INvariance of $V^\infty_K$ UNDER $K$ AND $g$

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We prove the following proposition:

**Proposition 0.1.** Let $(\pi, V)$ be a representation of a real Lie group $G$, and let $K$ be a compact subgroup of $G$. Then the space $V^\infty_K$ of smooth $K$-finite vectors is preserved by the actions of $K$ and $g$. These two actions satisfy the conditions

1) the representation of $K$ is a direct sum of finite-dimensional irreducible representations;

2) the differential of the action of $K$ is equal to the restriction to $\mathfrak{k}$ of the action of $g$; and

3) for $k \in K, X \in g, v \in V^\infty_K$, we have $k \cdot (X \cdot v) = (\text{Ad}(k)X) \cdot (k \cdot v)$.

**Proof.** First, we show that $V^\infty_K$ is invariant under the action of $K$. Let $v \in V^\infty_K, k \in K$. We claim that $k \cdot v \in V^\infty_K$. Indeed, $K \cdot (k \cdot v) = K \cdot v$, thus since $v$ is $K$-finite, $\dim(K \cdot (k \cdot v)) = \dim(K \cdot v) < \infty$, hence $k \cdot v$ is $K$-finite. To see that $k \cdot v$ is also smooth, observe that for $g \in G$ we have $g \cdot (k \cdot v) = (g \cdot k) \cdot v$ is the composition of right multiplication by a fixed $k$ and the action of $G$ on $V$, both of which are smooth maps. Thus we have that $V^\infty_K$ is $K$-invariant.

Next, we show that $V^\infty_K$ is invariant under the action of $g$. Again, let $v \in V^\infty_K$. We claim that the subspace of $V$ spanned by $v$ under the action of $\mathfrak{k}$ is finite dimensional. Indeed, by the definition of $K$-finite the subspace $W$ of $V$ spanned by $v$ under the action of $K$ is finite dimensional. We have, for $X \in g$,

$$X \cdot v = \frac{d}{dt}\pi(e^{tX})v|_{t=0}.$$

Each of the vectors $\pi(e^{tX})v$ lies in $W$, hence so does $X \cdot v$. Thus we have the space $W$ is invariant under $\mathfrak{k}$ and contains all $\mathfrak{k}$-translates of $v$, hence the span of $v$ under $\mathfrak{k}$ is finite dimensional. Call this span $W'$ (actually, it can be shown that $W = W'$, but this suffices for our purposes).

Let $U \subseteq V$ be the subspace spanned by all $Xw$ with $X \in g$ and $w \in W'$. Since $g$ and $W'$ are finite dimensional, so is $U$. We have that $U$ is invariant under $\mathfrak{g}$: for $Y \in k, (X \cdot w) \in U$ we have

$$Y \cdot (X \cdot w) = [Y, X] \cdot w + X \cdot (Y \cdot w).$$

Both terms on the right lie in $U$, hence $U$ is invariant under $\mathfrak{g}$. So, we now have a $\mathfrak{g}$-invariant subspace $U \subseteq V$ which is finite dimensional and contains the vector $v$. Since $K$ is compact, it has finitely many connected components. We have that the map $\text{exp} : \mathfrak{k} \to K$ is surjective onto the identity component $K_0$ of $K$. Let $u \in U, k \in K_0$. Write $k = \exp(X), X \in \mathfrak{k}$. We claim that $k \cdot u \in U$. First, note that
Since $U$ is finite dimensional we have, for any $X \in \mathfrak{k}$,
\[
\pi(\exp(X)) = \exp(d\pi(X)).
\]
Thus, we may write
\[
k \cdot u = \pi(k)u = \pi(\exp(X))u = \exp(\pi(X))u.
\]
Since $U$ is finite dimensional, $\pi(X) \in GL_n(\mathbb{C})$. Thus, the term on the right may be written as a convergent power series in $X$ acting on $u$, with the result being another vector in $U$. Therefore, we have shown that $U$ is invariant under the action of $K_0$. Since each connected component $K_i$ of $K$ is isomorphic to $K_0$, we have that $K_i u$ is a finite dimensional subspace of $V$ for each $i$. Taking the direct sum of these subspaces then gives us a finite dimensional subspace of $V$ containing $v$ which is invariant under $K$. Thus we have shown that $X \cdot v$ is $K$-finite.

To see that $X \cdot v$ is smooth, observe that since $v$ is smooth, $\pi(\exp(tX))v$ is smooth for any $t$, thus
\[
\pi(X) = \lim_{t \to 0} \frac{\pi(e^{tX}) - I}{t} v = \pi(k)^{-1} \left( \pi(\exp(tX)k^{-1}) - I \right) v
\]
is smooth as well. We have thus shown that for an arbitrary $X \in \mathfrak{g}$, $X \cdot v \in V^\infty_K$, thus $V^\infty_K$ is $\mathfrak{g}$-invariant, as needed.

Part 1) of the proposition follows directly from the Peter-Weyl theorem. Any representation of a compact Lie group is a direct sum of finite-dimensional irreducible (unitary) representations.

Part 2) of the proposition is automatic; by definition, the action of $K$ is the restriction of the action of $G$, thus the differential of the action of $K$ is the restriction to $\mathfrak{k}$ of the action of $\mathfrak{g}$.

To prove part 3), we show that, for $k \in K, X \in \mathfrak{g}, v \in V^\infty_K$,
\[
\pi(k)\pi(X)\pi(k)^{-1} = \pi(\text{Ad}(k)X).
\]
Observe that
\[
\lim_{t \to 0} \frac{\pi(\exp(tX)) - I}{t} \pi(k)^{-1} v = \pi(k)^{-1} \left( \frac{\pi(\exp(tX)k^{-1}) - I}{t} \right) v
\]
\[
= \pi(k)^{-1} \left( \frac{\pi(\text{Ad}(k)tX) - I}{t} \right) v.
\]
Taking the limit as $t \to 0$, we get $\pi(X)\pi(k)^{-1} v = \pi(k)^{-1} \pi(\text{Ad}(k)X)v$, hence
\[
\pi(k)\pi(X)\pi(k)^{-1} = \pi(\text{Ad}(k)X),
\]
so
\[
\pi(k)\pi(X) = \pi(\text{Ad}(k)X)\pi(k).
\]
That is,
\[
k \cdot (X \cdot v) = (\text{Ad}(k)X) \cdot (k \cdot v),
\]
and part 3) is proven. This completes the proof of the proposition.