pi} \frac{f(z_0 + \varepsilon e^{i\theta}) d(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{i\theta}) \varepsilon e^{i\theta} i d\theta}{\varepsilon e^{i\theta}} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta}) d\theta \end{aligned}$$

which is just the average value of  $f(z)$  along the circle  $\partial D_\varepsilon$ . As  $\varepsilon \rightarrow 0$ , this obviously approaches  $f(z_0)$ .  $\square$