1. OUTLINE

Let $G$ be a linear connected reductive group (hence a Lie group) with Lie algebra $\mathfrak{g}$, and let $K$ be a maximal compact subgroup. Let $(\pi, V)$ be a representation of $G$. We say that a vector $v \in V$ is $C^1$ if the limit
\[
\pi(X)v := \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}
\]
eexists for all $X \in \mathfrak{g}$. We say that a vector $v \in V$ is smooth if
\[
\pi(X_1)\pi(X_2)\cdots \pi(X_n)v
\]
exists for all choices of elements $X_1, \ldots, X_n$ in $\mathfrak{g}$, $n \in \mathbb{Z}_{\geq 0}$. We define the subspace $V^\infty \subseteq V$ to be the set of all smooth vectors in $V$. We also define the subspace $V_K \subseteq V$ to be the set of $K$-finite vectors; this is the set of all $v \in V$ such that the dimension of $\text{Span} \{\pi(k)v, k \in K\}$ is finite.

We will show the following:
1) $V^\infty$ is dense in $V$
2) if $\pi$ is unitary, then $V_K$ is dense in $V$
3) if $\pi$ is admissible, then $V_K \subseteq V^\infty$

Most of the arguments contained in this paper are taken from Anthony W. Knapp’s *Representation Theory of Semisimple Groups: an Overview Based on Examples*.

2. $V^\infty$ IS DENSE IN $V$

First, we define the *Garding subspace* for $\pi$ to be the subspace $S \subseteq V$ spanned by vectors of the form
\[
\pi(f)v := \int_G f(g)\pi(g)v \, dg,
\]
where $v \in V, f \in C^\infty_{\text{com}}(G)$, and $dg$ is a left-invariant Haar measure. We first show that this subspace is contained in $V^\infty$, then show that it is dense in $V$.

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*Proof that $S \subseteq V^\infty$.* We have an action of $G$ on $C^\infty_{\text{com}}$ given by
\[
g \cdot f(x) := f(g^{-1}x).
\]
This induces an *infinitesimal action* of $\mathfrak{g}$ on $C^\infty_{\text{com}}$:
\[
X \cdot f(x) := \lim_{t \to 0} \frac{f((\exp tX)^{-1}x) - f(x)}{t}.
\]
Let \( \pi(f)v \in S, \) and let \( X \in \mathfrak{g}. \) We show that \( \pi(X)\pi(f)v \) exists. We have
\[
\left( \frac{\pi(\exp tX) - 1}{t} \right) \pi(f)v = \frac{\pi(\exp tX) - 1}{t} \int_G f(g)\pi(g)v \, dg
\]
\[
= t^{-1} \int_G (\pi(\exp tX) - 1)(f(g)\pi(g)v) \, dg
\]
\[
= t^{-1} \int_G f(g) (\pi(\exp tX)\pi(g) - \pi(g))v \, dg.
\]
We make the substitution \( g \mapsto \exp(-tX)g \) and continue:
\[
t^{-1} \int_G f(\exp(-tX)g)\left[ \pi(\exp tX)\pi(\exp(-tX)g) - \pi(\exp(-tX)g) \right]v \, dg
\]
\[
= t^{-1} \int_G f(\exp(-tX)g)[1 - \pi(\exp(-tX))]\pi(g)v \, dg
\]
\[
= t^{-1} \int_G (f(\exp(-tX)g) - f(g))\pi(g)v \, dg
\]
\[
= - \int_G f(\exp(-tX)g) - f(g) \frac{-1}{t-1} \pi(g)v \, dg.
\]
So, we have shown that
\[
\left( \frac{\pi(\exp tX) - 1}{t} \right) \pi(f)v = - \int_G f(\exp(-tX)g) - f(g) \frac{-1}{t-1} \pi(g)v \, dg.
\]
By taking limits as \( t \to 0 \) on both sides (and by using dominated convergence on the right to move the limit inside the integral), we then have
\[
\pi(X)\pi(f)v = -\pi(\langle X \cdot f \rangle v).
\]
This shows that any \( X \in \mathfrak{g} \) stabilizes \( S, \) and by iterating the action of \( \mathfrak{g} \) on \( S \) we thus see that any \( \pi(f)v \in S \) is smooth. Thus we have shown that \( S \subseteq V^\infty. \)

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**Proof that \( S \) is dense in \( V^\infty. ** Let \( v \in V \) and fix some \( \epsilon > 0. \) By definition, \( \pi \) is continuous, so the set
\[
T = \{ g \in G \mid |\pi(g)v - v| < \epsilon \}
\]
is open. This means we can find a compact set contained in \( T, \) and thus a function \( f \geq 0 \) in \( C^\infty_{\text{com}}(G) \) supported in \( T. \) By scaling, we may assume \( \int_G f(g) \, dg = 1. \) Then we have
\[
|\pi(f)v - v| = \left| \int_G f(g)[\pi(g)v - v] \, dg \right| \leq \int_G f(g)|\pi(g)v - v| \, dg
\]
\[
\leq \epsilon \int_G f(g) \, dg = \epsilon.
\]
This shows that the Gårding subspace \( S \) is dense in \( V. \)

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We have now shown that \( S \) is contained in \( V^\infty, \) and that \( S \) is dense in \( V, \) thus we have that \( V^\infty \) is dense in \( V, \) as claimed.
3. If $\pi$ is unitary, then $V_K$ is dense in $V$

Suppose we are given a unitary representation $(\pi, V)$ of $G$. As a consequence of the Peter-Weyl Theorem, then, we have that under the restriction of $\pi$ to $K$, $V$ can be decomposed as a sum of orthogonal, finite dimensional subspaces which are irreducible under the action of $K$. Letting $\tau$ signify the equivalence class of such an irreducible representation, we can write

$$\pi|_K = \sum_{\tau \in \hat{K}} n_{\tau} \tau,$$

where each $n_{\tau}$ is the (possibly infinite) multiplicity of the irreducible representation.

Since each irreducible representation is finite dimensional, any vector in such a representation is automatically $K$-finite. We then have that all finite sums of vectors chosen from various irreducible subspaces are $K$-finite, and thus the space of $K$-finite vectors is dense in $V$.

4. If $\pi$ is admissible, then $V_K \subseteq V^\infty$

First, we prove the following lemma:

**Lemma 4.1.** The set of $K$-finite vectors which are smooth is dense in $V$.

**Proof.** We use an argument similar to that used to prove the density of the Gårding subspace. Let $f$ be a $K$-finite function on $K$. That is, let $f : K \to \mathbb{C}$ be such that

$$\dim \text{span} \{ \pi(k)f | k \in K \} < \infty,$$

where

$$\pi(k)f(x) := f(k^{-1}x).$$

Let $h \in C^\infty_{\text{com}}(\exp p)$. Since $G$ is linear connected reductive, we have that $G = K \exp p$. We then define, for $k \in K, X \in p$,

$$F(k\exp X) := f(k)h(\exp X).$$

This function is in $C^\infty_{\text{com}}$, since both $f$ and $h$ are compactly supported. We also have that $F$ is left $K$-finite: given $k \in K$ and $x = k_0 \exp X_0$ we have

$$\pi(k)F(x) = F(k^{-1}x) = F(k^{-1}k_0 \exp X_0) = f(k^{-1}k_0)h(\exp X_0).$$

Thus, since $f$ is $K$-finite and since $h$ is unaffected by $K$, we have that $F$ is left $K$-finite.

Now, for any $v \in V$, we have $\pi(F)v \in V^\infty$ by the discussion in section 2. We also claim that $\pi(F)v$ is $K$-finite. Observe that, given $k_0 \in K$, we have

$$\pi(k_0)\pi(F)v = \int_G F(x)\pi(k_0x)v \, dx = \int_G F(k_0^{-1}x)\pi(x)v \, dx.$$

Since $F$ is left $K$-invariant, then, the collection of vectors of the form on the right side of the equality is finite, hence $\pi(F)$ is $K$-finite, as claimed.
To see that the smooth $K$-finite vectors are smooth in $V$, we mirror the argument for the density of the Gårding subspace from section 2. Let $v \in V$ and fix $\epsilon > 0$. Consider the open set

$$T = \{g \in G \mid |\pi(g)v - v| < \epsilon\}.$$ 

We may choose $f$ and $h$ above in such a way that $F$ is supported on a compact set contained in $T$ and such that $\int_G F(x) \, dx = 1$. Then we have that

$$|\pi(F)v - v| = \left| \int_G F(g) |\pi(g)v - v| \, dg \right| \leq \int_G |F(g)||\pi(g)v - v| \, dg$$

$$\leq \epsilon \int_G F(g) \, dg = \epsilon.$$

So, we have shown that any given $v \in V$ is arbitrarily close to a smooth $K$-finite vector, thus the smooth $K$-finite vectors are dense in $V$.

By the above lemma, the smooth $K$-finite vectors are dense in $V$, hence the smooth $K$-finite vectors of a given $K$-type are dense among the vectors of that type. If $\pi$ is admissible, then these spaces are finite dimensional. Since a dense linear subspace of a finite dimensional vector space must be the whole space, and since the $K$-finite vectors are precisely the finite linear combinations of vectors from a finite collection of $K$-types, we then have that the all $K$-finite vectors are smooth, as needed.