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To
Bart Braden, Editor
College Mathematics Journal
Department of Mathematics
and Computer Science
Northern Kentucky University
Highland Hights,KY 41076–99

Dear Bart Braden:

I read with interest the article *Why Polynomials Have Roots* in the March issue of THE COLLEGE MATHEMATICS JOURNAL. However, there is a proof where all the ingredients are essentially known to any student having taken a first year of calculus. I learned of the proof when I was a graduate student at the University of Utah from one of my fellow graduate students, Tomasz Serbinowski, who is now at UC Irvine, and who, if my memory serves me right, learned about this proof while being an undergraduate student in Poland.

Yours sincerely,

Uwe F. Mayer

Fundamental Theorem of Algebra. *Every nonconstant polynomial in one variable with complex coefficients has a zero among the complex numbers.*

Proof. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be the polynomial under consideration, $a_n \neq 0$, $n > 0$. Then one factors out z^n which yields $p(z) = \left(\sum_{j=0}^{n-1} a_j \frac{1}{z^{n-j}} + a_n \right) \cdot z^n$, and therefore $\lim_{|z|\to\infty} |p(z)| = \infty$.

Hence for a sufficiently large disk $B_R(0)$ the estimate

$$|p(0)| < |p(z)|, \quad z \in \partial B_R(0)$$

holds. Together with the compactness of $\overline{B_R(0)}$ this implies that the continuous function $z \mapsto |p(z)|$ has an *interior* minimum on this disk, say, at z_0. Two cases arise. Either $p(z_0) = 0$, in which case the proof is complete, or $p(z_0) = b_0 \neq 0$, which will lead to a contradiction. The polynomial can be expanded at z_0,

$$p(z) = b_0 + \sum_{j=k}^{n} b_j (z - z_0)^j, \quad b_k \neq 0.$$

Let $re^{i\phi} = -\frac{b_0}{b_k}$ and define $\omega = \sqrt[k]{r} e^{i\frac{\phi}{k}} = \sqrt[k]{-\frac{b_0}{b_k}}$ and compute

$$\begin{aligned} p(z_0 + \omega\epsilon) &= b_0 + b_k \omega^k \epsilon^k + \sum_{j=k+1}^{n} b_j \omega^j \epsilon^j = b_0 - b_0 \epsilon^k + \sum_{j=k+1}^{n} b_j \omega^j \epsilon^j = b_0(1 - \epsilon^k) + \sum_{j=k+1}^{n} b_j \omega^j \epsilon^j \\ &= p(z_0)(1 - \epsilon^k) + \sum_{j=k+1}^{n} b_j \omega^j \epsilon^j. \end{aligned}$$

As $\lim_{\epsilon\to 0} \sum_{j=k+1}^{n} |b_j \omega^j| \epsilon^{j-k} = 0$ one has

$$|p(z_0 + \omega\epsilon)| \leq |p(z_0)|(1 - \epsilon^k) + \sum_{j=k+1}^{n} |b_j \omega^j| \epsilon^j = |p(z_0)| - \left(|p(z_0)| - \sum_{j=k+1}^{n} |b_j \omega^j| \epsilon^{j-k} \right) \epsilon^k < |p(z_0)|$$

for small enough positive ϵ. This contradicts the minimality of $|p(z_0)|$. □