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Self-intersections for the surface diffusion and the volume preserving mean curvature flow

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Abstract

We prove that the surface diffusion flow and the volume preserving mean curvature flow can drive embedded hypersurfaces to self-intersections.

Key words. Surface diffusion, mean curvature, loss of embeddedness, immersed manifold.

AMS subject classifications. 35R35, 35K55, 58G11, 58F39.

1 Introduction

In this paper we consider two geometric evolution laws: the surface diffusion flow and the volume preserving mean curvature flow. We prove that embedded hypersurfaces can be driven to a self-intersection in finite time. This situation is in strict contrast to the behavior of hypersurfaces under the mean curvature flow, where the maximum principle prevents self-intersections.

The surface diffusion flow is a geometric evolution law in which the normal velocity of a moving hypersurface equals the Laplace-Beltrami of the mean curvature. More precisely we assume in the following that Γ_0 is a closed embedded hypersurface in $\mathbb{R}^n$. Then the surface diffusion flow is governed by the law

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)}\,, \qquad \Gamma(0) = \Gamma_0\,. \tag{1.1}$$

Here $\Gamma = \{\Gamma(t)\,;\, t \geq 0\}$ is a family of smooth immersed orientable hypersurfaces, $V(t)$ denotes the velocity of Γ in the normal direction at time t, while $\Delta_{\Gamma(t)}$ and $H_{\Gamma(t)}$ stand for the Laplace-Beltrami operator and the mean curvature of $\Gamma(t)$, respectively. The volume preserving mean curvature flow is governed by the law

$$V(t) = -(H(t) - \overline{H}(t))\,, \qquad \Gamma(0) = \Gamma_0\,, \tag{1.2}$$

where $\overline{H}(t) := |\Gamma(t)|^{-1} \int_{\Gamma(t)} H(t)\, d\sigma$ denotes the average of the mean curvature.

The evolution laws (1.1) and (1.2) do not depend on the local choice of the orientation. However, if $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$ we always choose the outer normal, so that $V(t)$ is positive if $\Omega(t)$ grows, and so that $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$.

The surface diffusion flow (1.1) was first proposed by Mullins [20] to model the dynamics for the motion of the surface of a crystal when all mass transport is by curvature driven diffusion along the surface. It has also been examined in a more general mathematical and physical context by Davì and Gurtin [10], and by Cahn and Taylor [7]. The surface diffusion flow has recently attracted attention by various researchers, see [3, 4, 6, 8, 11, 12, 13, 16, 19, 21]. We refer to [5, 14, 15, 17] for work related to the volume preserving mean curvature flow.

The surface diffusion flow and the volume preserving mean curvature flow evolve hypersurfaces in such a way that the surface area decreases. Moreover, if Γ is embedded then both flows preserve the volume of the region $\Omega(t)$ enclosed by $\Gamma(t)$, see for instance [12, 14]. The results herein show that both flows can force $\Gamma(t)$ to lose embeddedness in order to decrease surface area.

Theorem 1. *Let $0<\beta<1$. There exist a closed embedded hypersurface $\Sigma_0 \in C^{2+\beta}$, a constant $T_0 > 0$, numbers $t_0, t_1 \in (0, T_0]$ with $t_0 < t_1$, and a $C^{2+\beta}$–neighborhood U_0 of Σ_0 such that the surface diffusion flow (1.1) has a unique classical solution $\Gamma = \{\Gamma(t)\,;\, t \in [0, T_0]\}$ for all $\Gamma_0 \in U_0$, and such that $\Gamma(t)$ ceases to be embedded for every $t \in (t_0, t_1)$ and every $\Gamma_0 \in U_0$. Each hypersurface $\Gamma(t)$ is of class C^∞ for $t \in (0, T_0]$ and smooth in $t \in (0, T_0)$.*

It was conjectured in [11] and later proved in [16] that the surface diffusion flow can drive a dumbbell curve of an appropriate shape to a self-intersection. Theorem 1 extends this result considerably: we can handle nonsymmetric hypersurfaces in any dimension, whereas the method of [16] seems restricted to (symmetric) curves. It should be noted that the neighborhood U_0 of Theorem 1 also contains C^∞-hypersurfaces that will be driven to a self-intersection in finite time. Our approach relies on results and techniques in [12].

Theorem 2. *Let $0<\beta<1$. There exist a closed embedded hypersurface $\Sigma_0 \in C^{1+\beta}$, a constant $T_0 > 0$, numbers $t_0, t_1 \in (0, T_0]$ with $t_0 < t_1$, and a $C^{1+\beta}$–neighborhood U_0 of Σ_0 such that the volume preserving mean curvature flow (1.2) has a unique classical solution $\Gamma = \{\Gamma(t)\,;\, t \in [0, T_0]\}$ for all $\Gamma_0 \in U_0$, and such that $\Gamma(t)$ ceases to be embedded for every $t \in (t_0, t_1)$ and every $\Gamma_0 \in U_0$. Each hypersurface $\Gamma(t)$ is of class C^∞ for $t \in (0, T_0]$ and smooth in $t \in (0, T_0)$.*

To the best of our knowledge, Theorem 2 provides the first rigorous proof for the occurrence of self-intersections for the volume preserving mean curvature flow. In particular, we give a proof for an example proposed by Gage [15] who considered a curve similar to our Fig. 3.

2 The surface diffusion flow

In this section we prove Theorem 1. We first introduce some notations. Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ. It is known that $BUC^t(\Sigma)$ is continuously embedded in $h^s(\Sigma)$ whenever $t > s$. In the following, we assume that Σ is a smooth compact closed immersed oriented hypersurface in $\mathbb{R}^n$. Let ν be the unit normal field on Σ commensurable with the chosen orientation. Then we can find $a > 0$ and an open covering $\{U_l \,;\, l = 1, \dots, m\}$ of Σ such that

$$X_l : U_l \times (-a, a) \to \mathbb{R}^n\,, \qquad X_l(s, r) := s + r\nu(s)$$

is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$, that is,

$$X_l \in \mathit{Diff}^\infty(U_l \times (-a, a), \mathcal{R}_l)\,, \qquad 1 \le l \le m\,.$$

This can be done by selecting the open sets $U_l \subset \Sigma$ in such a way that they are embedded in $\mathbb{R}^n$ instead of only immersed, and then taking $a > 0$ sufficiently small so that each of the U_l has a tubular neighborhood of radius a. It follows that $\mathcal{R} := \cup \mathcal{R}_l$ consists of those points in $\mathbb{R}^n$ with distance less than a to Σ. Let $\beta \in (0, 1)$ be fixed. Then we choose numbers α, $\beta_0 \in (0, 1)$ with $\alpha < \beta_0 < \beta$. Let

$$W := \{\rho \in h^{2+\beta_0}(\Sigma) \,;\, \|\rho\|_\infty < a\}\,. \tag{2.1}$$

Given any $\rho \in W$ we obtain a compact oriented immersed manifold Γ_ρ of class $h^{2+\beta_0}$ by means of the following construction:

$$\Gamma_\rho := \bigcup_{l=1}^{m} \mathrm{Im}\left(X_l : U_l \to \mathbb{R}^n\,,\ [s \mapsto X_l(s, \rho(s))]\right).$$

Thus Γ_ρ is a graph in normal direction over Σ and ρ is the signed distance between Σ and Γ_ρ. On the other hand, every compact immersed oriented manifold Γ that is a smooth graph over Σ and that is contained in $\mathcal{R}$ can be obtained in this way. For convenience we introduce the mapping

$$\theta_\rho \,:\, \Sigma \to \Gamma_\rho, \qquad \theta_\rho(s) := X_l(s, \rho(s)) \text{ for } s \in U_l\,, \quad \rho \in W\,.$$

It follows that θ_ρ is a well-defined global $(2+\beta_0)$–diffeomorphism from Σ onto Γ_ρ. The surface diffusion flow (1.1) can now be expressed as an evolution equation for the distance function ρ over the fixed reference manifold Σ,

$$\partial_t \rho = G(\rho)\,, \qquad \rho(0) = \rho_0\,. \tag{2.2}$$

Here $G(\rho) := L_\rho \theta^*_\rho (\Delta_{\Gamma_\rho} H_{\Gamma_\rho})$ for $\rho \in h^{4+\alpha}(\Sigma) \cap W$, while Δ_{Γ_ρ} and H_{Γ_ρ} are the Laplace-Beltrami operator and the mean curvature of Γ_ρ, respectively, and $L(\rho)$ is a factor that comes in by calculating the normal velocity in terms of ρ, see [12] for more details. We are now ready to state the following existence result for solutions of (2.2).

Proposition 2.1.

(a) *Let $\sigma \in W$ be given. Then there exist a positive constant $T_0 > 0$ and a neighborhood $W_0 \subset W$ of σ in $h^{2+\beta_0}(\Sigma)$ such that (2.2) has a unique solution*

$$\rho(\cdot, \rho_0) \in C([0, T_0], W) \cap C^\infty((0, T_0) \times \Sigma) \text{ for every } \rho_0 \in W_0 .$$

(b) *The map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a smooth local semiflow on W_0.*

(c) $\rho(\cdot, \rho_0) \in C([0, T_0], h^{4+\alpha}(\Sigma)) \cap C^1([0, T_0], h^\alpha(\Sigma))$ *for every* $\rho_0 \in h^{4+\alpha}(\Sigma) \cap W_0$.

Proof. (a) and (b) follow from [12, Theorem 2.2]. Moreover, [12, Lemma 2.1] shows that the mapping $[\rho \mapsto G(\rho)] : h^{4+\alpha}(\Sigma) \cap W \to h^\alpha(\Sigma)$ is smooth and that the derivative is given by $G'(\rho) = P(\rho) + B(\rho)$, where

$$P(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad B(\rho) \in L(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad \rho \in h^{4+\alpha}(\Sigma) \cap W .$$

In the following we fix $\rho \in h^{4+\alpha}(\Sigma) \cap W$. Lemma 2.1 in [12] also shows that $P(\rho)$ generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. A well-known perturbation result, see [1, Theorem I.1.3.1], then implies $G'(\rho) \in L(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ also generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$. It is known (see [1, Vol II], for instance) that the little Hölder spaces are stable under the continuous interpolation method [1, 2, 9, 18]. Therefore, the spaces $(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ form a pair of maximal regularity for $G'(\rho)$, see [1, Theorem III.3.4.1] or [2, 9, 18]. Part (c) follows now from maximal regularity results, for instance [2, Theorem 2.7]. $\square$

In order to provide a proof of Theorem 1 we now choose Σ to be any smooth compact closed immersed orientable hypersurface in $\mathbb{R}^n$ such that its image contains the flat $(n-1)$-dimensional disk $U := \{(s, 0) \in \mathbb{R}^{n-1} \times \mathbb{R} \,;\, |s| \le 1\}$ twice, and with opposite orientations. To be precise, let $i : \Sigma \to \mathbb{R}^n$ be the immersion under consideration, then we ask that

$$i^{-1}(U) = U^+ \cup U^-$$

with $U^+ \cap U^- = \emptyset$ and both U^+ and U^- are flat $(n-1)$-dimensional disks of radius 1. Additionally we ask that $\Sigma \setminus (U^+ \cup U^-)$ is embedded in $\mathbb{R}^n$. Identifying U^+ for the moment with its image U we ask that the normal on U^+ points upwards, that is, $\nu(\cdot)|_{U^+} = e_n$, the n^{th} basis vector of $\mathbb{R}^n$. It follows that $\nu(\cdot)|_{U^-} = -e_n$.

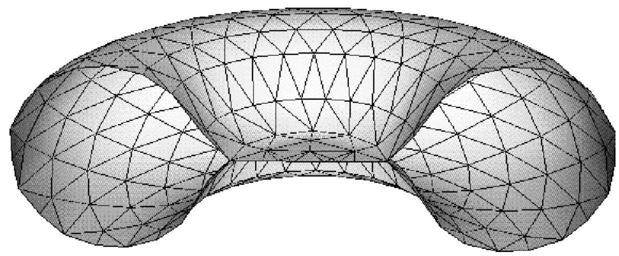

Fig. 1 This is a possible choice of Σ, cut in halves.

Let W be as in (2.1) and let $\sigma \in h^{4+\alpha} \cap W$ have the following local symmetry: $\sigma(-s) = \sigma(s)$ for every $s \in U^{\pm}$. This implies $\partial_j \sigma(0) = 0$ for $1 \leq j \leq n-1$. Observe that $\theta_\sigma(s) = (s, \pm\sigma(s))$ (these are coordinates in $\mathbb{R}^n$) for $s \in U^{\pm}$ and that $\theta_\sigma : U^{\pm} \to \theta_\sigma(U^{\pm})$ is an $h^{4+\alpha}$–diffeomorphism. It is straightforward to compute

$$G(\sigma)|_{U^{\pm}} := L(\rho)\theta_\sigma^*\big(\Delta_{\Gamma_\sigma} H_{\Gamma_\sigma}\big)|_{U^{\pm}}$$

in local coordinates, yielding

$$\begin{aligned}(n-1)G(\sigma)|_{U^{\pm}}(0) \quad &= \quad -\Delta_{n-1}^2\sigma(0) + \sum_{j,k=1}^{n-1}(\partial_j\partial_k\sigma(0))^2\Delta_{n-1}\sigma(0) \\ &\qquad +2\sum_{j,k,l=1}^{n-1}\partial_j\partial_k\sigma(0)\partial_j\partial_l\sigma(0)\partial_k\partial_l\sigma(0)\,,\end{aligned}$$

where Δ_{n-1} is the Laplacian in Euclidean coordinates of $\mathbb{R}^{n-1}$ (see [12, Section 2] for more details). We will now specify one more property of σ. We choose $r > 0$ small and we require that $\sigma(s) = |s|^4$ for $s \in U_r^{\pm} = \{s \in U^{\pm}\,;\, |s| < r\}$; if r is small enough then this is compatible with $\sigma \in h^{4+\alpha}(\Sigma) \cap W$. We conclude that

$$G(\sigma)|_{U^{\pm}}(0) = -24 < 0\,. \tag{2.3}$$

It follows from Proposition 2.1 that the evolution equation (2.2) with initial value $\rho(0) = \sigma$ has a unique solution

$$\rho(\cdot,\sigma) \in C([0,T_0], h^{4+\alpha}(\Sigma)) \cap C^1([0,T_0], h^{\alpha}(\Sigma))\,. \tag{2.4}$$

Next we consider the restriction $\rho^{\pm}(t,\sigma)$ on $U^{\pm}$ of the function $\rho(t,\sigma)$, that is, $\rho^{\pm}(t,\sigma) := \rho(t,\sigma)|_{U^{\pm}}$ for $0 \leq t \leq T_0$, and we set $d^{\pm}(t) := \rho^{\pm}(t,\sigma)(0)$, to track the position of the center. It follows from (2.4) that $d^{\pm} \in C^1([0,T_0])$. Moreover, using the local character of G, we conclude that $d^{\pm}$ satisfies the equation

$$(d^{\pm})'(t) = G(\rho(t,\sigma))|_{U^{\pm}}(0) \quad \text{for} \quad 0 \leq t \leq T_0\,, \qquad d^{\pm}(0) = 0\,. \tag{2.5}$$

Equations (2.3)–(2.5) and the mean value theorem yield

$$d^{\pm}(t) = -Mt + \Big(\int_0^1 \big((d^{\pm})'(\tau t) - (d^{\pm})'(0)\big)\, d\tau \Big) t\,, \tag{2.6}$$

where $M := 24$. It follows from (2.6) that there exists a positive constant $\mu > 0$ and an interval $(t_0, t_1) \subset (0, T_0]$ such that $\rho^{\pm}(t,\sigma)(0) = d^{\pm}(t) \le -\mu$ for $t \in (t_0, t_1)$. By Proposition 2.1(b) we can find a function $\sigma_0 \in W_0$ such that $\Sigma_0 := \Gamma_{\sigma_0}$ is embedded and such that $\Gamma(t) := \Gamma_{\rho(t,\sigma_0)}$ is immersed for at least $t \in (t_0, t_1)$. By employing Proposition 2.1(b) once more we conclude there is a neighborhood $W(\sigma_0) \subset W_0$ of σ_0 in $h^{2+\beta_0}(\Sigma)$ such that Γ_{ρ_0} is still embedded, whereas $\Gamma_{\rho(t,\rho_0)}$ is immersed for $t \in (t_0, t_1)$ and all $\rho_0 \in W(\sigma_0)$. We note that $C^{2+\beta}(\Sigma)$ is contained in $h^{2+\beta_0}(\Sigma)$ with continuous injection $j : C^{2+\beta}(\Sigma) \to h^{2+\beta_0}(\Sigma)$. Hence $U_0 := j^{-1}(W(\sigma_0))$ is an open $C^{2+\beta}$–neighborhood of σ_0 and Theorem 1 follows by choosing $\Sigma_0 := \Gamma_{\sigma_0}$ and $\Gamma_0 := \Gamma_{\rho_0}$ for $\rho_0 \in U_0$. $\Box$

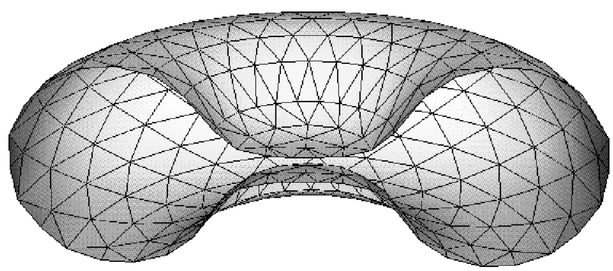

Fig. 2 This is half of Γ_0, a surface that loses embeddedness and becomes immersed. The gap might have to be much smaller than depicted.

Remark 2.2. The following is the essence of the construction: Γ_σ is an immersed hypersurface such that its image contains two opposing fourth-order paraboloids touching only at the vertex. The symmetry of Γ_σ is irrelevant. Locally we can compute the initial velocity of Γ_σ, and it is such as to create an overlapping of the paraboloids. A continuity argument then guarantees the same behavior for nearby embedded hypersurfaces, which do exist by construction of Γ_σ. We have chosen a fourth-order paraboloid in order to facilitate the computation of $G(\sigma)|_{U^{\pm}}$. Any other configuration that produces the same sign as in (2.3) will work as well.

3 The volume preserving mean curvature flow

As in the previous section Σ denotes a smooth compact closed immersed orientable hypersurface in $\mathbb{R}^n$ and we define $W := \{\rho \in h^{1+\beta_0}(\Sigma)\,;\, \|\rho\|_\infty < a\}$ for $a > 0$ appropriate. The volume preserving mean curvature flow (1.2) in $\mathcal{R}$ is equivalent to the following evolution equation for the distance function ρ:

$$\partial_t \rho = G(\rho), \qquad \rho(0) = \rho_0\,, \tag{3.1}$$

where $\rho_0 \in W$ is chosen such that $\Gamma_0 = \Gamma_{\rho_0}$, and where

$$G(\rho) := L(\rho)\Big(\overline{H}_{\Gamma_\rho} - \theta^*_\rho H_{\Gamma_\rho}\Big)\,, \qquad \rho \in h^{2+\alpha}(\Sigma) \cap W\,. \tag{3.2}$$

Here H_{Γ_ρ} is the mean curvature of Γ_ρ and $L(\rho)$ comes from calculating the normal velocity in coordinates of Σ, see [14] for more details. We have the following existence result for solutions of (3.1).

Proposition 3.1.
(a) *Let $\sigma \in W$ be given. Then there exists a positive constant $T_0 > 0$ and a neighborhood $W_0 \subset W$ of σ in $h^{1+\beta_0}(\Sigma)$ such that* (3.1) *has a unique solution*

$$\rho(\cdot, \rho_0) \in C([0,T_0], W) \cap C^\infty((0,T_0) \times \Sigma) \text{ for every } \rho_0 \in W_0\,.$$

(b) *The map $[(t,\rho_0) \mapsto \rho(t,\rho_0)]$ defines a smooth local semiflow on W_0.*
(c) $\rho(\cdot,\rho_0) \in C([0,T_0], h^{2+\alpha}(\Sigma)) \cap C^1([0,T_0], h^\alpha(\Sigma))$ *for every* $\rho_0 \in h^{2+\alpha}(\Sigma) \cap W_0$.

Proof. Part (a) and part (b) follow from the results in [14, Section 2]. To be more precise, in [14] only embedded surfaces are considered. However, a careful analysis of the proof shows that the existence, uniqueness, and semiflow results remain valid for immersed hypersurfaces, provided one defines the mappings X and Φ_ρ of [14, Section 2] by their local analogs as in [12, Section 2]. Part (c) can be established by similar arguments as in the proof of Proposition 2.1. $\square$

We proceed to prove Theorem 2. Our first goal is to construct a suitable reference manifold Σ. We take a positively oriented immersed curve in $\mathbb{R}^2$, such as the one in Fig. 3. The immersed image contains a line segment twice, with opposite orientations, and the image without this line segment is an embedded curve.

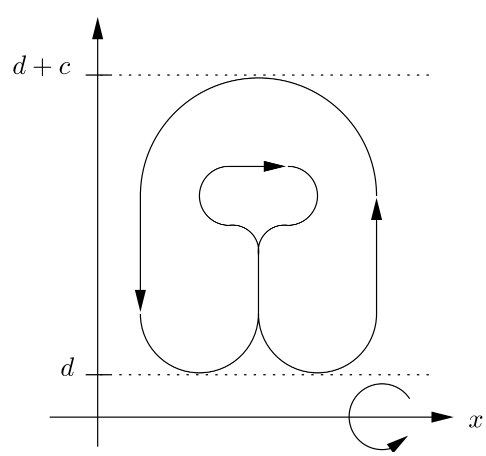

Fig. 3 Rotation of this curve in $\mathbb{R}^n$ yields the hypersurface Σ.

For the two-dimensional case this curve will be Σ, while for the higher dimensional case we rotate the curve to generate a hypersurface, as outlined below. Let $[s \mapsto (x(s), y(s))] : [0, L] \to \mathbb{R}^2$ be a parameterization by arc length of the curve and let $S^{n-2} \subset \mathbb{R}^{n-1}$ be the standard $(n-2)$-dimensional unit sphere. Then we set

$$\Sigma = \{(x(s), y(s)\omega)\ ;\ s \in [0, L], \omega \in S^{n-2}\}\,.$$

Let κ_1 denote the scalar curvature of the curve $[s \mapsto (x(s), y(s))]$. Then a standard computation yields the mean curvature of Σ as

$$H = \frac{1}{n-1}\Big(\kappa_1 - (n-2)\frac{x'}{y}\Big)\,. \tag{3.3}$$

Furthermore, the symmetry of Σ can be used to compute the average of H as

$$\overline{H} = \frac{|S^{n-2}|}{|\Gamma|}\int_0^L H(s)y(s)\,ds\,.$$

Using equation (3.3) and setting $d = \min\{y(s)\}$ one derives

$$(n-1)\int_0^L H(s)y(s)\,ds = d\int_0^L \kappa_1(s)\,ds + \int_0^L \kappa_1(s)(y(s)-d)\,ds - (n-2)\int_0^L x'(s)\,ds\,.$$

The theorem of the turning tangents implies that $\int \kappa_1(s)ds = 2\pi$, and hence it is clear that $\overline{H} > 0$ provided d is large enough, which amounts to shifting the curve far enough away from the axis of rotation. By continuity the average of the mean curvature of Γ_ρ is therefore also positive provided $\rho \in W$ is small enough. Finally, it is clear that by construction Σ contains a flat $(n-1)$-dimensional annulus U twice, with opposite orientations, and that Γ_ρ is in fact embedded in $\mathbb{R}^n$ provided $\rho < 0$ on the annulus.

We let $U^\pm$ be the two components of $i^{-1}(U)$, where $i : \Sigma \to \mathbb{R}^n$ is the immersion under consideration, and U is the flat annulus from above. We now choose $\sigma \in h^{2+\alpha}(\Sigma) \cap W$ with $\sigma \equiv 0$ on $U^\pm$ (in fact we could choose $\sigma \equiv 0$ on Σ so that $\Gamma_\sigma = \Sigma$), then by (3.2)

$$G(\sigma)|_{U^\pm} = \overline{H}_{\Gamma_\sigma} > 0\,. \tag{3.4}$$

Let $\rho(\cdot, \sigma)$ be the unique solution of (3.1) with initial value $\rho(0) = \sigma$ and note that

$$\rho(\cdot, \sigma) \in C([0, T_0], h^{2+\alpha}(\Sigma)) \cap C^1([0, T_0], h^\alpha(\Sigma)) \tag{3.5}$$

due to Proposition 3.1. As in Section 2 we let $\rho^\pm(t, \sigma)$ denote the restriction of $\rho(t, \sigma)$ on $U^\pm$. We set $d^\pm(t) := \rho^\pm(t, \sigma)(s_0, \omega)$ with s_0 any fixed point on the line segment that generated U and any fixed $\omega \in S^{n-2}$. It follows that $d^\pm \in C^1[0, T_0]$ and it is easy to see that $d^\pm$ solves the equation

$$(d^\pm)'(t) = G(\rho(t, \sigma))|_{U^\pm}(s_0, \omega) \quad \text{for} \quad 0 \le t \le T_0\,, \qquad d^\pm(0) = 0\,. \tag{3.6}$$

Equations (3.4)–(3.6) and the mean value theorem show that

$$d^{\pm}(t) = Mt + \Big(\int_0^1 \big((d^{\pm})'(\tau t) - (d^{\pm})'(0)\big)\, d\tau \Big) t$$

with $M := \overline{H}_{\Gamma_\sigma}$. Using Proposition 3.1(b) we can choose a function σ_0 in W_0 such that Γ_{σ_0} is embedded and such that $\Gamma_{\rho(t,\sigma_0)}$ ceases to be embedded on a time interval $(t_0, t_1) \subset (0, T_0]$. The idea is that the time derivative of $d^{\pm}$ is positive, and hence so will be $d^{\pm}$ for some later time even if it was initially negative, see Section 2 for more details. According to Proposition 3.1(b) the same behavior will still prevail for ρ_0 in a small enough $h^{1+\beta_0}(\Sigma)$–neighborhood $W(\sigma_0) \subset W_0$ of σ_0. Theorem 2 now follows by setting $U_0 := j^{-1}(W(\sigma_0))$ with $j := C^{1+\beta}(\Sigma) \to h^{1+\beta_0}(\Sigma)$ the natural injection. $\square$

Remark 3.2. The following is the essence of the construction: Γ_σ is an immersed hypersurface such that its image contains an $(n-1)$-dimensional flat piece twice, with opposite orientation, and Γ_σ has a positive average of its mean curvature. The symmetry of Γ_σ is irrelevant. Locally we can compute the initial velocity of Γ_σ, and it is such as to create an overlapping near the two flat pieces of the surface. We can find embedded hypersurfaces as close to Γ_σ as we want. A continuity argument then guarantees the same behavior for nearby embedded hypersurfaces.

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