ANOTHER ELEMENTARY PROOF OF THE INJECTIVITY OF
THE HARISH-CHANDRA HOMOMORPHISM FOR $\mathfrak{sl}(2,\mathbb{C})$

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1. Overview

This proof is taken directly from some lecture notes of Dragan Milicic. The idea of the proof is as follows: let $U_n(g)$ denote the set of elements of $\mathcal{U}(g)$ whose underlying monomials have degree at most $n$. Let $z \in \mathfrak{g} \cap U_n(g)$, $z \notin U_{n-1}(g)$. We let

$$H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \bar{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$ 

Hence we will prove some combinatorial lemmas and use them to show that (for some $c \neq 0$),

$$z = c H^n + \{ \text{linear combinations of monomials of the form } Y^i H^k \bar{Y}^i, (i \geq 1) \} + \{ \text{terms in } U_{n-1}(g) \}.$$ 

Since the image of $\rho$ lies in $\mathbb{C}$, we will thus have shown that $\gamma(z)$ is nonzero for any nonzero $z \in \mathfrak{g}$, proving the injectivity of $\gamma$.

2. Preliminary lemmas

Lemma 2.1. For arbitrary $A, B \in g$, we have

$$[A, B^n] = -\sum_{k=1}^{n} \binom{n}{k} ((\text{ad } B)^k A) B^{n-k} = \sum_{k=1}^{n} (-1)^k \binom{n}{k} B^{n-k} ((\text{ad } B)^k A).$$

Proof. We define operators $R_Y, L_Y : \mathcal{U}(g) \to \mathcal{U}(g)$ by

$$L_Y A = YA \quad \text{and} \quad R_Y A = AY, (A \in \mathcal{U}(g)),$$

where $Y$ is as defined in section 1.

We clearly have $R_Y L_Y = L_Y R_Y$. Furthermore,

$$(L_Y - R_Y)A = YA - AY = [Y, A] = (\text{ad } Y) A,$$

so

$$L_Y = R_Y + \text{ad } Y \quad \text{and} \quad R_Y = L_Y - \text{ad } Y.$$ 

We then have

$$L_Y^n = \sum_{k=0}^{n} \binom{n}{k} R_Y^{n-k} (\text{ad } Y)^k$$

and

$$R_Y^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_Y^{n-k} (\text{ad } Y)^k.$$
This gives us (for $A \in U(g)$)

\[
[A, Y^n] = AY^n - Y^n A = R^n(A) - Y^n A
\]

\[
= \left( \sum_{k=0}^{n} (-1)^k \binom{n}{k} Y^{n-k}((\text{ad} Y)^k A) \right) - Y^n A
\]

\[
= \sum_{k=1}^{n} (-1)^k \binom{n}{k} Y^{n-k}((\text{ad} Y)^k A),
\]

and

\[
[A, Y^n] = AY^n - \sum_{k=0}^{n} \binom{n}{k} R^{n-k}(\text{ad} Y)^k(A)
\]

\[
= -\sum_{k=1}^{n} \binom{n}{k} ((\text{ad} Y)^k A)Y^{n-k},
\]

as needed. □

Observe that

\[
(\text{ad} \bar{Y})Y = [\bar{Y}, Y] = 2H,
\]

\[
(\text{ad} \bar{Y})^2 Y = [\bar{Y}, 2H] = 2\bar{Y}, \text{ and}
\]

\[
(\text{ad} \bar{Y})^kY = 0 \text{ for } k \geq 3.
\]

Combining this result with the above lemma gives

\[
[Y, \bar{Y}^n] = -2nH\bar{Y}^{n-1} + n(n-1)\bar{Y}^{n-1}.
\]

For future use, we also note that $[H, Y] = Y$ and calculate

\[
[Y, H^n] = \sum_{k=1}^{n} (-1)^k \binom{n}{k} H^{n-k}((\text{ad} H)^k Y) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} H^{n-k}Y.
\]

**Lemma 2.2.** Let $X_1, \ldots, X_n \in \mathfrak{g}$, and let $\sigma \in S_n$. Then

\[
X_1X_2 \cdots X_n - X_{\sigma(1)}X_{\sigma(2)} \cdots X_{\sigma(n)} \in U_{n-1}(\mathfrak{g}).
\]

**Proof.** First, observe that

\[
X_1 \cdots X_iX_{i+1} \cdots X_n - X_1 \cdots X_{i+1}X_i \cdots X_n = X_1 \cdots [X_i, X_{i+1}] \cdots X_n \in U_{n-1}(\mathfrak{g}).
\]

Thus, the lemma holds for any transposition in $S_n$. Now, suppose that $\sigma$ and $\tau$ are any two permutations in $S_n$ for which the lemma holds. We have

\[
X_1 \cdots X_n - X_{(\sigma\tau)(1)} \cdots X_{(\sigma\tau)(n)}
\]

\[
= (X_1 \cdots X_n - X_{\tau(1)} \cdots X_{\tau(n)}) + (X_{\tau(1)} \cdots X_{\tau(n)} - X_{(\sigma\tau)(1)} \cdots X_{(\sigma\tau)(n)}) \in U_{n-1}(\mathfrak{g}).
\]

Therefore, the set of permutations for which the lemma holds is a subgroup of $S_n$. Since the lemma holds for transpositions (which generate $S_n$), the proof is complete. □
3. PROOF OF INJECTIVITY

We are ready to prove the injectivity of the Harish-Chandra homomorphism. As mentioned in the first section, we let \( z \in \mathfrak{z} \cap \mathcal{U}_n(\mathfrak{g}) \) such that \( z \notin \mathcal{U}_{n-1}(\mathfrak{g}) \) and \( z \neq 0 \). We show that \( \gamma(z) \neq 0 \), thus \( \gamma \) is injective. Using \( Y, H, \) and \( \bar{Y} \) as a basis for \( \mathfrak{sl}(2, \mathbb{R}) \), we have by Poincaré-Birkhoff-Witt that

\[
z = \sum c_{ij} Y^i H^j \bar{Y}^i, \quad (i, j \in \mathbb{Z}_{\geq 0}, i + j + i \leq n)
\]

Note that the exponents of \( Y \) and \( \bar{Y} \) are equal in any given monomial term of \( z \). This is because \( z \) is a zero weight of the adjoint representation of \( \mathcal{U}(\mathfrak{g}) \).

We let

\[
j_0 = \max \{ j \in \mathbb{Z}_{\geq 0} | c_{ij} \neq 0 \text{ and } 2i + j = n \}.
\]

Such a \( j_0 \) exists since \( z \notin \mathcal{U}_{n-1}(\mathfrak{g}) \).

We now apply the lemmata and observations from section 2 to arrive at the following series of congruences:

\[
0 = [Y, z] \equiv \sum_{2i+j=n} c_{ij} [Y, Y^i H^j \bar{Y}^i] \mod \mathcal{U}_{n-1}(\mathfrak{g})
\]

\[
= \sum_{2i+j=n} c_{ij} (Y^i [Y, H^j] \bar{Y}^i + Y^i H^j [Y, \bar{Y}^i])
\]

\[
\equiv \sum_{2i+j=n} c_{ij} (-jY^i+1 H^j-1 \bar{Y}^i - 2iY^i H^j+1 \bar{Y}^i-1) \mod \mathcal{U}_{n-1}(\mathfrak{g})
\]

\[
\equiv -2i_0 \cdot c_{i_0 j_0 i_0} (Y^{i_0} H^{j_0+1} \bar{Y}^{i_0-1}) \mod \mathcal{U}_{n-1}(\mathfrak{g})
\]

modulo terms involving lower powers of \( H \).

By hypothesis, \( c_{i_0 j_0 i_0} \neq 0 \), so the above implies that \( i_0 = 0 \), thus \( j_0 = n \).

Therefore,

\[
z = c_{0n0} H^n + \{ \text{other terms} \},
\]

so

\[
\gamma(z) = c_{0n0} H^n + \{ \text{lower powers of } H \} - \rho( \text{multiples of some powers of } H ) \neq 0,
\]

hence \( \gamma \) is injective, as claimed.