1. Linearity and its consequences

We are interested in solving a linear n^{th}-order differential equation with continuous coefficients and continuous right-hand side, that is

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)\,. \tag{1.1}$$

If $g(t)$ is not identically zero, then this equation is said to be inhomogeneous; if $g(t)$ is identically zero, then this equation is called homogeneous. If we have an inhomogeneous equation like (1.1) then we will associate with the homogeneous equation one obtains if one replaces the right-hand side with zero, that is

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0\,. \tag{1.2}$$

We will now call the left-hand side of (1.1) or (1.2) $L[y]$. In other words, L is a transformation into which we plug in a function y and which returns the function $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y$. It can easily be seen that L is a linear transformation, that is, L satisfies the following two properties:

1. $L[y_1 + y_2] = L[y_1] + L[y_2]$ for any pair of functions y_1 and y_2 ,
2. $L[\alpha y] = \alpha L[y]$ for any function y and any scalar α .

Proof.

$$\begin{aligned} L[y_1+y_2] &= (y_1+y_2)^{(n)} + p_1(t)(y_1+y_2)^{(n-1)} + p_2(t)(y_1+y_2)^{(n-2)} + \cdots \\ &\quad + p_{n-1}(t)(y_1+y_2)' + p_n(t)(y_1+y_2) \\ &= (y_1^{(n)} + y_2^{(n)}) + p_1(t)(y_1^{(n-1)} + y_2^{(n-1)}) + p_2(t)(y_1^{(n-2)} + y_2^{(n-1)}) + \cdots \\ &\quad + p_{n-1}(t)(y_1' + y_2') + p_n(t)(y_1+y_2) \\ &= y_1^{(n)} + p_1(t)y_1^{(n-1)} + p_2(t)y_1^{(n-2)} + \cdots + p_{n-1}(t)y_1' + p_n(t)y_1 \\ &\quad + y_2^{(n)} + p_1(t)y_2^{(n-1)} + p_2(t)y_2^{(n-2)} + \cdots + p_{n-1}(t)y_2' + p_n(t)y_2 \\ &= L[y_1] + L[y_2] \end{aligned}$$

$$\begin{aligned} L[\alpha y] &= (\alpha y)^{(n)} + p_1(t)(\alpha y)^{(n-1)} + p_2(t)(\alpha y)^{(n-2)} + \cdots + p_{n-1}(t)(\alpha y)' + p_n(t)(\alpha y) \\ &= \alpha y^{(n)} + \alpha p_1(t)y^{(n-1)} + \alpha p_2(t)y^{(n-2)} + \cdots + \alpha p_{n-1}(t)y' + \alpha p_n(t)y \\ &= \alpha L[y] \end{aligned}$$

□

Consequences of this are listed below.

1. If y_1 and y_2 satisfy the homogeneous equation (1.2) and if α and β are any numbers, then $\alpha y_1 + \beta y_2$ also satisfies the homogeneous equation (1.2).

   Proof. y solving (1.2) is the same as saying $L[y] = 0$. Thus if y_1 and y_2 solve (1.2), we have $L[y_1] = 0$ and $L[y_2] = 0$. The linearity of L now yields $L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2] = \alpha 0 + \beta 0 = 0$. □

2. If y_1 and y_2 satisfy the inhomogeneous equation (1.1), then $y_1 - y_2$ satisfies the homogeneous equation (1.2).

Proof. y solving (1.1) is the same as saying $L[y] = g$. Thus if y_1 and y_2 solve (1.1), we have $L[y_1] = g$ and $L[y_2] = g$. The linearity of L now yields $L[y_1 - y_2] = L[y_1] - L[y_2] = g - g = 0$. □

3. If y_1 satisfies the inhomogeneous equation (1.1) and y_2 satisfies the homogeneous equation (1.2) and if α is any number, then $y_1 + \alpha y_2$ also solves the inhomogeneous equation (1.1).

   Proof. We have $L[y_1] = g$ and $L[y_2] = 0$. The linearity of L now yields $L[y_1 + \alpha y_2] = L[y_1] + \alpha L[y_2] = g + \alpha 0 = g$. □

The upshot of this all is, that if we find just one solution of the inhomogeneous equation and all solutions to the homogeneous equation, then we get all solutions of the inhomogeneous equation as well by simply adding all solutions of the homogeneous equation to the one solution of the inhomogeneous equation. This is so, because we have seen that any two solutions to the inhomogeneous equation differ by a solution to the homogeneous equation (this is exactly what 2. from above states), and because adding a solution of the inhomogeneous equation and a solution of the homogeneous equation produces a solution of the inhomogeneous equation (this is exactly what 3. from above states).

An Example. We want to find all solutions for the inhomogeneous equation

$$y'' + y = t\,.$$

The associated homogeneous equation is

$$y'' + y = 0\,.$$

It can be shown (see your homework from this week) that the general solution to the homogeneous equation is given by

$$y_h = C_1 \sin t + C_2 \cos t\,.$$

Furthermore, it is easy to guess that $y = t$ is a solution for the inhomogeneous equation. Hence all solutions for the inhomogeneous are given by

$$y = t + C_1 \sin t + C_2 \cos t\,.$$

2. Transformation into a system

We introduce new variables to get rewrite the n^{th} order differential equation into a system of n first order differential equations. We let

$$u_1 = y, u_2 = y', u_3 = y'', \dots, u_n = y^{(n-1)},$$

and we rewrite the inhomogeneous equation (1.1) as

$$\begin{aligned}
u_1' &= u_2\,,\\
u_2' &= u_3\,,\\
&\cdots\\
u_{n-1}' &= u_n\,,\\
u_n' &= -p_1(t)u_{n-1} - p_2(t)u_{n-2} - \cdots - p_{n-1}(t)u_2 - p_n(t)u_1 + g(t)\,.
\end{aligned}$$

If we now introduce vector notation, that is

$$\mathbf{x} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & -p_{n-3} & \cdots & -p_1 \end{pmatrix}.$$

then this system can very compactly be rewritten as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \tag{2.1}$$

where $\mathbf{x}$ and $\mathbf{b}$ are vectors depending on time t, and $\mathbf{A}$ is a $n \times n$ matrix depending on time t. Of course, if we start with a homogeneous equation (1.2) we have $\mathbf{b} = \mathbf{0}$ and we get a simpler system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{2.2}$$

For further use we note

$$\mathbf{x} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}. \tag{2.3}$$

Using this it is fairly straight-forward to see that if we have a solution to the inhomogeneous ODE (1.1), we can define $\mathbf{x}$ as above, and we get a solution to the system (2.1). The converse is also true, if we have a solution to the system (2.1), then we can define y to be the first component of $\mathbf{x}$, and we have a solution to (1.1). The same can be reasoned for the equivalence of solutions of the homogeneous equation (1.2) and the reduced system (2.2).

Now we proceed to solve (2.1). For this we treat it as if it were a scalar first-order linear differential equation, which we know how to solve. We simply bring all the $\mathbf{x}$'s onto the left-hand side, multiply through with the correct integrating factor, integrate, and solve for $\mathbf{x}$. That is, we solve

$$\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{b}$$

by multiply with the integrating factor $e^{-\int \mathbf{A}(t)dt}$. Notice that this is the exponential of a matrix, which is itself a matrix. So we get

$$e^{-\int \mathbf{A}(t)dt}\mathbf{x}' - \mathbf{A}(t)e^{-\int \mathbf{A}(t)dt}\mathbf{x} = e^{-\int \mathbf{A}(t)dt}\mathbf{b}(t),$$

which is

$$\left(e^{-\int \mathbf{A}(t)dt}\mathbf{x}\right)' = e^{-\int \mathbf{A}(t)dt}\mathbf{b}(t).$$

We integrate to get

$$e^{-\int \mathbf{A}(t)dt}\mathbf{x} = \int e^{-\int \mathbf{A}(t)dt}\mathbf{b}(t)\,dt.$$

Finally we multiply by the inverse of the integrating factor, which of course is $e^{\int \mathbf{A}(t)dt}$, to get $\mathbf{x}$ alone,

$$\mathbf{x} = e^{\int \mathbf{A}(t)dt} \int e^{-\int \mathbf{A}(t)dt}\mathbf{b}(t)\,dt. \tag{2.4}$$

Note that the integration gives an integration constant, which is a vector, so that the general solution has a vector constant in it. That is to say, the general solution has n scalar constants in it.

With definite integrals we can rewrite the formula as

$$\mathbf{x} = e^{\int_{t_0}^{t} \mathbf{A}(s)ds} \left(\int_{t_0}^{t} e^{-\int_{t_0}^{\tau} \mathbf{A}(s)ds} \mathbf{b}(\tau)\, d\tau + \mathbf{x}_0 \right) , \tag{2.5}$$

and equation (2.5) gives the unique solution to

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b} , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 . \tag{2.6}$$

Here $\mathbf{x}_0$ is the initial condition, that is, $\mathbf{x}_0 = (y_0, y_1, \dots , y_{n-1})$ is an arbitrary vector in n-space. Recalling the relationship (2.3) between the vector $\mathbf{x}$ and the function y solving our original problem (1.1) we see that the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ can be read as $y(t_0) = y_0, y'(t_0) = y_1, \dots , y^{(n-1)}(t_0) = y_{n-1}$.

Finally, if we are dealing with the homogeneous equation then $g(t) = 0$, or equivalently $\mathbf{b} = \mathbf{0}$, and the formula is much easier, namely the general solution to (2.2) is given by

$$\mathbf{x} = e^{\int \mathbf{A}(t)dt} \mathbf{c} , \tag{2.7}$$

where $\mathbf{c} = (C_1, C_2, \dots , C_n)^T$ is an arbitrary vector constant. Of course, we can also use the formula with the definite integrals, and we obtain

$$\mathbf{x} = e^{\int_{t_0}^{t} \mathbf{A}(s)ds} \mathbf{x}_0 , \tag{2.8}$$

and equation (2.8) gives the unique solution to

$$\mathbf{x}' = \mathbf{A}\mathbf{x} , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 . \tag{2.9}$$

We summarize this in the following theorem.

Theorem 1. *The inhomogeneous n^{th}-order linear ODE*

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

with initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_1,\ \dots ,\ y^{(n-1)}(t_0) = y_{n-1}$$

has exactly one solution that exists on the same interval on which the coefficients and the right-hand side are continuous. The solution can be computed by computing (2.5) *(or* (2.8) *in case $g(t) = 0$) and taking the first component of this vector-valued function.*

If no initial condition is specified, then the general solution can be computed using (2.4) *(or* (2.7) *in case $g(t) = 0$) and again taking the first component of this vector-valued function. This (or any other) general solution will have n constants in it.*

3. An example

In this section we want to use the method from above solve

$$y'' - y = 0 , \quad y(0) = 1 ,\ y'(0) = 0 .$$

That is, in the above notation,

$$n = 2 ,\ p_1(t) = 0 ,\ p_2(t) = -1 ,\ g(t) = 0 ,\ t_0 = 0 ,\ y_0 = 1 ,\ y_1 = 0 ,$$

so that the matrix and the vectors are given as

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The system corresponding to our ODE is therefore

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now we proceed to use formula (2.7) to find the general solution, so we first need to integrate $\mathbf{A}$. A matrix is integrated by integrating each of its entries separately, so

$$\int \mathbf{A} dt = \int \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dt = \begin{pmatrix} c_1 & t + c_2 \\ t + c_3 & c_4 \end{pmatrix},$$

where we will choose all the integration constants to be zero, just as we did when we found an integrating factor in the scalar case. Next we need to compute $e^{\int \mathbf{A}(t)dt}$, that is we need to compute

$$e^{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}}.$$

For this recall the Taylor series of the natural exponential function, which states that for any real number x one has

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots.$$

This series is used to define the exponential of a matrix as well, that is for any square matrix $\mathbf{M}$ we define

$$e^{\mathbf{M}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{M}^k = \mathbf{I} + \mathbf{M} + \frac{1}{2!} \mathbf{M}^2 + \frac{1}{3!} \mathbf{M}^3 + \frac{1}{4!} \mathbf{M}^4 + \cdots.$$

Here $\mathbf{I}$ stands for the identity matrix, which is the matrix that has ones on the diagonal and zeros elsewhere, so for $n = 2$,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we need to compute the powers of the matrix:

$$\begin{aligned}
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^2 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \\
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^3 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^3 \\ t^3 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^4 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^3 \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^3 \\ t^3 & 0 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} \\
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^5 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^4 \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^5 \\ t^5 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^6 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^5 \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^5 \\ t^5 & 0 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^6 & 0 \\ 0 & t^6 \end{pmatrix} \\
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^7 &= \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^6 \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^6 & 0 \\ 0 & t^6 \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^7 \\ t^7 & 0 \end{pmatrix}
\end{aligned}$$

There is an obvious pattern, and we have

$$
\begin{aligned}
e^{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & t^3 \\ t^3 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} \\
&\quad + \frac{1}{5!}\begin{pmatrix} 0 & t^5 \\ t^5 & 0 \end{pmatrix} + \frac{1}{6!}\begin{pmatrix} t^6 & 0 \\ 0 & t^6 \end{pmatrix} + \frac{1}{7!}\begin{pmatrix} 0 & t^7 \\ t^7 & 0 \end{pmatrix} + \cdots \\
&= \begin{pmatrix} 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \frac{1}{6!}t^6 + \cdots & t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \frac{1}{7!}t^7 + \cdots \\ t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \frac{1}{7!}t^7 + \cdots & 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \frac{1}{6!}t^6 + \cdots \end{pmatrix} \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},
\end{aligned}
$$

where for the last equality we use the Taylor series for cosh and sinh, which are

$$
\begin{aligned}
\cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots, \\
\sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots.
\end{aligned}
$$

Finally we get the general solution for the system using (2.7),

$$
\mathbf{x}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \cosh t + C_2 \sinh t \\ C_1 \sinh t + C_2 \cosh t \end{pmatrix}.
$$

Remember that the first component of $\mathbf{x}$ gives use the general solution y for the ODE, and that's after all what we really want, so

$$
y(t) = C_1 \cosh t + C_2 \sinh t.
$$

Now we find the constants by plugging in the initial conditions,

$$
\begin{aligned}
y(0) = 1 &= C_1 \cosh 0 + C_2 \sinh 0 = C_1, \\
y'(0) = 0 &= C_1 \sinh 0 + C_2 \cosh 0 = C_2.
\end{aligned}
$$

We are lucky here, we don't need to solve a linear system, but the values $C_1 = 1$ and $C_2 = 0$ pop right out without any further work. So the solution to the original differential equation with the specified initial conditions is

$$
y(t) = \cosh t.
$$

This is the end of the example.

However, we could have also used (2.8) to find the desired solution right away, instead of using (2.7) to first find the general solution and then finding the constants. If we do this, then we use

$$
\int_0^t \mathbf{A}(s)ds = \int_0^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ds = \begin{pmatrix} c_1 & s + c_2 \\ s + c_3 & c_4 \end{pmatrix}\Bigg|_{s=0}^{s=t} = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix},
$$

and we need to compute

$$
e^{\int_0^t \mathbf{A} ds} = e^{\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}}.
$$

This is by chance exactly the matrix we had above. This is pure coincidence, choosing all constants in the indefinite integral equal to zero does not always result exactly in the definite integral one needs. So, lucky as we are, we already know

$$e^{\int_0^t \mathbf{A}(s)ds} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

and hence the solution to the system is given according to formula (2.8) as being

$$\mathbf{x}(t) = e^{\int_0^t \mathbf{A}(s)ds}\mathbf{x}_0 = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.$$

Taking the first component of $\mathbf{x}$ we obtain y, and so we get, as before,

$$y(t) = \cosh t\,.$$

4. Linear dependence and independence

Definition 2. A collection of vectors $\mathbf{v}_1, \dots \mathbf{v}_k$ is called linear dependent, if there are scalars $c_1, \dots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_k\mathbf{v}_k = \mathbf{0}$. If there are no such numbers then the collection of vectors is called linearly independent.

This definition implies the following statement: A collection of vectors $\mathbf{v}_1, \dots \mathbf{v}_k$ is linear independent, if whenever $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_k\mathbf{v}_k = \mathbf{0}$ for some scalars $c_1, \dots, c_k$, then necessarily $c_1 = 0, \dots, c_k = 0$.

Examples. The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly dependent, since

$$(1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent, since

$$c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

can only be true if $c_1 = 0, c_2 = 0, c_3 = 0$. Just solve the linear system

$$\begin{array}{ccccl} c_1 & + & c_2 & & = 0 \\ & & & c_3 & = 0 \\ & & c_2 & & = 0 \end{array}$$

to see that.

In the vector space of functions, the functions e^x, e^{-x}, and $\cosh x$ are linearly dependent, because we know that $\cosh x = \frac{1}{2}(e^x + e^{-x})$, so that

$$1\cosh x - \frac{1}{2}e^x - \frac{1}{2}e^{-x} = 0$$

for all x.

In the vector space of functions, the functions e^x, e^{-x}, and x are linearly independent. To see this, start out with

$$c_1e^x + c_2e^{-x} + c_3x = 0$$

for all x. Now simply plug in some numbers for x, for example plug in $x = 0$, $x = 1$, $x = 2$, and you get the linear system

$$\begin{array}{ccccccc} c_1 & + & c_2 & & & = 0 \\ ec_1 & + & \frac{1}{e}c_2 & + & c_3 & = 0 \\ e^2c_1 & + & \frac{1}{e^2}c_2 & + & 2c_3 & = 0 \end{array}$$

Now solve this linear system, and you will get $c_1 = 0, c_2 = 0, c_3 = 0$.

Linear independence of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ be vector functions with n components each, and assume each of them solves $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\overline{\mathbf{x}}_1 = \mathbf{x}_1(t_0), \ldots, \overline{\mathbf{x}}_k = \mathbf{x}_k(t_0)$ be the initial values. Using formula (2.8) we know that

$$\mathbf{x}_1(t) = e^{\int_{t_0}^t \mathbf{A}(s)ds}\overline{\mathbf{x}}_1\,, \;\ldots\;, \;\mathbf{x}_k(t) = e^{\int_{t_0}^t \mathbf{A}(s)ds}\overline{\mathbf{x}}_k\,. \tag{4.1}$$

Theorem 3. *The vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly independent for all t if and only if the vectors $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_k$ are linearly independent.*

Proof. One direction of this "if and only if" proof is easy. Namely, if $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly independent for all t then they are in particular linearly independent for t_0, which gives exactly that the vectors $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_k$ are linearly independent.

For the other direction we assume that the vectors $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_k$ are linearly independent. We start out with $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}$ and we need to show that $c_1 = 0, \ldots, c_k = 0$. Using (4.1) we get

$$c_1 e^{\int_{t_0}^t \mathbf{A}(s)ds}\overline{\mathbf{x}}_1 + \cdots + c_k e^{\int_{t_0}^t \mathbf{A}(s)ds}\overline{\mathbf{x}}_k = e^{\int_{t_0}^t \mathbf{A}(s)ds}\Big(c_1\overline{\mathbf{x}}_1 + \cdots + c_k\overline{\mathbf{x}}_k\Big) = \mathbf{0}\,,$$

where we can multiply with with the inverse of $e^{\int_{t_0}^t \mathbf{A}(s)ds}$ (which is $e^{-\int_{t_0}^t \mathbf{A}(s)ds}$), and we get $c_1\overline{\mathbf{x}}_1 + \cdots + c_k\overline{\mathbf{x}}_k = \mathbf{0}$. By the assumptions this implies $c_1 = 0, \ldots, c_k = 0$. $\square$

It is important to notice that the proof shows in fact more: If $c_1\overline{\mathbf{x}}_1 + \cdots + c_k\overline{\mathbf{x}}_k = \mathbf{0}$ for some $c_1, \ldots, c_k$, then with the same constants $c_1, \ldots, c_k$ one has for all times t the equality $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}$. That is, if solutions to a homogeneous linear system of first order differential equations satisfy a linear relationship at the starting time, then they satisfy the same linear relationship for all times.

Another way of stating the theorem is the following.

Theorem 4. *If the solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are linearly independent for one t, then they are linearly independent for all t.*

If the solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are linearly dependent for one t, then they are linearly dependent for all t, and there are constants $c_1, \ldots, c_k$ not all zero such that for all times t we have $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}$.

As any maximal set of independent vectors in $\mathbb{R}^n$ contains exactly n vectors, we can at most choose n linear independent starting vectors $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_n$. This tells us, that we can find exactly n linearly independent solutions to our linear system. In particular, if we have such a collection $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ of solutions belonging to our chosen collection of initial conditions, then any other solution $\mathbf{x}(t)$ to the same system must be expressible as a linear combination of the solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ we already have, that is, there are constants $c_1, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

for all t. The constants can be found by expressing the starting value $\overline{\mathbf{x}}$ of $\mathbf{x}(t)$ as a linear combination of the starting values of our maximal collection of linear independent solutions, that is,

$$\overline{\mathbf{x}} = c_1\overline{\mathbf{x}}_1 + \cdots + c_n\overline{\mathbf{x}}_n \,.$$

This is so important that we restate it as a theorem.

Theorem 5. *If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are a collection of n linearly independent solution for $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then any other solution $\mathbf{x}(t)$ is a linear combination of those n solutions. That is, there are constants $c_1, \ldots, c_n$ such that for all t we have*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) \,.$$

What does this have to do with the linear differential equation of order n that we started out with? As we have seen before, any solution $\mathbf{x}$ to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ gives us a solution y to (1.2) by just taking the first component of $\mathbf{x}$. Conversely, if we have a solution y of (1.2), then $\mathbf{x} = (y, y', \ldots, y^{(n-1)})^T$ is a solution to $x' = \mathbf{A}\mathbf{x}$.

Theorem 6. *Assume $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and let $y_1, \ldots, y_k$ be the corresponding solutions to* (1.2). *Then $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly independent vectors for one (and hence for all) t if and only if $y_1, \ldots, y_k$ are linearly independent as functions.*

Proof. We will actually prove the equivalent statement "$\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly dependent for one (and hence for all) t if and only if $y_1, \ldots, y_k$ are linearly dependent as functions".

Assume first that the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$ are linearly dependent for some t, then there are constants $c_1, \ldots, c_k$ not all zero such that for all t (by Theorem 4) we have $c_1\mathbf{x}_1(t)+\cdots+c_k\mathbf{x}_k(t) = \mathbf{0}$. Looking only at the first coordinates of the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_k(t)$, this clearly implies $c_1y_1(t) + \cdots + c_ky_k(t) = 0$ for all t.

Now assume that $y_1, \ldots, y_k$ are linearly dependent as functions, that is, there are constants $c_1, \ldots, c_k$ not all zero such that for all t we have $c_1y_1(t) + \cdots + c_ky_k(t) = 0$. Differentiate this equation $n-1$ times, this results in

$$\begin{array}{ccccccc}
c_1y_1(t) & + & \cdots & + & c_ky_k(t) & = & 0\,, \\
c_1y_1'(t) & + & \cdots & + & c_ky_k'(t) & = & 0\,, \\
c_1y_1''(t) & + & \cdots & + & c_ky_k''(t) & = & 0\,, \\
 & & & & & \cdots & \\
c_1y_1^{(n-1)}(t) & + & \cdots & + & c_ky_k^{(n-1)}(t) & = & 0\,.
\end{array}$$

Recalling that $\mathbf{x}_1 = (y_1, y_1', \ldots, y_1^{(n-1)})^T, \ldots, \mathbf{x}_k = (y_k, y_k', \ldots, y_k^{(n-1)})^T$ we obtain $c_1\mathbf{x}_1(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}$. □

Corollary 7. *There are exactly n linearly independent solutions of* (1.2).

This corollary follows, because we know that we can have exactly n linearly independent solutions to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. We can now just restate Theorem 5 for y's instead of $\mathbf{x}$'s.

Theorem 8. *If $y_1(t), \ldots, y_n(t)$ are a collection of n linearly independent solutions for the homogeneous equation* (1.2), *then any other solution $y(t)$ is a linear combination of those n solutions. That is, there are constants $c_1, \ldots, c_n$ such that for all t we have*

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t) \,.$$

5. The Wronskian

Definition 9. The Wronskian of a collection $y_1, \ldots, y_n$ of n solutions to the homogeneous ODE (1.2) is defined as the function

$$W(y_1, y_2, \ldots, y_n)(t) = \det \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & y_3'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & y_3^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix}.$$

Here is the theorem that shows why this determinant is important.

Theorem 10. *The Wronskian $W(y_1, y_2, \ldots, y_n)(t)$ is either zero for all t or for no value of t. If the Wronskian is zero, then the solutions $y_1, \ldots, y_n$ are linearly dependent. If the Wronskian is not zero, then the solutions $y_1, \ldots, y_n$ are linearly independent.*

Proof. Once again we will go over to the vector solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ associated to the solutions $y_1, \ldots, y_n$. In fact, by the very definition of the Wronskian we have

$$W(y_1, y_2, \ldots, y_n)(t) = \det(\mathbf{x}_1(t)|\mathbf{x}_2(t)| \cdots |x_n(t))$$

where this last determinant simply denotes the determinant of the matrix one obtains if one writes all the column vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ into a single $n \times n$-matrix. From linear algebra we know that this determinant is zero if and only if the vectors are linearly dependent. From Theorem 4 we know that the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are either linearly dependent for all t, in which case the determinant is zero for all t, or the vectors are linearly independent for all t, in which case the determinant is nonzero for all t. Theorem 6 now tells us that linear dependence of the vectors $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ is exactly the same as linear dependence of the functions $y_1, \ldots, y_n$, and we have the desired result. $\square$

An example. We have seen before that $y_1(t) = \cosh t$ and $y_2(t) = \sinh t$ are solutions for the homogeneous second order linear ODE $y'' - y = 0$. The associated Wronskian is

$$W(\cosh t, \sinh t)(t) = \det \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \cosh^2 t - \sinh^2 t = 1\,.$$

(To get this to be equal to 1 we have used an identity from calculus.) So Theorem 10 tells us that $y_1(t) = \cosh t$ and $y2(t) = \sinh t$ are linearly independent, and as the ODE is second order, we know by Corollary 7 that there there are exactly two linearly independent solutions. Hence $y_1(t) = \cosh t$ and $y2(t) = \sinh t$ form a maximal system of linearly independent solutions. By Theorem 8 any other solution must be a linear combination of the two, that is, any other solution is of the form

$$y(t) = C_1 \cosh t + C_2 \sinh t\,.$$

Of course, this is exactly what we found before.