Self-intersections for the Willmore flow

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Abstract

We prove that the Willmore flow can drive embedded surfaces to self-intersections in finite time.

1 Introduction

In this paper we consider the Willmore flow in three space dimensions. We prove that embedded surfaces can be driven to a self-intersection in finite time. This situation is in strict contrast to the behavior of hypersurfaces under the mean curvature flow, where the maximum principle prevents self-intersections, but very much analogous to the surface diffusion flow.

The Willmore flow is a geometric evolution law in which the normal velocity of a moving surface equals the Laplace-Beltrami of the mean curvature plus some lower order terms. More precisely, we assume in the following that Γ_0 is a closed compact immersed and orientable surface in $\mathbb{R}^3$. Then the Willmore flow is governed by the law

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)} + 2H_{\Gamma(t)}(H^2_{\Gamma(t)} - K_{\Gamma(t)})\,, \qquad \Gamma(0) = \Gamma_0\,. \tag{1.1}$$

Here $\Gamma = \{\Gamma(t)\,;\, t \geq 0\}$ is a family of smooth immersed orientable surfaces, $V(t)$ denotes the velocity of Γ in the normal direction at time t, while $\Delta_{\Gamma(t)}$, $H_{\Gamma(t)}$, and $K_{\Gamma(t)}$ stand for the Laplace-Beltrami operator, the mean curvature, and the Gauß curvature of $\Gamma(t)$, respectively.

The evolution law (1.1) does not depend on the local choice of the orientation. However, if $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$ we always choose the outer normal, so that $V(t)$ is positive if $\Omega(t)$ grows, and so that $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$.

Any equilibrium of (1.1), that is, any closed smooth surface that satisfies the equation

$$\Delta H + 2H(H^2 - K) = 0 \tag{1.2}$$

is called a Willmore surface [18, p. 282]. There has been much interest over the last years in characterizing Willmore surfaces, see for instance [15, 18] and the references cited therein. Willmore surfaces arise as the critical points of the functional

$$W(f) := \int_{f(M)} H^2 \, dS, \tag{1.3}$$

see [18, Section 7.4]. Here, M denotes a smooth closed orientable surface and $f : M \to \mathbb{R}^3$ is a smooth immersion of M into $\mathbb{R}^3$. Associated with this functional is a variational problem: Given a smooth closed orientable surface M_g of genus g determine the infimum $W(M_g)$ of $W(f)$ over all immersions $f : M_g \to \mathbb{R}^3$ and classify all manifolds $f(M_g)$ which minimize W. We refer to [4, 8, 14, 15, 17, 18] and the references therein for more details and interesting results.

The Willmore flow is the L^2-gradient flow for the functional (1.3) on the moving boundary, see for example [7], and also [10] for related work on gradient flows. Thus the Willmore flow has the distinctive property that it evolves surfaces in such a way as to reduce the total quadratic curvature. To be more precise, we show that the flow decreases the total quadratic curvature for any $C^{2+\beta}$ initial surface Γ_0.

Proposition 1. *Let $0<\beta<1$ and let Γ_0 be a closed compact immersed orientable surface that is $C^{2+\beta}$-smooth. Then*

$$\int_{\Gamma(t)} H^2(t) \, d\mu \le \int_{\Gamma_0} H^2(0) \, d\mu, \qquad 0 \le t \le T,$$

where $[0,T]$ denotes the interval of existence guaranteed in the existence theorem of [16], and where $H(t)$ denotes the mean curvature of $\Gamma(t)$.

To the best of our knowledge, the result of Proposition 1 is new (under the given assumptions).

Next we show that the flow can force $\Gamma(t)$ to lose embeddedness in order to decrease the total quadratic curvature.

Theorem 2. *Let $0<\beta<1$ be fixed.*

There exist a closed embedded surface $\Sigma_0 \in C^{2+\beta}$, a constant $T_0 > 0$, numbers $t_0, t_1 \in (0, T_0]$ with $t_0 < t_1$, and a $C^{2+\beta}$–neighborhood U_0 of Σ_0 such that

- **(a)** *the Willmore flow (1.1) has a unique classical solution $\Gamma = \{\Gamma(t); t \in [0, T_0]\}$ for all $\Gamma_0 \in U_0$,*
- **(b)** $\Gamma(t)$ *ceases to be embedded for every $t \in (t_0, t_1)$ and every $\Gamma_0 \in U_0$.*
- **(c)** *each surface $\Gamma(t)$ is of class C^∞ for $t \in (0, T_0]$ and smooth in $t \in (0, T_0)$.*

It should be noted that the neighborhood U_0 of Theorem 2 also contains C^∞-surfaces that will be driven to a self-intersection in finite time. Our approach relies on results and techniques in [6, 12, 16], and we follow closely the original argument in [12].

Lastly we mention that numerical simulations [13] seem to indicate that the Willmore flow can drive immersed surfaces to topological changes in finite time.

2 The mathematical setting

We first introduce some notations. Given an open set $U \subset \mathbb{R}^3$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^3$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$. In the following, we assume that Σ is a smooth compact closed immersed oriented surface in $\mathbb{R}^3$. Let ν be the unit normal field on Σ commensurable with the chosen orientation. Then we can find $a > 0$ and an open covering $\{U_l\,;\, l = 1, \dots, m\}$ of Σ such that

$$X_l : U_l \times (-a, a) \to \mathbb{R}^3\,, \qquad X_l(s, r) := s + r\nu(s)\,,$$

is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$, that is,

$$X_l \in \mathit{Diff}^{\,\infty}(U_l \times (-a, a), \mathcal{R}_l)\,, \qquad 1 \le l \le m\,.$$

This can be done by selecting the open sets $U_l \subset \Sigma$ in such a way that they are embedded in $\mathbb{R}^3$ instead of only immersed, and then taking $a > 0$ sufficiently small so that each of the U_l has a tubular neighborhood of radius a. It follows that $\mathcal{R} := \cup \mathcal{R}_l$ consists of those points in $\mathbb{R}^3$ with distance less than a to Σ. Let $\beta \in (0, 1)$ be fixed. Then we choose numbers α, $\beta_1 \in (0, 1)$ with $\alpha < \beta_1 < \beta$. Let

$$W := \{\rho \in h^{2+\beta_1}(\Sigma)\,;\, \|\rho\|_\infty < a\}\,. \tag{2.1}$$

Given any $\rho \in W$ we obtain a compact oriented immersed manifold Γ_ρ of class $h^{2+\beta_1}$ by means of the following construction:

$$\Gamma_\rho := \bigcup_{l=1}^{m} \mathrm{Im}\left(X_l : U_l \to \mathbb{R}^3\,,\ [s \mapsto X_l(s, \rho(s))]\right). \tag{2.2}$$

Thus Γ_ρ is a graph in normal direction over Σ and ρ is the signed distance between Σ and Γ_ρ. On the other hand, every compact immersed oriented manifold Γ that is a smooth graph over Σ and that is contained in $\mathcal{R}$ can be obtained in this way. For convenience we introduce the mapping

$$\theta_\rho\,:\,\Sigma \to \Gamma_\rho, \qquad \theta_\rho(s) := X_l(s, \rho(s)) \text{ for } s \in U_l\,, \quad \rho \in W\,.$$

It follows that θ_ρ is a well-defined global $(2+\beta_1)$–diffeomorphism from Σ onto Γ_ρ. The Willmore flow (1.1) can now be expressed as an evolution equation for the distance function ρ over the fixed reference manifold Σ,

$$\partial_t \rho = G(\rho)\,, \qquad \rho(0) = \rho_0\,. \tag{2.3}$$

Here $G(\rho) := L_\rho \theta_\rho^*(\Delta_{\Gamma_\rho} H_{\Gamma_\rho} + 2H_{\Gamma_\rho}(H_{\Gamma_\rho}^2 - K_{\Gamma_\rho}))$ for $\rho \in h^{4+\alpha}(\Sigma) \cap W$, while Δ_{Γ_ρ}, H_{Γ_ρ}, and K_{Γ_ρ} are the Laplace-Beltrami operator, the mean curvature, and the Gauss curvature of Γ_ρ, respectively, and $L(\rho)$ is a factor that comes in by calculating the normal velocity in terms of ρ, see [6] for more details. We are now ready to state the following existence result for solutions of (2.3).

Proposition 2.1. *Let $\sigma \in W$ be given.*

- **(a)** *There exist a positive constant $T_0 > 0$ and a neighborhood $W_0 \subset W$ of σ in $h^{2+\beta_1}(\Sigma)$ such that (2.3) has a unique solution*

$$\rho(\cdot, \rho_0) \in C([0, T_0], W) \cap C^\infty((0, T_0) \times \Sigma) \text{ for every } \rho_0 \in W_0.$$

- **(b)** *The map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a smooth local semiflow on W_0.*
- **(c)** *$\rho(\cdot, \rho_0) \in C([0, T_0], h^{4+\alpha}(\Sigma)) \cap C^1([0, T_0], h^\alpha(\Sigma))$ for all $\rho_0 \in h^{4+\alpha}(\Sigma) \cap W_0$.*

Proof. (a) and (b) follow from [16, Proposition 2.2]. Moreover, [16, Lemma 2.1] shows that the mapping $[\rho \mapsto G(\rho)] : h^{4+\alpha}(\Sigma) \cap W \to h^\alpha(\Sigma)$ is smooth and that the derivative is given by $G'(\rho) = P(\rho) + B(\rho)$, where

$$P(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad B(\rho) \in L(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad \rho \in h^{4+\alpha}(\Sigma) \cap W\,.$$

In the following we fix $\rho \in h^{4+\alpha}(\Sigma) \cap W$. [16, Lemma 2.1] also shows that $P(\rho)$ generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. A well-known perturbation result, see [1, Theorem I.1.3.1], then implies $G'(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ also generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. It is known that the little Hölder spaces are stable under the continuous interpolation method [1, 2, 5, 9]. Therefore, the spaces $(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ form a pair of maximal regularity for $G'(\rho)$, see [1, Theorem III.3.4.1] or [2, 5, 9]. Part (c) follows now from maximal regularity results, for instance [2, Theorem 2.7]. □

3 The proof of Proposition 1

We first note that any function in $C^{2+\beta}$ is also in $h^{2+\beta_1}$ for $\beta_1 \in (0, \beta)$. Let Γ_0 be a given surface in $\mathbb{R}^3$ that satisfies the assumptions of Proposition 1. We can find a smooth surface Σ as in Section 2 and a function $\rho_0 \in W$ such that $\Gamma_0 = \Gamma_{\rho_0}$, where Γ_{ρ_0} is defined in (2.2). According to Proposition 2.1(a) there exists a number $T = T(\rho_0) > 0$ such that equation (2.3) has a unique solution $\rho(\cdot, \rho_0)$ with the smoothness properties stated in the proposition. It follows from the construction in Section 2 that the family $\Gamma := \{\Gamma(t)\ ;\ 0 \le t \le T\}$, where $\Gamma(t) := \Gamma_{\rho(t)}$ for $0 \le t \le T$, is the unique classical solution for the Willmore flow (1.1). In particular, we conclude that

$$[t \mapsto \int_{\Gamma(t)} H^2(t)\, d\mu] \in C^\infty((0, T), \mathbb{R}).$$

Given $x \in \Gamma(t)$, let $\{z(\tau, x) \in \mathbb{R}^3\,;\, \tau \in (-\varepsilon, \varepsilon)\}$ be an orthogonal flow line through x, that is, $z(\cdot, x)$ satisfies

$$
\begin{aligned}
&z(\tau, x) \in \Gamma(t+\tau) \text{ for } \tau \in (-\varepsilon, \varepsilon), \\
&\dot{z}(\tau) = (VN)(t+\tau, z(\tau)) \text{ for } \tau \in (-\varepsilon, \varepsilon), \;\; z(0) = x,
\end{aligned}
$$

where $N(t, \cdot)$ denotes the unit normal field on $\Gamma(t)$, and $V(t, \cdot)$ is the normal velocity of $\Gamma(t)$. A proof for the existence of a unique trajectory $\{z(\tau, x) \in \mathbb{R}^3\,;\, \tau \in (-\varepsilon, \varepsilon)\}$ with the above properties can for instance be found in [11, Lemma 2.1]. For further use we introduce the manifold $\mathcal{M} := \bigcup_{t \in (0,T)} \{t\} \times \Gamma(t)$. Given any smooth function u on $\mathcal{M}$ we define

$$
\frac{d}{dt} u(t, x) := \frac{d}{d\tau} u(t+\tau, z(\tau, x))\Big|_{\tau=0}, \quad (t, x) \in \mathcal{M}.
$$

The following differentiation rule is well-known in differential geometry,

$$
\frac{d}{dt} \int_{\Gamma(t)} u(t, x)\, d\mu(x) = \int_{\Gamma(t)} \frac{d}{dt} u(t, x)\, d\mu(x) + 2 \int_{\Gamma(t)} (uHV)(t, x)\, d\mu(x). \tag{3.1}
$$

Let $(t, x) \in \mathcal{M}$ be fixed and let $\{z(\tau, x)\,;\, \tau \in (-\varepsilon, \varepsilon)\}$ be a flow line trough x. Then one can show that

$$
\frac{d}{d\tau} H^2(t+\tau, z(\tau, x))\Big|_{\tau=0} = -H[\Delta_{\Gamma(t)} V + (4H^2 - 2K)V](t, x), \tag{3.2}
$$

see for instance [18, Section 7.4]. If follows from (3.1)–(3.2), from the divergence theorem, and from (1.1) that

$$
\frac{d}{dt} \int_{\Gamma(t)} H^2(t)\, d\mu = -\int_{\Gamma(t)} [\Delta H + 2H(H^2 - K)]V\, d\mu \leq 0. \tag{3.3}
$$

This is true for any $t \in (0, T)$. The mean value theorem now implies that

$$
\int_{\Gamma(t)} H^2(t)\, d\mu - \int_{\Gamma(\tau)} H^2(\tau)\, d\mu \leq 0 \qquad \text{for } \; 0 < \tau \leq t < T.
$$

Taking the limit as $\tau \to 0$ and using that $[\tau \mapsto \int_{\Gamma(\tau)} H^2(\tau)\, d\mu] \in C([0, T], \mathbb{R})$, see Proposition 2.1(b), yields the assertion of Proposition 1. $\square$

4 The proof of Theorem 2

In order to provide a proof of Theorem 2 we now choose Σ to be any smooth compact closed immersed orientable surface in $\mathbb{R}^3$ such that its image contains the flat 2–dimensional disk $U := \{(s, 0) \in \mathbb{R}^2 \times \mathbb{R}\,;\, |s| \leq 1\}$ twice, and with opposite orientations. To be precise, let $i : \Sigma \to \mathbb{R}^3$ be the immersion under consideration, then we ask that

$$i^{-1}(U) = U^+ \cup U^-$$

with $U^+ \cap U^- = \emptyset$ and both U^+ and U^- are flat 2–dimensional disks of radius 1. Additionally we ask that $\Sigma \setminus (U^+ \cup U^-)$ is embedded in $\mathbb{R}^3$. Identifying U^+ for the moment with its image U we ask that the normal on U^+ points upwards, that is, $\nu(\cdot)|_{U^+} = e_3$, the 3rd basis vector of $\mathbb{R}^3$. It follows that $\nu(\cdot)|_{U^-} = -e_3$.

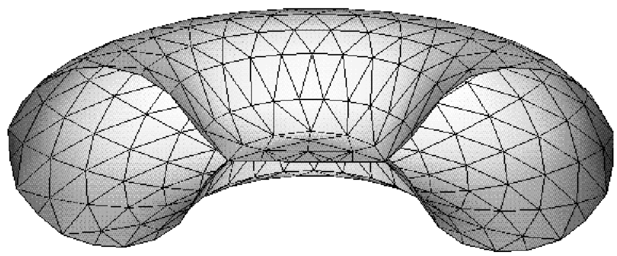

Fig. 1 This is a possible choice of Σ, cut in halves.

Let W be as in (2.1) and let $\sigma \in h^{4+\alpha} \cap W$ locally be radially symmetric with regards to the centers of $U^\pm$. This implies $\partial_j \sigma(0) = 0$ for $j = 1, 2$. Observe that $\theta_\sigma(s) = (s, \pm\sigma(s))$ (these are coordinates in $\mathbb{R}^3$) for $s \in U^\pm$ and that $\theta_\sigma : U^\pm \to \theta_\sigma(U^\pm)$ is an $h^{4+\alpha}$–diffeomorphism. It is straightforward to compute

$$G(\sigma)|_{U^\pm} := L(\sigma)\theta_\sigma^*\big(\Delta_{\Gamma_\sigma} H_{\Gamma_\sigma} + 2H_{\Gamma_\sigma}(H_{\Gamma_\sigma}^2 - K_{\Gamma_\sigma})\big)|_{U^\pm}$$

in local coordinates, yielding

$$\begin{aligned} 2G(\sigma)|_{U^\pm}(0) \quad &= \quad -\Delta^2\sigma(0) + \sum_{j,k=1}^{2} (\partial_j\partial_k\sigma(0))^2 \Delta\sigma(0) \\ &\quad +2 \sum_{j,k,l=1}^{2} \partial_j\partial_k\sigma(0)\partial_j\partial_l\sigma(0)\partial_k\partial_l\sigma(0)\,, \end{aligned}$$

where Δ is the Laplacian in Euclidean coordinates of $\mathbb{R}^2$ (see [6, Section 2] for more details). Because of the radial symmetry of σ we have $H_{\Gamma_\sigma}^2 = K_{\Gamma_\sigma}$ at the center of the disks $U^\pm$, so that lower order term $\theta_\sigma^*(2H_{\Gamma_\sigma}(H_{\Gamma_\sigma}^2 - K_{\Gamma_\sigma}))$ vanishes at the center of $U^\pm$. We will now specify one more property of σ. We choose $r > 0$ small and we require that $\sigma(s) = |s|^4$ for $s \in U_r^\pm = \{s \in U^\pm\,;\, |s| < r\}$. If r is small enough then this is compatible with $\sigma \in h^{4+\alpha}(\Sigma) \cap W$. We conclude that

$$G(\sigma)|_{U^\pm}(0) = -16 < 0\,. \tag{4.1}$$

It follows from Proposition 2.1 that the evolution equation (2.3) with initial value $\rho(0) = \sigma$ has a unique solution

$$\rho(\cdot, \sigma) \in C([0, T_0], h^{4+\alpha}(\Sigma)) \cap C^1([0, T_0], h^\alpha(\Sigma)) . \tag{4.2}$$

Next we consider the restriction $\rho^\pm(t, \sigma)$ on $U^\pm$ of the function $\rho(t, \sigma)$, that is, $\rho^\pm(t, \sigma) := \rho(t, \sigma)|_{U^\pm}$ for $0 \leq t \leq T_0$, and we set $d^\pm(t) := \rho^\pm(t, \sigma)(0)$, to track the position of the center. It follows from (4.2) that $d^\pm \in C^1([0, T_0])$. Moreover, using the local character of G, we conclude that $d^\pm$ satisfies the equation

$$(d^\pm)'(t) = G(\rho(t, \sigma))|_{U^\pm}(0) \quad \text{for} \quad 0 \leq t \leq T_0 , \qquad d^\pm(0) = 0 . \tag{4.3}$$

Equations (4.1)–(4.3) and the mean value theorem yield

$$d^\pm(t) = -Mt + \Big(\int_0^1 \big((d^\pm)'(\tau t) - (d^\pm)'(0) \big) \, d\tau \Big) t , \tag{4.4}$$

where $M := 16$. It follows from (4.4) that there exists a positive constant $\mu > 0$ and an interval $(t_0, t_1) \subset (0, T_0]$ such that $\rho^\pm(t, \sigma)(0) = d^\pm(t) \leq -\mu$ for $t \in (t_0, t_1)$. By Proposition 2.1(b) we can find a function $\sigma_0 \in W_0$ such that $\Sigma_0 := \Gamma_{\sigma_0}$ is embedded and such that $\Gamma(t) := \Gamma_{\rho(t, \sigma_0)}$ is immersed for at least $t \in (t_0, t_1)$. By employing Proposition 2.1(b) once more we conclude there is a neighborhood $W(\sigma_0) \subset W_0$ of σ_0 in $h^{2+\beta_1}(\Sigma)$ such that Γ_{ρ_0} is still embedded, whereas $\Gamma_{\rho(t, \rho_0)}$ is immersed for $t \in (t_0, t_1)$ and all $\rho_0 \in W(\sigma_0)$. We note that $C^{2+\beta}(\Sigma)$ is contained in $h^{2+\beta_1}(\Sigma)$ with continuous injection $j : C^{2+\beta}(\Sigma) \to h^{2+\beta_1}(\Sigma)$. Hence $U_0 := j^{-1}(W(\sigma_0))$ is a $C^{2+\beta}$–neighborhood of σ_0 and Theorem 2 follows by choosing $\Sigma_0 := \Gamma_{\sigma_0}$ and $\Gamma_0 := \Gamma_{\rho_0}$ for $\rho_0 \in U_0$. $\square$

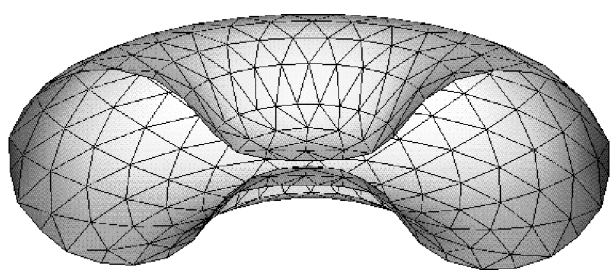

Fig. 2 This is half of Γ_0, a surface that loses embeddedness and becomes immersed. The gap might have to be much smaller than depicted.

Remark 4.2. The following is the essence of the construction: Γ_σ is an immersed surface such that its image contains two opposing fourth-order paraboloids touching only at the vertex. The global symmetry of Γ_σ is irrelevant, we only need the local symmetry at the center. Locally we can compute the initial velocity of Γ_σ, and it is such as to create an overlapping of the fourth-order paraboloids. A continuity argument then guarantees the same behavior for nearby embedded surfaces, which do exist by construction of Γ_σ. We have chosen a fourth-order paraboloid in order to facilitate the computation of $G(\sigma)|_{U^\pm}$. Any other configuration that produces the same sign as in (4.1) will work as well.

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