

THE DIMENSION OF THE TORELLI GROUP

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September 3, 2007

ABSTRACT. We prove that the cohomological dimension of the Torelli group for a closed connected orientable surface of genus $g \geq 2$ is equal to $3g - 5$. This answers a question of Mess, who proved the lower bound and settled the case of $g = 2$. We also find the cohomological dimension of the Johnson kernel (the subgroup of the Torelli group generated by Dehn twists about separating curves) to be $2g - 3$. For $g \geq 2$, we prove that the top dimensional homology of the Torelli group is infinitely generated. Finally, we give a new proof of the theorem of Mess that gives a precise description of the Torelli group in genus 2. The main tool is a new contractible complex, called the “complex of cycles”, on which the Torelli group acts.

1. INTRODUCTION

Let $S = S_g$ be a closed surface of genus g (unless specified otherwise, we take all surfaces to be connected and orientable). Let $\text{Mod}(S)$ be the *mapping class group* of S , defined as $\pi_0(\text{Homeo}^+(S))$, where $\text{Homeo}^+(S)$ is the group of orientation preserving homeomorphisms of S . The *Torelli group* $\mathcal{I}(S)$ is the kernel of the natural action of $\text{Mod}(S)$ on $H_1(S, \mathbb{Z})$. This action is symplectic—it preserves the algebraic intersection number—and it is a classical fact that $\text{Mod}(S)$ surjects onto $\text{Sp}(2g, \mathbb{Z})$. All of this information is encoded in the following short exact sequence:

$$1 \rightarrow \mathcal{I}(S) \rightarrow \text{Mod}(S) \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1.$$

Cohomological dimension. For a group G , we denote by $\text{cd}(G)$ its *cohomological dimension*, which is the supremum over all n so that there exists a G -module M with $H^n(G, M) \neq 0$.

Theorem A. *For $g \geq 2$, we have $\text{cd}(\mathcal{I}(S_g)) = 3g - 5$.*

Since $\text{Mod}(S_0) = 1$, we have $\mathcal{I}(S_0) = 1$. Also, it is a classical fact that $\text{Mod}(S_1) \cong \text{Sp}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})$, and so $\mathcal{I}(S_1)$ is also trivial.

The lower bound for Theorem A was already given by Mess in 1990 [30].

Theorem 1.1 (Mess). *For $g \geq 2$, we have $\text{cd}(\mathcal{I}(S_g)) \geq 3g - 5$.*

2000 *Mathematics Subject Classification.* Primary: 20F34; Secondary: 57M07.

Key words and phrases. mapping class group, Torelli group, Johnson kernel, cohomological dimension, complex of cycles.

The first and third authors gratefully acknowledge support by the National Science Foundation.

This theorem is proven by constructing a subgroup of $\mathcal{I}(S_g)$ which is a Poincaré duality subgroup of dimension $3g - 5$. We recall Mess's proof in Section 8. Mess explicitly asked if the statement of Theorem A is true [30].

We also study the *Johnson kernel*, denoted $\mathcal{K}(S_g)$, which is the subgroup of $\mathcal{I}(S_g)$ generated by Dehn twists about separating curves. Powell showed that $\mathcal{K}(S_2)$ is equal to $\mathcal{I}(S_2)$ [33]. On the other hand, Johnson proved that the index of $\mathcal{K}(S_g)$ in $\mathcal{I}(S_g)$ is infinite when $g \geq 3$ [23], answering a question of Birman.

The following theorem answers a question of Farb [14, Problem 5.9].

Theorem B. *For $g \geq 2$, we have $\text{cd}(\mathcal{K}(S_g)) = 2g - 3$.*

$\mathcal{K}(S_g)$ has a free abelian subgroup of rank $2g - 3$ when $g \geq 2$ (see Section 8). From the inclusions $\mathbb{Z}^{2g-3} < \mathcal{K}(S_g) < \mathcal{I}(S_g)$, and Theorem A, it immediately follows that $2g - 3 \leq \text{cd}(\mathcal{K}(S_g)) \leq 3g - 5$ (see Fact 5.5 below).

We can think of Theorems A and B as giving the smallest possible dimensions of Eilenberg–MacLane spaces for $\mathcal{I}(S)$ and $\mathcal{K}(S)$ —the so-called *geometric dimension*. Indeed, it is a theorem of Eilenberg–Ganea, Stallings, and Swan that if $\text{cd}(G) \neq 2$ for some group G , then $\text{cd}(G)$ is the same as the geometric dimension [13, 37, 38].

For a group with torsion, such as $\text{Mod}(S)$, the cohomological dimension is infinite. However, if a group G has a torsion free subgroup H of finite index, then we can define $\text{vcd}(G)$, the *virtual cohomological dimension* of G , to be $\text{cd}(H)$. It is a theorem of Serre that $\text{vcd}(G)$ does not depend on the choice of H [35, Théorème 1]. In 1986, Harer proved that $\text{vcd}(\text{Mod}(S_g)) = 4g - 5$ [17], so we see that there is a gap of g between $\text{cd}(\mathcal{I}(S_g))$ and $\text{vcd}(\text{Mod}(S_g))$.

Let $\mathcal{I}^k(S_g)$ be the subgroup of $\text{Mod}(S_g)$ consisting of elements that act trivially on a fixed $2k$ -dimensional symplectic subspace of $H_1(S, \mathbb{Z})$.

Conjecture 1.2. *For $g \geq 2$ and $0 \leq k \leq g$, $\text{vcd}(\mathcal{I}^k(S_g)) = 4g - 5 - k$.*

We discuss this conjecture further in Section 8.

The groups $\text{Mod}(S_g)$, $\mathcal{I}(S_g)$, and $\mathcal{K}(S_g)$ are the first three groups in a series of groups associated to S_g . Let $\pi = \pi_1(S_g)$, and let π^k be the k^{th} term of the lower central series of π , that is, $\pi^0 = \pi$ and $\pi^k = [\pi, \pi^{k-1}]$. The *Johnson filtration* of $\text{Mod}(S)$ is the sequence of groups $\{\mathcal{N}_k(S_g)\}$ given by

$$\mathcal{N}_k(S_g) = \ker\{\text{Mod}(S) \rightarrow \text{Out}(\pi/\pi^k)\}.$$

It follows from definitions that $\mathcal{N}_0(S_g) = \text{Mod}(S_g)$ and $\mathcal{N}_1(S_g) = \mathcal{I}(S_g)$, and it is a deep theorem of Johnson that $\mathcal{N}_2(S_g) = \mathcal{K}(S_g)$ [26].

For $g \geq 2$ and any k , Farb gives the lower bound $g - 1$ for $\text{cd}(\mathcal{N}_k(S_g))$ by constructing a free abelian subgroup of $\mathcal{N}_k(S_g)$ of rank $g - 1$ [14]. From this, and Theorem B, it follows that, for $g \geq 2$ and any $k \geq 2$, we have $g - 1 \leq \text{cd}(\mathcal{N}_k(S_g)) \leq 2g - 3$ (again apply Fact 5.5). The following question was already asked by Farb [14, Problem 5.9].

Question 1.3. *What is $\text{cd}(\mathcal{N}_k(S_g))$ for $g \geq 3$ and $k \geq 3$?*

Infinite generation of top homology. Besides the question of cohomological dimension, one would also like to know in which dimensions the homology is (in)finitely generated. For instance, if the second homology of a group is infinitely generated, then that group is not finitely presented. As of this writing, it is a central open problem to determine whether or not $\mathcal{I}(S_g)$ is finitely presented or not. We have the following theorem.

Theorem C. *For $g \geq 2$, the group $H_{3g-5}(\mathcal{I}(S_g), \mathbb{Z})$ is infinitely generated.*

In Kirby’s problem list, Problem 2.9(B) (attributed to Mess) is to find the largest $k = k_g$ so that $\mathcal{I}(S_g)$ has an Eilenberg–MacLane space with finite k -skeleton [1]. An immediate consequence of Theorem C is that $k_g \leq 3g - 6$. It follows from a theorem of Mess (see below) that $k_3 \leq 2$.

Theorem C also solves a problem of Farb [14, Problem 5.14].

We do not know if the top dimensional homology of $\mathcal{K}(S_g)$ is finitely generated for $g \geq 3$.

Genus 2. It is a theorem of Stallings and Swan that a group of cohomological dimension 1 is free [37, 38]. Thus, Theorem A implies that $\mathcal{I}(S_2) = \mathcal{K}(S_2)$ is a free group. A celebrated theorem of Mess, Theorem D below, gives a much more precise picture of this group.

For the statement, note that each separating simple closed curve in S_2 gives rise to an algebraic splitting of $H_1(S_2, \mathbb{Z})$: the two subspaces are the ones spanned by the curves on the two different sides of the separating curve. Each component of the splitting is a 2-dimensional symplectic subspace of $H_1(S_2, \mathbb{Z})$, and so we call such a splitting a *symplectic splitting* of $H_1(S_2, \mathbb{Z})$. Note that two separating curves give rise to the same symplectic splitting if they differ by an element of $\mathcal{I}(S_2)$.

Theorem D (Mess). *$\mathcal{I}(S_2)$ is an infinitely generated free group, with one Dehn twist generator for each symplectic splitting of $H_1(S_2, \mathbb{Z})$.*

We give a new proof of this theorem in Section 6. While Mess’s original proof is rooted in algebraic geometry, our proof is confined entirely to the realm of geometric group theory.

Strategy. All of our theorems cited above are proven by studying the actions of $\mathcal{I}(S_g)$ and $\mathcal{K}(S_g)$ on a new complex $\mathcal{B}(S_g)$, called the “complex of cycles”, which we construct in Section 2. Vertices of the complex correspond to integral representatives in S_g of some fixed element of $H_1(S_g, \mathbb{Z})$. The key feature is the following fact, proven in Sections 3 and 4.

Theorem E. *The complex $\mathcal{B}(S_g)$ is contractible for $g \geq 2$.*

The basic idea is to use a nerve-cover argument to show that $\mathcal{B}(S_g)$ is homotopy equivalent to Teichmüller space.

In light of Theorem 1.1 (and the discussion after Theorem B), Theorems A and B reduce to the following:

$$\text{cd}(\mathcal{I}(S_g)) \leq 3g - 5 \quad \text{and} \quad \text{cd}(\mathcal{K}(S_g)) \leq 2g - 3.$$

For a group G acting on a contractible cell complex X , Quillen proved that

$$\mathrm{cd}(G) \leq \sup\{\mathrm{cd}(\mathrm{Stab}_G(\sigma)) + \dim(\sigma)\}$$

where σ ranges over the cells of X [35, Proposition 11]. We take X to be $\mathcal{B}(S_g)$, and we take G to be either $\mathcal{I}(S_g)$ or $\mathcal{K}(S_g)$. We then show in Section 5 that the right hand side is bounded above by $3g - 5$ for $\mathcal{I}(S_g)$ and by $2g - 3$ for $\mathcal{K}(S_g)$.

In genus 2, the picture is very explicit—we give a picture of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ in Figure 7. Thus, we are able to deduce Theorem D by a careful analysis of this quotient. For a free group, infinite generation of the group is the same as infinite generation of its first homology. As such, Theorem D gives the genus 2 case of Theorem C.

For the proof of Theorem C we again appeal to the action of $\mathcal{I}(S_g)$ on $\mathcal{B}(S_g)$. It is a general fact that if a group G acts on a contractible cell complex X and satisfies the condition

$$\sup\{\mathrm{cd}(\mathrm{Stab}_G(\sigma)) + \dim(\sigma)\} = d$$

as above, then the d^{th} integral homology of the stabilizer in G of a vertex of X injects into $H_d(G, \mathbb{Z})$ (see Fact 7.2 in Section 7.1). In Section 7, we use an inductive argument—with base case Theorem D—to prove that the $(3g - 5)^{\mathrm{th}}$ homology of the stabilizer in $\mathcal{I}(S_g)$ of a particular kind of vertex of $\mathcal{B}(S_g)$ is infinitely generated.

History. The classical fact that $\mathcal{I}(S_1)$ is trivial was proven by Dehn in the 1920's [12]. In 1983, Johnson proved that $\mathcal{I}(S_g)$ is finitely generated for $g \geq 3$ [25], and in 1986, McCullough–Miller proved that $\mathcal{I}(S_2)$ is not finitely generated [28]. Mess improved on this in 1992 by proving Theorem D [31]. At the same time, Mess showed that $H_3(\mathcal{I}(S_3), \mathbb{Z})$ is not finitely generated (Mess credits the argument to Johnson–Millson) [31]. In 2001, Akita proved that $H_*(\mathcal{I}(S_g), \mathbb{Z})$ is not finitely generated for $g \geq 7$ [2].

Mess proved Theorem 1.1, the lower bound for $\mathrm{cd}(\mathcal{I}(S_g))$, in 1990 [30]. The authors recently proved that $\mathrm{cd}(\mathcal{I}(S_g)) < 4g - 5 = \mathrm{vcd}(\mathrm{Mod}(S_g))$ [3] (Harer's result already implied that $\mathrm{cd}(\mathcal{I}(S_g)) \leq 4g - 5$).

In the case of the Johnson kernel, less is known. As $\mathcal{K}(S_2) = \mathcal{I}(S_2)$, we have the theorem of Mess in this case. Biss–Farb proved in 2006 that $\mathcal{K}(S_g)$ is not finitely generated for $g \geq 3$ [8]. An open question of Morita asks whether or not $H_1(\mathcal{K}(S_g))$ is finitely generated for $g \geq 3$.

In an earlier paper, the authors studied the Torelli subgroup of $\mathrm{Out}(F_n)$, that is, the group \mathcal{T}_n of outer automorphisms of a free group of rank n that act trivially on the first homology of the free group [3]. The main results of that paper—that $\mathrm{cd}(\mathcal{T}_n) = 2n - 4$ and that $H_{2n-4}(\mathcal{T}_n, \mathbb{Z})$ is infinitely generated—are analogous to Theorems A and C of the present paper. While proofs for corresponding theorems about $\mathrm{GL}(n, \mathbb{Z})$, $\mathrm{Out}(F_n)$, and $\mathrm{Mod}(S)$ often run parallel, at least on a philosophical level, we encounter here a situation where this paradigm does not hold. For instance, the construction of the complex of cycles in this paper is predicated on the fact that there can be more than one shortest representative for a first homology class in a hyperbolic surface; in a metric graph there is a exactly one shortest representative for a first homology class. Also, the proof of the upper bound for $\mathrm{cd}(\mathcal{T}_n)$ does not (in any obvious way) give the correct upper bound for $\mathrm{cd}(\mathcal{I}(S))$.

Acknowledgements. We would like to thank Ken Bromberg for many discussions on Teichmüller theory. We are also grateful to Tara Brendle, Benson Farb, Chris Leininger, Justin Malestein, and Andy Putman for helpful comments and conversations.

2. THE COMPLEX OF CYCLES

After introducing some definitions and terminology, we will define the complex of cycles for $S = S_g$.

Cycles and curves. A 1-cycle in S is a finite formal sum

$$\sum k_i c_i$$

where $k_i \in \mathbb{R}$, and each c_i is an oriented simple closed curve in S ; the set $\{c_i : k_i \neq 0\}$ is called the *support*. We say that the 1-cycle is *simple* if the curves of the support are pairwise disjoint, and we say that it is *positive* if each k_i is positive. If we want to emphasize that the k_i are in \mathbb{R} , \mathbb{Q} , or \mathbb{Z} , we can call a cycle *real*, *rational*, or *integral*.

A *basic cycle* is a simple positive 1-cycle $\sum k_i c_i$ with the property that its support forms a linearly independent subset of $H_1(S, \mathbb{R})$. If x is an element of $H_1(S, \mathbb{Z})$, a basic cycle for x (that is, a basic cycle representing x) must be integral.

Let \mathcal{S} be the set of isotopy classes of oriented curves in S . By taking isotopy classes, there is a natural map from the set of 1-cycles in S to $\mathbb{R}^{\mathcal{S}}$. Formally, a point of $\mathbb{R}^{\mathcal{S}}$ is a function from \mathcal{S} to \mathbb{R} , and the image of the 1-cycle $\sum k_i c_i$ in $\mathbb{R}^{\mathcal{S}}$ is the function given by

$$s \mapsto \sum_{c_i \in s} k_i.$$

A *multicurve* in the surface S is a nonempty collection of disjoint simple closed curves in S that are isotopically nontrivial and isotopically distinct.

The construction. The first step is to fix an arbitrary nontrivial element x of $H_1(S, \mathbb{Z})$. Next, let \mathcal{M} be the set of isotopy classes of oriented multicurves M in S with the property that each oriented curve of M appears in the support of some basic cycle for x supported in M . The set \mathcal{M} has a natural partial ordering under inclusion (the orientation is important here). As such, we can think of \mathcal{M} as a category where the morphisms are inclusions.

Given $M \in \mathcal{M}$, let P_M be the polytope in $\mathbb{R}^{\mathcal{S}}$ given by the convex hull of the images of the basic cycles for x supported (with orientation) in M . The collection $\{P_M\}$ is a category under inclusion, which we denote \mathcal{F} . The functor given by

$$M \mapsto P_M$$

is an isomorphism from \mathcal{M} to \mathcal{F} . We define the *complex of cycles* $\mathcal{B}(S)$ as the geometric realization of the colimit of this functor:

$$\mathcal{B}(S) = \left| \operatorname{colim}_{M \in \mathcal{M}} \{P_M\} \right|$$

(the choice of x is suppressed in the notation). We can thus regard \mathcal{F} as the category of cells of $\mathcal{B}(S)$.

Informally, $\mathcal{B}(S)$ is obtained from the disjoint union of the elements of \mathcal{F} by identifying faces that are equal in \mathbb{R}^S .

Since $\mathcal{I}(S)$ acts trivially on $H_1(S, \mathbb{Z})$, it acts on $\mathcal{B}(S)$. In the next two sections, we show that $\mathcal{B}(S)$ is homotopy equivalent to Teichmüller space, and so it is contractible.

Examples of cells. Consider the picture on the left hand side of Figure 1. Say that the homology classes of the curves a , b , and c , are x , y , and $x - y$, respectively. There are exactly two basic cycles for x supported on this multicurve, namely a and $b + c$. Thus, the associated cell of $\mathcal{B}(S_2)$ is an edge.

The three curves in the picture in the middle of Figure 1 are all homologous. Thus, we may assume that they are all in the homology class x . It follows that the basic cycles are simply a , b , and c , and the corresponding cell of $\mathcal{B}(S_4)$ is a 2-simplex.

Finally, we can think of the situation in the right hand side of the figure as two independent copies of the first example. Therefore, we get the product of an edge with itself, or, a square.

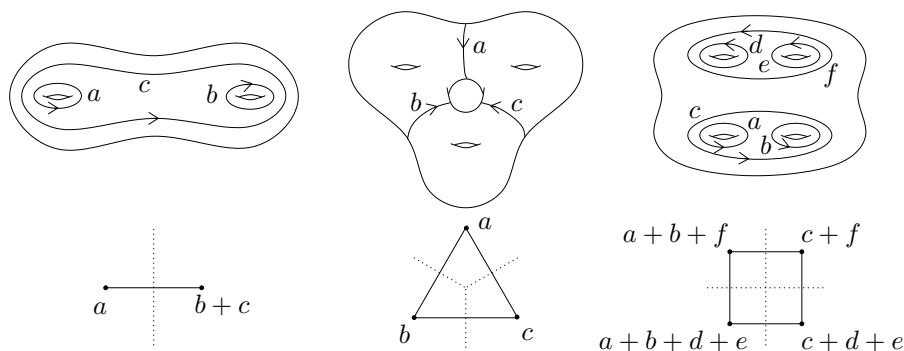


FIGURE 1. Multicurves with their associated cells. The dotted lines indicate the (local) subdivision of $\mathcal{T}(S)$ into chambers (see Section 3.4).

While the examples of Figure 1 are, in a sense, canonical examples, the cells of $\mathcal{B}(S)$ come in a wide variety of shapes. Let a , b , c , d , e , and f be the oriented curves in S_6 shown in the left hand side of Figure 2, and consider, for instance, the homology class $x = [d] + 2[e] + [f]$. There are relations $[a] + [b] + [c] = [e]$ and $[a] + [b] + [d] = [f]$. One can check that the resulting cell is a pentagon, as shown in the right hand side of Figure 2.

Dimension. We now characterize the dimension of a cell of $\mathcal{B}(S)$, and compute the dimension of the complex.

Given $M \in \mathcal{M}$, let $D = D(M)$ be the dimension of the span of M in $H_1(S, \mathbb{R})$, let N be the number of components of $S - M$, and let $B = B(M)$ be the dimension of the associated cell of $\mathcal{B}(S)$.

Lemma 2.1. *We have $B = |M| - D = N - 1$.*

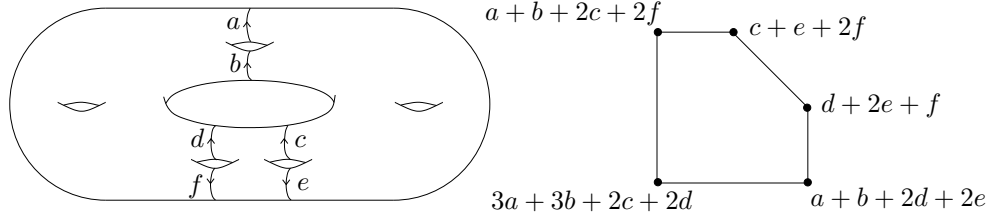


FIGURE 2. Curves that give rise to a pentagonal cell of $\mathcal{B}(S_6)$. The picture is drawn in the “ ef -plane”.

Proof. There is a natural map from the real vector space spanned by the curves of M to $H_1(S, \mathbb{Z})$. The number B is exactly the dimension of this kernel, which is the same as $|M| - D$. The second equality follows from basic algebraic topology. \square

The following proposition is not used anywhere in the paper, but it is a basic fact about the complex of cycles.

Proposition 2.2. *For $g \geq 2$, the dimension of $\mathcal{B}(S_g)$ is $2g - 3$.*

Proof. By Lemma 2.1, the dimension of the cell of $\mathcal{B}(S_g)$ associated to the oriented multicurve M is one less than the number of connected components of $S - M$. Since each component of $S - M$ has negative Euler characteristic, the largest possible number of components is $-\chi(S) = 2g - 2$; this is when M is a pants decomposition. Thus, the dimension of a cell of $\mathcal{B}(S_g)$ is at most $2g - 3$. On the other hand, by choosing a pants decomposition of S that supports x and consists only of nonseparating curves, we obtain a cell of dimension $2g - 3$. \square

3. THE COMPLEX OF CYCLES VIA MINIMIZING CYCLES

The goal of this section is to give an alternate characterization of $\mathcal{B}(S)$ that is related to Teichmüller space. The idea is to decompose Teichmüller space into regions indexed by the shortest representatives of the homology class x chosen in Section 2.

Teichmüller space. For a surface S with $\chi(S) < 0$ and no boundary, we think of Teichmüller space $\mathcal{T}(S)$ as the space of equivalence classes of marked hyperbolic surfaces $\{(X, f)\}$, where X is a complete hyperbolic surface, $f : S \rightarrow X$ is a homeomorphism, and $(X, f) \sim (Y, g)$ if $f \circ g^{-1}$ is isotopic to an isometry; see [20] for an introduction.

There is a natural action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ given by the formula $g \cdot (X, f) = (X, f \circ g^{-1})$.

It is a classical theorem that $\mathcal{T}(S)$ is diffeomorphic to $\mathbb{R}^{-3\chi(S)}$ when $\chi(S) < 0$. In particular, and most important for us, $\mathcal{T}(S)$ is contractible.

Lengths of curves. Given a 1-cycle $c = \sum k_i c_i$ in S , and a point $X = (X, f)$ in $\mathcal{T}(S)$, we can define the *length of c in X* as

$$\ell_X(c) = \sum |k_i| \ell_X(c_i)$$

where $\ell_X(c_i)$ is the length of the geodesic representative of $f(c_i)$ in the hyperbolic metric of X . It is an important and basic fact that the function $\ell.(c) : \mathcal{T}(S) \rightarrow \mathbb{R}_+$ is continuous.

3.1. Minimizing cycles. Let $S = S_g$. As in the definition of $\mathcal{B}(S)$, there is a fixed element x of $H_1(S, \mathbb{Z})$, chosen once and for all. Given $X \in \mathcal{T}(S)$, we denote by $|x|_{\mathbb{R}}$ the infimum of lengths in X of all real cycles in S representing x , and we define a (real) *minimizing cycle* to be a cycle that realizes $|x|_{\mathbb{R}}$.

The goal of this subsection is to prove that minimizing cycles exist (this is not obvious!), and to describe them in terms of multicurves. In the next subsection, we will see that, for various points of $\mathcal{T}(S)$, the associated spaces of minimizing cycles are polytopes that exactly correspond to cells of $\mathcal{B}(S)$.

In what follows, integral minimizing cycles, rational minimizing cycles, $|x|_{\mathbb{Z}}$, and $|x|_{\mathbb{Q}}$ are defined analogously to the real case.

Integral minimizing cycles. We start by proving the necessary facts about minimizing cycles in the integral case (see also [29]), then proceed to the rational case, and finally the real case.

Lemma 3.1. *Integral minimizing cycles exist, that is, $|x|_{\mathbb{Z}}$ is realized by an integral cycle. There are finitely many integral minimizing cycles.*

Proof. Let c be any integral cycle representing x . There are finitely many geodesics in X with length less than or equal to $\ell_X(c)$, and hence there are finitely many candidates for the support of an integral minimizing cycle. Given any choice of support, there are finitely many choices of integral coefficients that result in a cycle of length less than or equal to $\ell_X(c)$. The lemma follows. \square

Lemma 3.2. *An integral minimizing cycle is simple.*

Proof. Consider an integral cycle that is not simple, and assume its self intersections are transverse. By resolving the intersections as in Figure 3, we get a new, shorter, cycle in the class x . \square

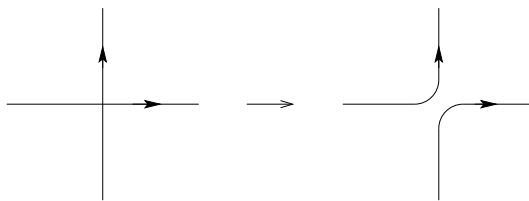


FIGURE 3. The resolution of an intersection.

We now want to prove that any two integral minimizing cycles have no transverse intersections. This will follow from the next lemma, which appears, for instance, in work of Thurston [39, Lemma 1].

Lemma 3.3. *Let y be a nontrivial element of $H_1(S, \mathbb{Z})$, and let k be a positive integer. Any simple integral 1-chain in S representing ky can be written as the sum of k integral representatives for y .*

Proof. Via Poincaré duality, we can think of ky as an element of $H^1(S, \mathbb{Z})$, and as such, we obtain a map $f : S \rightarrow S^1$ with the properties that $f^*([S^1]) = ky$ and, for some regular value $p \in S^1$, $f^{-1}(p)$ is equal to the given simple chain representing ky . Since the image of f is $k\mathbb{Z}$, we can lift f as follows

$$\begin{array}{ccc} & & S^1 \\ & \nearrow \tilde{f} & \downarrow \pi \\ S & \xrightarrow{f} & S^1 \end{array}$$

where π is multiplication by k . Now $\pi^{-1}(p) = \{p_1, \dots, p_k\}$ is a set of k regular values for \tilde{f} , and by construction, $\cup \tilde{f}^{-1}(p_i) = f^{-1}(p)$ is the original representative for ky , and each $\tilde{f}^{-1}(p_i)$ is an integral cycle that represents y . \square

Lemma 3.4. *For $q \in \mathbb{Z}$, we have $|qx|_{\mathbb{Z}} = q|x|_{\mathbb{Z}}$.*

Proof. Let c' be an integral minimizing cycle for qx . By Lemma 3.2, c' is simple, and so by Lemma 3.3, it splits into q representatives of x . Each of these representatives has length at least $|x|$, and so the lemma follows. \square

Lemma 3.5. *Two integral minimizing cycles have no transverse intersections.*

Proof. Suppose that two integral minimizing cycles for x have transverse intersections. By Lemma 3.4, their sum would then be an integral minimizing cycle for $2x$ that is not simple, contradicting Lemma 3.2. \square

Rational minimizing cycles. The next lemma will help bridge the gap between integral minimizing cycles and real minimizing cycles.

Lemma 3.6. *We have $|x|_{\mathbb{Q}} = |x|_{\mathbb{Z}}$; in particular, rational minimizing cycles exist. Also, any rational minimizing cycle is simple, and any two rational minimizing cycles are disjoint.*

Proof. Suppose c is a rational cycle for x with length less than $|x|_{\mathbb{Z}}$. Choose $q \in \mathbb{Z}$ so that qc is integral. Then qc is an integral cycle representing qx with length less than $q|x|_{\mathbb{Z}}$, contradicting Lemma 3.4

Consider any rational minimizing cycle c that is not simple. Choose $q \in \mathbb{Z}$ so that qc is integral. By Lemma 3.4, the cycle qc is then an integral minimizing cycle for qx that is not simple, contradicting Lemma 3.2.

If we have two rational minimizing cycles that intersect, then we can take the “average” of the two cycles to get a rational minimizing cycle for x that is not simple, violating the second conclusion of the lemma. \square

The borrowing lemma. The following lemma is the key technical statement which will allow us to relate the complex of cycles to the concept of a minimizing cycle.

Lemma 3.7 (Borrowing lemma). *Let $c = \sum_{i=1}^N k_i c_i$ be a real positive cycle representing $x \in H_1(S, \mathbb{Z})$. Suppose that (after reindexing) there is a relation*

$$v_1[c_1] + \cdots + v_m[c_m] = v_{m+1}[c_{m+1}] + \cdots + v_{m+n}[c_{m+n}]$$

in $H_1(S, \mathbb{R})$, where each v_i is positive. Let

$$L_1 = \sum_{i=1}^m v_i \ell_X(c_i) \quad \text{and} \quad L_2 = \sum_{i=m+1}^{m+n} v_i \ell_X(c_i)$$

and assume (without loss of generality) that $L_1 \leq L_2$. Let δ be a positive real number less than or equal to $\min\{k_i/v_i\}$, where the minimum is taken over $m+1 \leq i \leq m+n$. The cycle

$$c_\delta = \sum_{i=1}^m (k_i + \delta v_i) c_i + \sum_{i=m+1}^{m+n} (k_i - \delta v_i) c_i + \sum_{i=m+n+1}^N k_i c_i$$

is a representative for x . Furthermore, $\ell_X(c_\delta) = \ell_X(c)$ if and only if $L_1 = L_2$ and $\ell_X(c_\delta) < \ell_X(c)$ if and only if $L_1 < L_2$. Finally, the supports of c and c_δ are equal if $\delta < \min\{k_i/v_i\}$.

The proof of the Borrowing lemma is a simple calculation, which we leave to the reader. As a basic example in genus 2, consider the curves a , b , and c in Figure 4. If the homology classes are x , y , and $x - y$, respectively, and the lengths ℓ_a , ℓ_b , and ℓ_c are close to zero and satisfy $\ell_a = \ell_b + \ell_c$, then the cycle $\epsilon a + (1 - \epsilon)b + (1 - \epsilon)c$ is minimizing for any $\epsilon \in [0, 1]$. This interval's worth of minimizing cycles exactly corresponds to the edge in the left hand side of Figure 1.

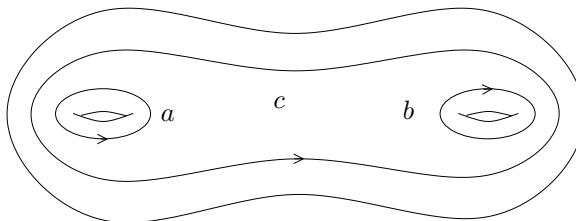


FIGURE 4. Curves in S_2 that give rise to “borrowing”.

Real minimizing cycles. Finally, we have the desired result about real minimizing cycles.

Lemma 3.8. *We have $|x|_{\mathbb{R}} = |x|_{\mathbb{Q}} = |x|_{\mathbb{Z}}$; in particular, real minimizing cycles exist. Also, any real minimizing cycle is simple, and any two real minimizing cycles are disjoint.*

Proof. Let $c = \sum k_i c_i$ be a nonrational real cycle. Since x is rational, it follows that the set of c_i with k_i irrational are linearly dependent. An application of the Borrowing lemma allows us to reduce the number of irrational coefficients without increasing the length of the representative. In the end, we get a rational cycle. Thus, $|x|_{\mathbb{R}} = |x|_{\mathbb{Q}}$, and these are equal to $|x|_{\mathbb{Z}}$ by Lemma 3.6.

If c is a real cycle that is not simple, we can, as above, use the Borrowing lemma to get a rational cycle of the same length that is also not simple. Since rational minimizing cycles are simple (Lemma 3.6), it follows that c is not a real minimizing cycle.

The proof of the last statement follows similarly. \square

3.2. The space of minimizing cycles. The goal of this subsection is to describe the parameter space of minimizing cycles for any one point of $\mathcal{T}(S)$. We will see that each such parameter space corresponds to a cell of $\mathcal{B}(S)$. As above, $x \in H_1(S, \mathbb{Z})$ and $X \in \mathcal{T}(S)$ are fixed.

Minimizing multicurves. A consequence of Lemma 3.8 is that there is a canonical multicurve $M = M_X$ in S , called the *minimizing multicurve*, which is the smallest multicurve with the property that every minimizing cycle for x is supported on M . The minimizing multicurve M is obtained by taking the union of the supports of all minimizing cycles for x . Note that M is a union of geodesics in X .

Lemma 3.9. *There is a unique orientation of the minimizing multicurve $M = M_X$ so that each minimizing cycle for x is a nonnegative linear combination of curves of M .*

Proof. Suppose M is the collection of oriented curves $\{c_1, \dots, c_n\}$, and that $c = \sum k_i c_i$ and $c' = \sum k'_i c_i$ are minimizing cycles for x with $k'_i < 0 < k_i$ for some particular i . In this case, $c + c'$ is a representative for $2x$ with length strictly less than twice that of c . This contradicts Lemma 3.4. \square

As a consequence of the lemma, it makes sense to focus our attention on positive minimizing cycles for x .

Borrowing relations. Let $M = \{c_i\}$ be an oriented multicurve. With the Borrowing lemma in mind, we call any relation amongst the $[c_i]$ in $H_1(S, \mathbb{Z})$ a *borrowing relation*. If we cut S along M , then each resulting connected component gives rise to a borrowing relation, which we call a *subsurface borrowing relation*.

Lemma 3.10. *Let $X \in \mathcal{T}(S)$, and let $M = \{c_i\}$ be an oriented multicurve that is the union of (oriented) supports of positive cycles that represent x . The following are equivalent.*

- (1) *All positive cycles for x supported in M have the same length.*
- (2) *For every subsurface borrowing relation $\sum v_i [c_i] = 0$, the $\ell_X(c_i)$ satisfy $\sum v_i \ell_X(c_i) = 0$.*

In particular, if M is a minimizing multicurve, then both conditions are satisfied.

Proof. By the Borrowing lemma, statement (1) is equivalent to the statement: (3) for every borrowing relation $\sum v_i [c_i] = 0$, the $\ell_X(c_i)$ satisfy $\sum v_i \ell_X(c_i) = 0$. By

basic algebraic topology, any borrowing relation between the $[c_i]$ is a consequence of the subsurface borrowing relations between the $[c_i]$. Hence (3) is equivalent to statement (2). \square

Borrowing cells. We now explain how the minimizing multicurve $M = M_X$ gives a natural way to parameterize the space of minimizing cycles in X . As in Lemma 3.9, we orient M so that all minimizing cycles are positive.

As in Section 2, $D = D(M)$ is the dimension of the span of M in $H_1(S, \mathbb{R})$ and $B = B(M)$ is $|M| - D$. There is a natural map $\mathbb{R}^{|M|} \rightarrow H_1(S, \mathbb{R})$, and the preimage of the subspace spanned by x is a B -dimensional affine subspace V_X of $\mathbb{R}^{|M|}$. By taking lengths of cycles in X , we get a map $L_X : V_X \rightarrow \mathbb{R}_+$.

We can think of the minset P_X of L_X in V_X as the parameter space of minimizing cycles for x in X . As P_X is the minset of a sum of convex functions it is a convex set P_X , which we call a *borrowing cell*. Since each of these (finitely many) functions is proper, it follows that P_X is a compact. Finally, since each of the functions is the absolute value of a linear function, it follows that P_X is a polytope (it is cut out by the zero sets of the individual functions).

Lemma 3.11. *Each borrowing cell P_X is a compact convex polytope in V_X .*

We now prove that the borrowing cell P_X is the convex hull of its basic cycles for x . This gives the key connection between minimizing cycles and the complex $\mathcal{B}(S)$. Recall from Section 2 that P_M is the cell of $\mathcal{B}(S)$ associated to a multicurve M .

Lemma 3.12. *There is a bijective correspondence between the set of basic cycles for x that are supported in $M = M_X$ and the set of vertices of P_X . In particular, $P_X = P_{M_X}$.*

Proof. We use the following characterization: a point of a polytope in Euclidean space is a vertex if and only if every open line segment through that point leaves the polytope.

First, let p be a point of P_X corresponding to a basic minimizing cycle. If we move away from p in any direction of V_X , some coordinate must change from zero to nonzero, and it follows that, in any open line segment of V_X through p , there must be a point with at least one negative coordinate. By Lemma 3.9, such points are not contained in P_X , and so p is a vertex.

Next, let p be a point of P_X that corresponds to a minimizing cycle that is not basic. In this case, there is an open line segment of V_X through p so that all points on the line segment have positive coordinates. Since M is a minimizing cycle, it follows from the last sentence of Lemma 3.10 that statement (1) of the same lemma holds, and we see that the line segment is contained in P_X . Thus, p is not a vertex. \square

The reader is invited to revisit the examples of Section 2, and reinterpret them as borrowing cells in the context of minimizing cycles.

Recall from Section 2 that \mathcal{M} is the category of oriented multicurves that are made up of basic cycles for x , and \mathcal{F} is the category of cells of $\mathcal{B}(S)$. The set of borrowing cells also forms a category under inclusion, if we view each borrowing cell as a subset of $\mathbb{R}^{\mathcal{S}}$.

Lemma 3.13. *The following categories are isomorphic: \mathcal{M} , \mathcal{F} , and the category of borrowing cells.*

Proof. The equivalence of \mathcal{M} and \mathcal{F} was already stated in Section 2. Also, by Lemma 3.12, the functor $P_X \mapsto M_X$ is an isomorphism from the category of borrowing cells to its image in the category \mathcal{M} . It remains to show that this functor is surjective.

In other words, we need to produce, for a given $M \in \mathcal{M}$, a marked hyperbolic surface $X \in \mathcal{T}(S)$ with $M_X = M$. This is done by choosing the lengths of the curves of M to be much smaller than all other curves in X , and inductively choosing the lengths of the curves of M to satisfy the subsurface borrowing relations, as in Lemma 3.10. \square

3.3. Chambers. To each borrowing cell (equivalently, to each element of \mathcal{M}), we now associate a subset of $\mathcal{T}(S)$, which we call a “chamber”. We will show that these chambers are contractible (Proposition 3.14), and that the cover of $\mathcal{T}(S)$ by chambers is, in a sense, dual to the cover of $\mathcal{B}(S)$ by cells. Since $\mathcal{T}(S)$ is contractible, it will follow that $\mathcal{B}(S)$ is contractible.

Let $M \in \mathcal{M}$. We define the *closed chamber* associated to M as

$$Y_M = \{X \in \mathcal{T}(S) : M \subseteq M_X\}.$$

The following proposition is proved in Section 4, and is the key technical part of the paper.

Proposition 3.14. *Each closed chamber Y_M is contractible.*

The next lemma is used in the proof of Proposition 3.14. See the appendix of [9] for an introduction to manifolds with corners. For the statement, we define the *open chamber* associated to the oriented multicurve M as follows:

$$Y_M^o = \{X \in \mathcal{T}(S) : M = M_X\}.$$

The proof of the lemma uses the Fenchel–Nielsen coordinates for $\mathcal{T}(S)$ (see [20, Section 3.2]), and the following basic fact from Teichmüller theory.

Fact 3.15. *Let $X_0 \in \mathcal{T}(S)$, and $D > 0$. There is a neighborhood U of X_0 so that the set $\{c \in \mathcal{S} : \ell_X(c) < D \text{ for some } X \in U\}$ is finite.*

Lemma 3.16. *Each closed chamber Y_M is a manifold with corners. Its interior is the open chamber Y_M^o .*

Proof. Let $X \in Y_M$. Choose a pants decomposition for S containing the curves of M , and consider the resulting Fenchel–Nielsen coordinates on $\mathcal{T}(S)$. By Fact 3.15, there is a finite set of linear inequalities among the length coordinates, so that, in a neighborhood U of X , a point is in Y_M if and only if those inequalities are satisfied. Since the solution set of a system of linear inequalities in \mathbb{R}^n is a manifold with corners, and Fenchel–Nielsen coordinates form a smooth chart for $\mathcal{T}(S)$, it follows that Y_M is a manifold with corners.

In the previous paragraph, the locus of points in \mathbb{R}^n where all of the inequalities are strict is exactly where M is the minimizing multicurve; but this is the definition of Y_M^o . \square

We now relate the partial orderings on multicurves and closed chambers.

Lemma 3.17. $Y_M \supseteq Y_{M'}$ if and only if $M \subseteq M'$.

Proof. By the definition of closed chambers, the statement $Y_M \supseteq Y_{M'}$ is equivalent to the statement that, for any $X \in \mathcal{T}(S)$, if $M_X \supseteq M'$, then $M_X \supseteq M$. The latter is equivalent to the inclusion $M \subseteq M'$. \square

Given two multicurves M and M' with no transverse intersections, the multicurve $M \cup M'$ is defined by taking the union of the two multicurves and replacing each pair of isotopic curves with a single curve.

Corollary 3.18. *The intersection of two closed chambers Y_M and $Y_{M'}$ is nonempty if and only if the multicurve $M \cup M'$ is defined and has an orientation compatible with those of M and M' . In this case, $Y_M \cap Y_{M'} = Y_{M \cup M'}$.*

From the above corollary we can deduce that the category of closed chambers (under inclusion), and the opposite of the category \mathcal{M} are isomorphic. We will not use this fact directly, but the contravariance belies the main difficulty in the proof of Theorem E in Section 3.4.

Corollary 3.19. *The nerve of the cover of $\mathcal{T}(S)$ by the closed chambers $\{Y_M\}$ is finite dimensional.*

Proof. Suppose $\{Y_{M_i}\}$ is a collection of closed chambers with nontrivial intersection. By Corollary 3.18, this intersection is a closed chamber Y_M , and each M_i is a submulticurve of M . Now, M consists of at most $3g-3$ curves, and so the set $\{Y_{M_i}\}$ contains at most 2^{3g-3} elements. This gives an upper bound for the dimension of the nerve. \square

3.4. The complex of cycles is contractible. In this section, we prove Theorem E by making a comparison between the cover of $\mathcal{B}(S)$ by cells and the cover of $\mathcal{T}(S)$ by chambers. Roughly, since both chambers and cells of $\mathcal{B}(S)$ are contractible, and since the gluing patterns are the same (up to contravariance), it will follow that $\mathcal{B}(S)$ is homotopy equivalent to $\mathcal{T}(S)$, and so $\mathcal{B}(S)$ is contractible.

Proof of Theorem E. Let $\mathcal{B}'(S)$ denote the barycentric subdivision of $\mathcal{B}(S)$; it suffices to show that $\mathcal{B}'(S)$ is contractible. For a vertex of $\mathcal{B}(S)$ corresponding to a multicurve M , denote by C_M the star of the corresponding vertex in $\mathcal{B}'(S)$.

The proof is in three steps.

- (1) The geometric realization of the nerve of the cover of $\mathcal{T}(S)$ by closed chambers is homotopy equivalent to $\mathcal{T}(S)$.
- (2) The geometric realization of the nerve of the cover of $\mathcal{B}'(S)$ by the $\{C_M\}$ is homotopy equivalent to $\mathcal{B}'(S)$.
- (3) The two nerves are the same.

Step 1. It is a special case of a result of Borel–Serre that if a topological space is covered by a set of closed, contractible manifolds with corners, and the dimension of the nerve of this cover is finite, then the nerve is homotopy equivalent to the original space [9, Theorems 8.2.1, 8.3.1]. Thus, it suffices to apply Proposition 3.14, Lemma 3.16, and Corollary 3.19.

Step 2. We apply the same theorem of Borel–Serre. In general, the closed star of a vertex of a simplicial complex is contractible ($\mathcal{B}(S)$ is not simplicial, but its barycentric subdivision is). In the category of simplicial complexes, the phrase “closed manifold with corners” used in Step 1 can be replaced with “subcomplex” (both types of spaces are examples of absolute neighborhood retracts). That the dimension of the nerve is finite follows from Step 3 (which is independent of the other steps). Thus, their theorem applies.

Step 3. A collection $\{C_{M_i}\}$ has nonempty intersection if and only if the vertices M_i are vertices of a common cell of $\mathcal{B}(S)$. By Lemma 3.13, this is equivalent to the condition that the M_i are vertices of some common borrowing cell, that is, if there is a hyperbolic structure X with $M_i \subseteq M_X$ for all i . But this is the same as the condition that the intersection of the Y_{M_i} is nonempty (Corollary 3.18).

Combining Steps 1, 2, and 3, plus the fact that $\mathcal{T}(S)$ contractible, we see that $\mathcal{B}'(S)$, and hence $\mathcal{B}(S)$, is contractible. \square

4. CHAMBERS ARE CONTRACTIBLE

In this section we prove Proposition 3.14, which says that each closed chamber Y_M is contractible. We take x to be the element of $H_1(S, \mathbb{Z})$ chosen in Section 2. As in Section 3, Y_M is the closed chamber associated to some multicurve M whose isotopy class is in \mathcal{M} .

4.1. The thick part of Teichmüller space is contractible. The goal is to reduce Proposition 3.14 to the following theorem of Ivanov [21, Theorem 3]. The ϵ -*thick part* of Teichmüller space $\mathcal{T}_\epsilon(S)$ is the subset consisting of hyperbolic surfaces where every nontrivial closed geodesic that is not a boundary component has length at least ϵ (in the case of a surface with boundary, we define Teichmüller space as the space of marked hyperbolic surfaces where each boundary is a geodesic of fixed length).

Also, we make use of the Margulis constant $\epsilon_{\mathbb{H}^2}$ for the hyperbolic plane. This number has the property that, in any hyperbolic surface, two nonisotopic curves of length less than $\epsilon_{\mathbb{H}^2}$ are disjoint.

Theorem 4.1. *Let S be any compact surface and let $\epsilon \in (0, \epsilon_{\mathbb{H}^2})$. The space $\mathcal{T}_\epsilon(S)$ is contractible.*

For any $\epsilon \in (0, \epsilon_{\mathbb{H}^2})$ and $p \in \mathbb{R}_+^{|M|}$, we define the subset $\mathcal{A}_{\epsilon,p}$ of $\mathcal{T}(S)$ to be the set of points with the following properties.

- (1) Each curve in S that is disjoint from the curves of M and is not isotopic to a curve of M has length at least ϵ .
- (2) The lengths of the curves of M are given by p .

Lemma 4.2. *For any $\epsilon < \epsilon_{\mathbb{H}^2}$, there is a choice of p so that $\mathcal{A}_{\epsilon,p}$ is a subset of the open chamber Y_M^o .*

Proof. Let $\epsilon < \epsilon_{\mathbb{H}^2}$ be given. We choose $p \in \mathbb{R}_+^{|M|}$ so that every positive cycle for x supported (with orientation) in M has the same length less than ϵ (take any point in Y_M^o , and shrink the curves of M by the same factor until this condition holds).

In $\mathcal{A}_{\epsilon,p}$, a basic minimizing cycle for x cannot have support on any curve which is disjoint from (and not isotopic to) M , since all such curves have length at least ϵ (by the definition of $\mathcal{A}_{\epsilon,p}$). Also, no geodesic intersecting the geodesic representative of M transversely can be in the support of a basic minimizing cycle, since all such curves have length greater than ϵ by the hypothesis on ϵ . Thus, $\mathcal{A}_{\epsilon,p} \subset Y_M^o$. \square

Lemma 4.3. *For any $\epsilon < \epsilon_{\mathbb{H}^2}$ and any p , the space $\mathcal{A}_{\epsilon,p}$ is contractible.*

Proof. Take a pants decomposition P of S containing M , and consider the associated Fenchel–Nielsen coordinates.

Let $\{S_i\}$ be the set of connected components of $S - M$. There is a natural map

$$\mathcal{A}_{\epsilon,p} \rightarrow \mathcal{T}_{\epsilon}(S_i)$$

for each i , since each S_i has an induced marking and an induced hyperbolic structure (we make sure to cut along the geodesic representative of M).

We also consider the $|M|$ maps

$$\mathcal{A}_{\epsilon,p} \rightarrow \mathbb{R}$$

obtained by taking the twist parameters at the curves of M .

Combining these maps, we get a bijective map

$$\mathcal{A}_{\epsilon,p} \rightarrow \prod \mathcal{T}_{\epsilon}(S_i) \times \mathbb{R}^{|M|}.$$

This map is a homeomorphism—there is a continuous inverse, obtained by gluing the pieces back together according to the prescribed twist parameters. Thus, applying Theorem 4.1, we see that $\mathcal{A}_{\epsilon,p}$ is contractible. \square

4.2. Chamber flow. The goal of this section is to prove the following technical statement, which, combined with Lemmas 4.2 and 4.3, completes the proof that closed chambers are contractible (Proposition 3.14).

Lemma 4.4. *Given any compact subset K of the closed chamber Y_M , there is a choice of $\mathcal{A}_{\epsilon,p} \subset Y_M^o$, and a deformation of a subset of Y_M which takes K into $\mathcal{A}_{\epsilon,p}$.*

For the proof, we need two lemmas from Teichmüller theory. For a pants decomposition P of a surface $S = S_g$, let FN_P be the map $\mathbb{R}^{6g-6} \rightarrow \mathcal{T}(S)$ corresponding to the Fenchel–Nielsen coordinates adapted to P , and let FN_P^* be the differential. In the statement of the lemma, we endow \mathbb{R}^{6g-6} with the Euclidean metric, and $\mathcal{T}(S)$ with the Teichmüller metric.

Lemma 4.5. *Let $S = S_g$ and $D_{\min}, D_{\max} \in \mathbb{R}_+$. There is a constant $C = C(S, D_{\min}, D_{\max})$ so that if P is any pants decomposition of S , and*

$$W = \{X \in \mathcal{T}(S) : D_{\min} \leq \ell_X(c) \leq D_{\max} \text{ for all } c \in P\},$$

then FN_P^ is C -Lipschitz on $FN_P^{-1}(W)$.*

Proof. Let P be given. The subset $FN_P^{-1}(W)$ of \mathbb{R}^{6g-6} is the infinite “strip”

$$[D_{\min}, D_{\max}]^{3g-3} \times \mathbb{R}^{3g-3},$$

that is, all length parameters are in $[D_{\min}, D_{\max}]$, and twist parameters are unconstrained. The free abelian group $G(P)$ generated by Dehn twists in the curves of

P acts on W , since $G(P)$ preserves the lengths of the curves of P . Moreover, this action is by isometries (in general, $\text{Mod}(S)$ acts on $\mathcal{T}(S)$ by isometries).

In the Fenchel–Nielsen coordinates, the Dehn twist about a curve of P is translation by 2π in the corresponding twist coordinate. Therefore, the quotient of $FN_P^{-1}(W)$ by $G(P)$ is

$$[D_{\min}, D_{\max}]^{3g-3} \times (S^1)^{3g-3}.$$

Again, this action is by isometries, so the stretch factor of FN_P^* is determined on this quotient. As this quotient is compact, and FN_P is smooth, it follows that FN_P^* is Lipschitz on $FN_P^{-1}(W)$.

To get the Lipschitz constant C which only depends on the surface S and the constants D_{\min} and D_{\max} (and not on the pants decomposition P), we note that there are only finitely many topological types of pants decompositions on S . Since $\text{Mod}(S)$ acts by isometries on $\mathcal{T}(S)$, and since $\text{Mod}(S)$ preserves the infinite strips $[D_{\min}, D_{\max}]^{3g-3} \times \mathbb{R}^{3g-3}$, the Lipschitz constant is the same for two pants decompositions which are topologically equivalent. Therefore, the constant $C = C(S, D_{\min}, D_{\max})$ is the maximum of finitely many Lipschitz constants obtained as above. \square

The next lemma is the technical statement which implies the existence of the Bers constant—the universal constant L_g with the property that every hyperbolic surface of genus g has a pants decomposition of total length at most L_g (cf. the inductive step of [11, Theorem 5.2.3]).

Lemma 4.6. *Let $g \geq 2$ and $L > 0$. There is a constant $D = D(g, L)$ so that, if M is a geodesic multicurve of $X \in \mathcal{T}(S_g)$ where each curve of M has length at most L , then there is a pants decomposition P containing M so that each curve of P has length less than $D(g, L)$.*

We now construct the deformation of K into $\mathcal{A}_{\epsilon, p}$. The argument is modelled on Ivanov’s proof of Theorem 4.1.

Proof of Lemma 4.4. We will define a smooth vector field \mathcal{V} on an open subset of Y_M , and the resulting flow will give the required deformation. The basic idea is to use Fenchel–Nielsen coordinates to define a vector field that, at a given point, points in a direction where the following conditions hold.

- The curves of M are getting shorter, and their lengths are approaching the point $p \in \mathbb{R}_+^{|M|}$.
- Any curve that is not in M , is disjoint from M , and has length ϵ , stays the same length; in particular, no curve outside of M gets shorter than ϵ (we will choose $\epsilon > 0$ so that all curves begin with length greater than ϵ).
- Any minimizing cycle for x not supported in M has at least one curve that stays the same length (in particular, by the first condition, minimizing cycles not supported in M immediately become nonminimizing).

In short, vectors should point towards the interior of Y_M (third condition), and in particular towards $\mathcal{A}_{\epsilon, p}$ (first two conditions). To accomplish this, there are three steps.

Step 1: Make a choice of $\mathcal{A}_{\epsilon, p}$.

Step 2: Define the vector field \mathcal{V} locally.

Step 3: Check that \mathcal{V} has the desired properties.

Step 1. Picking an $\mathcal{A}_{\epsilon,p}$ involves choosing ϵ and p . Let L_K be the length of the longest minimizing cycle for x in K . We choose $\epsilon < \epsilon_{\mathbb{H}^2}$ small enough so that the following two conditions are satisfied.

- Every nontrivial geodesic in each surface of K has length at least ϵ .
- If c is a geodesic that crosses another geodesic of length at most ϵ , then c has length greater than L_K .

To see that both conditions can be satisfied for ϵ small enough, we use the fact that the length of a particular curve in S is a continuous function on $\mathcal{T}(S)$, and for the second condition, we apply the collar lemma, which states that, in any hyperbolic surface, as the length of a geodesic d tends to zero, the length of the shortest geodesic intersecting d transversely tends to infinity.

The point of the first condition on ϵ is that, in order to prevent curves from getting shorter than ϵ , they must be longer than ϵ to begin with. The second condition implies that, in K , and at any point of Y_M where the length of a minimizing cycle is at most L_K , curves of length ϵ (or shorter) have trivial geometric intersection with the curves of M .

In the proof of Lemma 4.2, the choice of p only depended on the choice of ϵ and on M . We choose p in the same way here, using the ϵ defined here. As a result, we have $\mathcal{A}_{\epsilon,p} \subset Y_M^o$.

Step 2. We will define the vector field \mathcal{V} on open sets indexed by pants decompositions. First, let D_{\max} be the constant $D(g, L_K)$ from Lemma 4.6, where L_K is the constant from Step 1. Note that $D_{\max} > L_K$. Choose D_{\min} to be smaller than the smallest coordinate of p in $\mathbb{R}^{|M|}$. Note that $D_{\min} < |p| < \epsilon$, and, in particular, D_{\min} is smaller than the length of any curve in any surface of K .

Let \mathcal{P} be the set of pants decompositions of S that contain M . For $P \in \mathcal{P}$, let U_P be the open subset of Y_M given by

$$U_P = \{X \in Y_M \quad : \quad \begin{array}{l} \text{if } X \in Y_{M'} \text{ then } M' \subseteq P, \\ \text{if } \ell_X(c) \leq \epsilon \text{ then } c \in P, \\ D_{\min} < \ell_X(c) < D_{\max} \text{ for all } c \in P \}. \end{array}$$

We remark that the last two conditions in the definition of U_P are not contradictory since D_{\max} is automatically at least ϵ . Also, the second condition is indeed an open condition: restate it as $c \notin P \Rightarrow \ell_X(c) > \epsilon$ and apply Fact 3.15.

Let $U = \cup_{P \in \mathcal{P}} U_P$. The compact set K is contained in U ; this follows from the choices of D_{\max} and D_{\min} , Lemma 4.6, and the fact that there are no curves of length less than or equal to ϵ in surfaces of K .

For a given pants decomposition P , we now define a vector field \mathcal{V}_P on U_P . Let $d : \mathbb{R}^{|M|} \rightarrow \mathbb{R}_{\geq 0}$ be the square of the Euclidean distance from the point $p \in \mathbb{R}^{|M|}$. Write \mathbb{R}^{6g-6} as $\mathbb{R}^{|M|} \times \mathbb{R}^{6g-6-|M|}$, and let \mathcal{V}'_P be the vector field on $FN_P^{-1}(U_P) \subset \mathbb{R}^{6g-6}$ given by $(-\nabla d, 0)$. We then define \mathcal{V}_P on U_P via $\mathcal{V}_P = FN_P^*(\mathcal{V}'_P)$.

Consider the map

$$\Phi_P : U_P \rightarrow \mathbb{R}^{|M|}$$

obtained by taking the lengths of the curves of M . By construction, $\Phi_P^*(\mathcal{V}_P) = -\nabla d$.

There is a partition of unity subordinate to the cover of U by the U_P which we use to combine the local vector fields \mathcal{V}_P into a global vector field \mathcal{V} defined on all of U . Since \mathcal{V} is obtained by averaging the \mathcal{V}_P , we still have $\Phi^*(\mathcal{V}) = -\nabla d$, where $\Phi : U \rightarrow \mathbb{R}^{|M|}$ is the map which records the lengths of the curves of M . In other words, even after using the partition of unity, we still know what happens to the lengths of the curves of M along flow lines of \mathcal{V} .

Step 3. We will prove the following statements.

- (1) All flow lines starting in K stay in Y_M .
- (2) All flow lines starting in K stay in U .
- (3) There is a universal constant C so that $\|v\| \leq C\|\Phi^*(v)\|$ for all v .
- (4) All flow lines starting in K converge as time goes to infinity.

The first three statements imply that the flow induced by \mathcal{V} is defined for all positive time. From the fourth statement (plus the definition of \mathcal{V}), we see that, at infinity, all points of K end up in $\mathcal{A}_{\epsilon,p}$. The resulting map $K \rightarrow \mathcal{A}_{\epsilon,p}$ is automatically continuous. As $\mathcal{A}_{\epsilon,p}$ is contractible (Lemma 4.3), the lemma will follow.

To prove statement (1), there are two things to show: the positive cycles for x supported in M all remain the same length, and no other cycles for x get shorter than those. The first holds because the subset of $\mathbb{R}^{|M|}$ where the positive cycles for x have the same length is convex (by Lemma 3.10 it is cut out by the linear equations coming from the subsurface borrowing relations); since we are flowing along straight lines towards p , we stay in this convex set. For the second, we note that, by construction, all vectors on the boundary of Y_M point into the interior of Y_M , i.e. the open chamber Y_M^o .

For statement (2), let X be a point on a flow line which emanates from K . By the construction of \mathcal{V} , the image of Y_M under the flow induced by \mathcal{V} is contained in Y_M^o for all positive time. Therefore, since we already showed that $K \subset U$, we may assume that $X \in Y_M^o$. We need to argue that there is a pants decomposition P of X so that $X \in U_P$. To construct P , we start with M , add all curves of length at most ϵ , and (if needed) complete to a pants decomposition using curves of length at most D_{\max} . We now check that $X \in U_P$; the only nontrivial point is the condition $D_{\min} < \ell_X(c) < D_{\max}$ for all $c \in P$. Each $\ell_X(c)$ is less than D_{\max} by the definition of D_{\max} and the fact that the lengths of the curves of M get shorter along flow lines. Each $\ell_X(c)$ is greater than D_{\min} because each curve of M stays longer than $|p| > D_{\min}$ along flow lines, and curves disjoint from M do not get shorter than $\epsilon > D_{\min}$ along flow lines.

For statement (3), we apply Lemma 4.5. By the definition of U_P , for any $P \in \mathcal{P}$, and any curve c of P , we have $D_{\min} < \ell_X(c) < D_{\max}$. Let $C = C(S, D_{\min}, D_{\max})$ be the constant given by Lemma 4.5. By the lemma, each map $FN_P : FN_P^{-1}(U_P) \rightarrow U_P$ is C -Lipschitz. In other words, the length of a vector of \mathcal{V}_P is at most C times the length of the corresponding vector of \mathcal{V}'_P . The latter is the same as the length of the corresponding vector of $-\nabla d$. Since \mathcal{V} is obtained from the \mathcal{V}_P by averaging,

it follows that each vector of \mathcal{V} has length at most C times the length of the corresponding vector of $-\nabla d$ (the correspondence is given by Φ^*), which is what we wanted to show.

From statement (3), it follows that if γ is an integral path in \mathbb{R}^{6g-6} of length L , then the lift of this path to $\mathcal{T}(S)$ has length at most CL . Therefore, since flow lines of \mathcal{V}'_p in \mathbb{R}^{6g-6} all converge, the corresponding flow lines in $\mathcal{T}(S)$ also converge (statement (4)). This completes the proof. \square

5. STABILIZER DIMENSION

Recall from the Introduction that, to prove Theorems A and B, it suffices to show, for each cell σ of the complex of cycles $\mathcal{B}(S_g)$, that the following inequalities hold.

$$\begin{aligned} \text{cd}(\text{Stab}_{\mathcal{I}(S_g)}(\sigma)) + \dim(\sigma) &\leq 3g - 5 \\ \text{cd}(\text{Stab}_{\mathcal{K}(S_g)}(\sigma)) + \dim(\sigma) &\leq 2g - 3 \end{aligned}$$

By Lemma 3.13, a cell σ of $\mathcal{B}(S_g)$ is associated to a multicurve M consisting of nonseparating curves. The stabilizer in $\mathcal{I}(S_g)$ of the cell σ is exactly the stabilizer of M . Since an element of $\mathcal{I}(S_g)$ cannot reverse the orientation of a nonseparating curve, the orientation of M will play no role in this section.

As in Section 2, the dimension of the cell σ is the number $B(M)$, which only depends on M (without orientation).

In summary, the proof of Theorem A is reduced to the following.

Proposition 5.1. *For $g \geq 2$ and M as above, $\text{cd}(\text{Stab}_{\mathcal{I}(S_g)}(M)) + B(M) \leq 3g - 5$.*

And Theorem B is implied by the following proposition.

Proposition 5.2. *For $g \geq 2$ and M as above, $\text{cd}(\text{Stab}_{\mathcal{K}(S_g)}(M)) + B(M) \leq 2g - 3$.*

Before proving Propositions 5.1 and 5.2 in Sections 5.2 and 5.3, respectively, we need some preliminaries related to the Birman exact sequence.

5.1. The Birman exact sequence and dimension. In order to state the classical Birman exact sequence, we need one definition. The *pure mapping class group* of a surface S , denoted $\text{PMod}(S)$, is the subgroup of $\text{Mod}(S)$ consisting of elements that fix each puncture and boundary component of S individually.

Theorem 5.3 (Birman exact sequence). *Let S be a surface of negative Euler characteristic, and let S' be $S - p$. We have*

$$1 \rightarrow \pi_1(S, p) \rightarrow \text{PMod}(S') \rightarrow \text{PMod}(S) \rightarrow 1.$$

The map $\text{PMod}(S') \rightarrow \text{PMod}(S)$ is obtained by “forgetting” the puncture p , and the image of an element of $\pi_1(S)$ in $\text{PMod}(S')$ is realized by “pushing” the puncture p around that element of $\pi_1(S)$; see [6, Section 4.1].

We will use the Birman exact sequence to describe the effect of punctures on the cohomological dimension of subgroups of the mapping class group. There are three statements; one for higher genus ($g \geq 2$), one for genus 1, and one for genus 0. We use $S_{g,p}$ to denote a surface of genus g with p punctures (and no boundary).

The three corollaries below follow, in a straightforward way, by inducting on the number of punctures, and applying the following facts; see [10, Chapter VIII, Proposition 2.4].

Fact 5.4 (Subadditivity of cohomological dimension). *If we have a short exact sequence of groups*

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

then $\text{cd}(G) \leq \text{cd}(K) + \text{cd}(Q)$.

Fact 5.5 (Monotonicity of cohomological dimension). *If H is a subgroup of G , then $\text{cd}(H) \leq \text{cd}(G)$.*

One also needs the basic facts that $\text{cd}(\pi_1(S_g)) = 2$ for $g \geq 1$, and $\text{cd}(\pi_1(S_{g,p})) = 1$ if $p > 0$, and $(g, p) \neq (0, 1)$.

We start with the case $g \geq 2$. For the application of Corollary 5.6 in Section 5.2, we will take $H = \mathcal{I}(S_g)$.

Corollary 5.6. *Let $g \geq 2$, and let G be a subgroup of $\text{PMod}(S_{g,p})$. Let $F : \text{PMod}(S_{g,p}) \rightarrow \text{Mod}(S_g)$ be the map induced by forgetting all of the punctures, and suppose $F(G)$ is contained in a subgroup H of $\text{Mod}(S_g)$. We have*

$$\text{cd}(G) \leq \text{cd}(H) + (p + 1).$$

Since the torus has Euler characteristic zero, we will instead apply Theorem 5.3 to the once punctured torus. We have the following analog of Corollary 5.6 for this scenario. In Section 5.2, we will apply Corollary 5.7 with $g = 1$, $k = 1$, and $H = \mathcal{I}(S_{1,1}) = 1$.

Corollary 5.7. *Let $g \geq 1$, let $p \geq k > 0$, and let $F : \text{PMod}(S_{g,p}) \rightarrow \text{Mod}(S_{g,k})$ be the map induced by forgetting $p - k$ of the punctures of $S_{g,p}$. If $G < \text{PMod}(S_{g,p})$ and $F(G) < H$, then*

$$\text{cd}(G) \leq \text{cd}(H) + (p - k).$$

In the genus 0 case, we only need the following.

Corollary 5.8. *For $p \geq 3$, we have $\text{cd}(\text{PMod}(S_{0,p})) \leq p - 3$.*

To prove the last corollary, one needs the fact that $\text{PMod}(S_{0,3}) = 1$.

In Section 5.3, we will need a specialized version of Corollary 5.6. For the statement, a *bounding pair map* is a product $T_c T_d^{-1}$, where c and d are disjoint, homologous nonseparating curves.

Lemma 5.9. *Let $g \geq 2$, and let G be a subgroup of $\text{PMod}(S_{g,p})$. Let $F_1 : \text{PMod}(S_{g,p}) \rightarrow \text{Mod}(S_{g,1})$ and $F : \text{PMod}(S_{g,p}) \rightarrow \text{Mod}(S_g)$ be the maps obtained by forgetting all but one of the punctures and all of the punctures, respectively. Suppose that $F_1(G)$ contains no power of any bounding pair maps, and that $F(G)$ is contained in a subgroup H of $\text{Mod}(S_g)$. We have*

$$\text{cd}(G) \leq \text{cd}(H) + p.$$

Proof. As in the case of Corollary 5.6, we apply Theorem 5.3 inductively. However, to improve from $p + 1$ to p , we need an extra argument in the last step.

By the same straightforward applications of Fact 5.4 and Theorem 5.3 used for Corollary 5.6, we have that

$$\mathrm{cd}(G) \leq \mathrm{cd}(F_1(G)) + (p - 1).$$

It remains to show

$$\mathrm{cd}(F_1(G)) \leq \mathrm{cd}(H) + 1.$$

We consider the exact sequence

$$1 \rightarrow A \rightarrow F_1(G) \rightarrow F(G) \rightarrow 1.$$

By Theorem 5.3, the kernel A is a subgroup of $\pi_1(S_g)$. We claim that A has infinite index; indeed, under the map $\pi_1(S_{g,1}) \rightarrow \mathrm{Mod}(S_{g,1})$ from Theorem 5.3, an element α of $\pi_1(S_{g,1})$ represented by a simple nonseparating loop maps to a bounding pair map (cf. [6, Figure 14]), and so by assumption no power of α is an element of A . It follows that A is a free group (the corresponding cover of S_g is noncompact). This implies that $\mathrm{cd}(A) \leq 1$. It remains to apply Facts 5.4 and 5.5 (subadditivity and monotonicity). \square

5.2. Stabilizer dimensions for $\mathcal{I}(S_g)$. For the duration of this section, we fix some $g \geq 2$. Also, throughout the section, let M be a fixed multicurve in $S = S_g$ consisting entirely of nonseparating curves. Suppose that $S - M$ has P components of positive genus S_1, \dots, S_P and Z components of genus zero S_{P+1}, \dots, S_{P+Z} . Say that S_i is homeomorphic to a surface of genus g_i with p_i punctures.

Let $G(M)$ be the free abelian group generated by the Dehn twists in the curves of M . It is a theorem of Vautaw that $G(M) \cap \mathcal{I}(S_g)$ is generated by bounding pair maps [40, Theorem 3.1]. Let BP be the number of curves of M minus the number of distinct homology classes represented by the curves of M . We will use the following consequence of Vautaw's theorem.

Theorem 5.10. *For M as above, we have $G(M) \cap \mathcal{I}(S_g) \cong \mathbb{Z}^{BP}$.*

Lemma 5.11. *Assume Proposition 5.1 for all genera between 2 and $g-1$, inclusive. In the notation of the preceding paragraphs, we have*

$$\mathrm{cd}(\mathrm{Stab}_{\mathcal{I}(S)}(M)) \leq \sum_{i=1}^P (3g_i + p_i - 4) + \sum_{i=P+1}^{P+Z} (p_i - 3) + BP.$$

Proof. Birman–Lubotzky–McCarthy proved that we have the following short exact sequence [7, Lemma 2.1]:

$$1 \rightarrow G(M) \rightarrow \mathrm{Stab}_{\mathrm{Mod}(S)}(M) \rightarrow \mathrm{Mod}(S - M) \rightarrow 1.$$

Here, the group $\mathrm{Stab}_{\mathrm{Mod}(S)}(M)$ consists of those mapping classes that fix the isotopy class of M , and, as above, $G(M) \cong \mathbb{Z}^{|M|}$ is the group generated by the Dehn twists about the curves of M .

If we restrict to the Torelli group, work of Ivanov shows that the map from $\mathrm{Stab}_{\mathcal{I}(S)}(M)$ to $\mathrm{Mod}(S - M)$ has image in $\mathrm{PMod}(S - M)$, and so, if S_i is a component of $S - M$, there is a map $\mathrm{Stab}_{\mathcal{I}(S)}(M) \rightarrow \mathrm{Mod}(S_i)$, i.e., the components of $S - M$ are preserved [22, Corollary 1.8].

Applying Theorem 5.10, we arrive at the appropriate sequence for the Torelli group:

$$1 \rightarrow \mathbb{Z}^{BP} \rightarrow \mathrm{Stab}_{\mathcal{I}(S)}(M) \rightarrow \mathrm{PMod}(S - M).$$

By Fact 5.4 (subadditivity), Theorem 5.10, and the fact that $\text{cd}(\mathbb{Z}^{BP}) = BP$, it remains to show that the image of $\text{Stab}_{\mathcal{I}(S)}(M)$ in $\text{PMod}(S - M)$ has dimension at most

$$\sum_{i=1}^P (3g_i + p_i - 4) + \sum_{i=1}^Z (p_i - 3).$$

Let G_i be the image of $\text{Stab}_{\mathcal{I}(S)}(M)$ in each $\text{PMod}(S_i)$, and let G be the direct product of the G_i . Since the image of $\text{Stab}_{\mathcal{I}(S)}(M)$ in $\text{PMod}(S - M)$ is a subgroup of G , it suffices to show that if $g_i = 0$, then $\text{cd}(G_i) \leq p_i - 3$, and if $g_i > 0$, then $\text{cd}(G_i) \leq 3g_i + p_i - 4$ (apply Facts 5.5 and 5.4 (monotonicity and subadditivity)).

The genus zero case follows immediately from Fact 5.5 (monotonicity) and the inequality $\text{cd}(\text{PMod}(S_{0,p})) \leq p - 3$ (Corollary 5.8).

Now suppose S_i has genus 1. Let \hat{S}_i be the once-punctured torus obtained from S_i by forgetting all of the punctures except for one, and let $\Phi : S_i \rightarrow \hat{S}_i$ and $G_i \rightarrow \text{Mod}(\hat{S}_i)$ be the induced maps. We have the following key observation.

Observation: The image of G_i is a subgroup of $\mathcal{I}(\hat{S}_i)$.

The point is that, given any curve c in S_i , and any $f \in \text{Stab}_{\mathcal{I}(S)}(M)$, we have that $f(c)$ lies in S_i , and if Σ is the surface that proves c and $f(c)$ homologous, then $\Phi(\Sigma)$ shows that $\Phi(c)$ and $\Phi(f(c))$ are homologous.

Now, we know that $\mathcal{I}(S_{1,1}) = 1$, and so the observation combined with Corollary 5.7 gives $\text{cd}(G_i) \leq 0 + (p_i - 1)$. The latter happens to be equal to $3g_i + p_i - 4$.

The case $g_i \geq 2$ is similar. Here, we take \hat{S}_i to be the closed surface obtained from S_i by forgetting all of the punctures. The observation that G_i is a subgroup of $\mathcal{I}(\hat{S}_i)$ is still valid for the same reason as above. Since $g_i < g$, we can apply Proposition 5.1, which tells us that $\text{cd}(\mathcal{I}(\hat{S}_i)) = 3g_i - 5$. Applying the observation and Corollary 5.6, we arrive at $\text{cd}(G_i) \leq (3g_i - 5) + (p_i + 1) = 3g_i + p_i - 4$. This completes the proof. \square

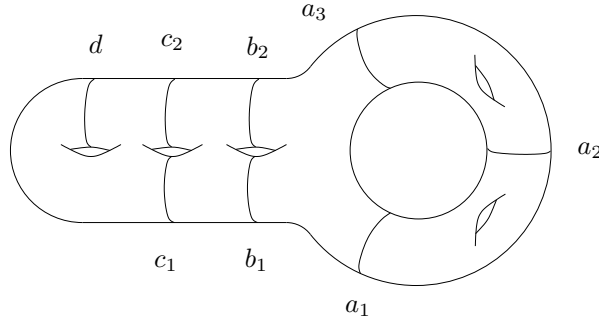


FIGURE 5. A multicurve in S_6 with $BP = 4$, $D = 4$, and $P = 2$.

We now give another version of Lemma 5.11, obtained by algebraic manipulation of the upper bound.

Lemma 5.12. *With the same assumptions and notation as Lemma 5.11, we have*

$$\text{cd}(\text{Stab}_{\mathcal{I}(S)}(M)) \leq 3g - 3 - P - |M| + BP.$$

Since the quantity BP is equal to $|M| - C$, where C is the number of distinct homology classes represented by the curves of M , the upper bound of the lemma can be replaced with the equivalent one, $3g - 3 - P - C$.

Proof. By Lemma 5.11, it suffices to show that

$$\sum_{i=1}^P (3g_i + p_i - 4) + \sum_{i=P+1}^{P+Z} (p_i - 3) + BP = 3g - 3 - P - |M| + BP.$$

We have

$$\begin{aligned} \sum_{i=1}^P (3g_i + p_i - 4) + \sum_{i=P+1}^{P+Z} (p_i - 3) + BP &= 3 \sum g_i + 2|M| - 4P - 3Z + BP \\ &= 3(g - D) + 2|M| - 4P - 3Z + BP \\ &= 3g - 3(|M| + 1) + 2|M| - P + BP \\ &= 3g - 3 - P - |M| + BP. \end{aligned}$$

For the first equality, we used the fact that $\sum p_i = 2|M|$. Then, we applied the fact that $D = g - \sum g_i$. The third equality is a consequence of Lemma 2.1, which says that $D + P + Z = |M| + 1$, and the last equality is obtained by simple algebra. \square

The next lemma is a basic fact about multicurves made up of nonseparating curves.

Lemma 5.13. *For the multicurve M , we have $BP + 2 \leq D + P$.*

Proof. We start with an observation. For any multicurve in S_g with $g \geq 2$, it is straightforward to see that $D + P \geq 2$ (recall that multicurves are nonempty).

We now proceed by induction on genus, with the base case $g = 2$. In that case, $BP = 0$ (there are no bounding pairs in S_2). Since $D + P \geq 2$, we are done in this case.

Now assume $g \geq 3$, and that the lemma is true for all genera between 2 and $g - 1$, inclusive. Let M be a multicurve in $S = S_g$. If there are no bounding pairs, then, as above, there is nothing to do. So assume $BP > 0$.

We can find an “innermost bounding pair” of M , by which we mean a bounding pair $\{c, d\} \subseteq M$ so that one component of $S - (c \cup d)$ contains no other bounding pairs of M . For example, in Figure 5, the bounding pairs $\{a_1, a_2\}$, $\{a_2, a_3\}$, and $\{c_1, c_2\}$ are innermost, whereas the bounding pairs $\{a_1, a_3\}$ and $\{b_1, b_2\}$ are not innermost.

When there is at least one bounding pair, then there is an innermost one: if there is a bounding pair on one side of a given bounding pair, then then the new bounding pair bounds a strictly smaller subsurface (as measured by Euler characteristic), and so the process of finding increasingly innermore bounding pairs must terminate.

Let $\{c, d\}$ be an innermost bounding pair of M , and let S_1 be a component of $S - (c \cup d)$ that contains no bounding pairs of M . We obtain a new surface S' from S by replacing S_1 with an annulus. There is an induced multicurve M' on S' , after removing the image of either c or d (they become homotopic by construction).

We claim that the genus of S' lies in $[2, g-1]$. Indeed, the closures of S_1 and $S-S_1$ have two boundary components each, and so if either had genus zero, we would see that c and d were homotopic.

By induction, M' satisfies

$$BP' + 2 \leq D' + P'$$

where BP' , D' , and P' are defined for M' in the same way as BP , D , and P for M . Since $\{c, d\}$ was innermost in S , we have

$$BP' = BP - 1.$$

Also we have

$$D' + P' + 1 \leq D + P$$

because when we replace S_1 with the annulus, we are either deleting a positive genus component of $S-M$, or we see that there is at least one nonseparating curve in S_1 , which is automatically homologically independent of the other curves of M . In the latter case, replacing S_1 by an annulus reduces D . Thus, we have

$$BP + 2 = BP' + 3 \leq D' + P' + 1 \leq D + P$$

and the proof is complete. \square

We now prove Proposition 5.1, which completes the proof of Theorem A.

Proof of Proposition 5.1. We proceed by strong induction. Assume that Proposition 5.1 holds for genera between 2 and $g-1$ inclusive (if $g=2$ this is an empty assumption). We want to prove the following inequality:

$$\text{cd}(\text{Stab}_{\mathcal{I}(S_g)}(M)) + B(M) \leq 3g - 5.$$

We can replace $B(M)$ with $|M| - D$ (Lemma 2.1). Applying Lemma 5.12, it is enough to show that

$$(3g - 3 - P - |M| + BP) + (|M| - D) \leq 3g - 5.$$

But this inequality is equivalent to the inequality $BP + 2 \leq D + P$, and so an application of Lemma 5.13 completes the proof. \square

5.3. Stabilizer dimensions for $\mathcal{K}(S_g)$. Throughout this section we again fix some $g \geq 2$ and some multicurve M in S_g consisting entirely of nonseparating curves.

We will require the analogue of Theorem 5.10 for $\mathcal{K}(S_g)$, recently proven by the authors [4].

Theorem 5.14. *For M as above, we have $G(M) \cap \mathcal{K}(S_g) = 1$.*

For once punctured surfaces, we will only need the following much weaker statement, proven by Johnson [23, Lemmas 4A and 4B].

Lemma 5.15. *The group $\mathcal{K}(S_{g,1})$ contains no nontrivial powers of any bounding pair maps.*

As before, let $S_{g,p}$ denote a surface of genus g with p punctures. For any surface $S = S_{g,p}$, we denote by $\mathcal{K}(S)$ the subgroup of $\text{Mod}(S)$ generated by Dehn twists about separating curves.

Lemma 5.16. *Let $G < \mathcal{K}(S_{g,p})$ with $g \geq 2$. We have*

$$\text{cd}(G) \leq \text{cd}(\mathcal{K}(S_g)) + p.$$

Proof. It suffices to show that Lemma 5.9 applies. First, since the generators for $\mathcal{K}(S_{g,p})$ do not permute punctures, we have that $G < \text{PMod}(S_{g,p})$. Also, since separating curves remain separating when punctures are forgotten, we see that the images of G in $\text{Mod}(S_{g,1})$ and $\text{Mod}(S_g)$ are subgroups of $\mathcal{K}(S_{g,1})$ and $\mathcal{K}(S_g)$, respectively. By Lemma 5.15, $\mathcal{K}(S_{g,1})$ contains no bounding pair maps. Thus, Lemma 5.9 applies and immediately gives the desired statement. \square

We recall the notation from the start of Section 5.2. We label the positive genus components of $S-M$ by S_1, \dots, S_P , and the genus zero components by S_{P+1}, \dots, S_{P+Z} , and we say that S_i is a surface of genus g_i with p_i punctures. We have the following analogue of Lemma 5.11.

Lemma 5.17. *Assume Proposition 5.2 for all genera between 2 and $g-1$, inclusive. In the notation of the preceding paragraph, we have*

$$\text{cd}(\text{Stab}_{\mathcal{K}(S_g)}(M)) \leq \sum_{i=1}^P (2g_i + p_i - 3) + \sum_{i=P+1}^{P+Z} (p_i - 3).$$

Proof. The proof is essentially the same as the proof of Lemma 5.11, with lemmas about $\mathcal{I}(S_g)$ replaced by lemmas about $\mathcal{K}(S_g)$.

As in the proof of Lemma 5.11, we cut S along M and obtain

$$1 \rightarrow G(M) \cap \mathcal{K}(S) \rightarrow \text{Stab}_{\mathcal{K}(S)}(M) \rightarrow \text{PMod}(S-M)$$

where $G(M)$ is the group generated by Dehn twists in the curves of M . By Theorem 5.14, the term $G(M) \cap \mathcal{K}(S)$ is trivial, and so $\text{Stab}_{\mathcal{K}(S)}(M)$ is isomorphic to its image in $\text{PMod}(S-M)$. So we must show that this image has cohomological dimension bounded above by

$$\sum_{i=1}^P (2g_i + p_i - 3) + \sum_{i=1}^Z (p_i - 3).$$

As in the proof of Lemma 5.11, we denote by G_i the image of $\text{Stab}_{\mathcal{K}(S)}(M)$ in $\text{PMod}(S_i)$.

Like before, it suffices to show that if $g_i = 0$, then $\text{cd}(G_i) \leq p_i - 3$, and if $g_i > 0$, then $\text{cd}(G_i) \leq 2g_i + p_i - 3$.

As in Lemma 5.11, the genus zero case follows from Corollary 5.8. For $g_i = 1$, note that $2g_i + p_i - 3 = 3g_i + p_i - 4 = p_i - 1$, and so, by Fact 5.5 (monotonicity), the upper bound of $p_i - 1$ obtained from Corollary 5.7 in the proof of Proposition 5.2 is sufficient.

It remains to deal with the case $g_i \geq 2$. Let \hat{S}_i be the closed surface obtained from S_i by forgetting all of the punctures. Since separating curves in S remain separating in S_i , we have $G_i < \mathcal{K}(S_i)$, and so Lemma 5.16 gives that $\text{cd}(G_i) \leq \text{cd}(\mathcal{K}(\hat{S}_i)) + p_i$. By induction, $\text{cd}(\mathcal{K}(\hat{S}_i)) \leq 2g_i - 3$, and this completes the proof. \square

We can simplify the upper bound of Lemma 5.17 as follows. The proof is analogous to the argument for Lemma 5.12, and is left to the reader.

Lemma 5.18. *With the assumptions and notation of Lemma 5.17, we have*

$$\mathrm{cd}(\mathrm{Stab}_{\mathcal{K}(S_g)}(M)) \leq 2g - 3 + D - |M|.$$

We are now ready to prove the upper bound for Theorem B.

Proof of Proposition 5.2. We proceed as in the proof of Proposition 5.1. Assume by induction that Proposition 5.2 is true for all genera between 2 and $g - 1$, inclusive. The goal is to show the following:

$$\mathrm{cd}(\mathrm{Stab}_{\mathcal{K}(S_g)}(M)) + B(M) \leq 2g - 3.$$

By Lemma 5.18 and Lemma 2.1 it suffices to show

$$(2g - 3 + D - |M|) + (|M| - D) \leq 2g - 3.$$

But this inequality simplifies to $0 \leq 0$. \square

Remark. We point out that in the case of $\mathcal{K}(S_g)$ we did not need an analog of Lemma 5.13. In this sense, the upper bound for Theorem B is slightly easier than that for Theorem A. Also recall that the lower bound was also easier for Theorem B.

6. THE MESS DESCRIPTION OF $\mathcal{I}(S_2)$

We now turn to the proof of Theorem D. As discussed in the introduction, Theorem A implies that $\mathcal{I}(S_2)$ is a free group. However, the statement of Theorem D is much stronger, and the infinite generation is needed for the base case of Theorem C.

The proof breaks up into three parts: we first analyze the stabilizer in $\mathcal{I}(S_2)$ of a pair of nonseparating curves, then we analyze the stabilizer of a single nonseparating curve, and finally we piece these together to get the desired description of $\mathcal{I}(S_2)$. We can think of these steps as the genus 0, genus 1, and genus 2 versions of the Mess theorem.

Genus 0. We start by investigating the stabilizer in $\mathcal{I}(S_2)$ of a vertex of $\mathcal{B}(S_2)$ corresponding to a pair of disjoint nonseparating curves. Since the complement in S_2 has genus zero, we think of this as a genus zero version of Theorem D.

In the statement of the lemma, we say that a symplectic splitting of $H_1(S_2, \mathbb{Z})$ is *compatible* with a pair of nonseparating curves $\{a, b\}$ (or the associated homology classes) if the homology class $[a]$ lies in one subspace determined by the splitting, and $[b]$ lies in the other.

Lemma 6.1. *Let M be the multicurve consisting of two nonseparating curves a and b in S_2 . The group $\mathrm{Stab}_{\mathcal{I}(S_2)}(M)$ is an infinitely generated free group, with one Dehn twist generator for each symplectic splitting compatible with M .*

Proof. We cut S_2 along a and b and obtain a sphere S' with 4 punctures. Let X be the flag complex with a vertex for each isotopy class of curves in S' that comes from a separating curve in S_2 , and an edge for each pair of isotopy classes with geometric intersection number 4.

The complex X is contractible: if we think of the vertices as isotopy classes of arcs in S' connecting the two punctures coming from a , then edges correspond to disjointness of arcs, and the required statement is given by Harer [16, Theorem 1.6]

(the argument of [19] also applies, with minor modifications). We also see from this point of view that X is a graph (so it is a tree)—any three arcs in S' that represent distinct separating curves must intersect.

The group $\text{Stab}_{\mathcal{I}(S_2)}(M)$ acts on X . The quotient has one vertex for each homology splitting of $H_1(S_2, \mathbb{Z})$ compatible with M ; in other words, if two separating curves c and c' are disjoint from M and induce the same homology splitting, then there is an element of $\mathcal{I}(S_2)$ that fixes a and b and takes c to c' . To construct such a mapping class, cut one copy of S along c and another copy of S along c' , choose the unique maps on the punctured tori that act trivially on homology, and then choose any element of $\text{Mod}(S_2)$ that restricts to the chosen maps on the punctured tori (cf. [24, Theorem 1A] and [34, Lemma A.3]).

Our goal is to show that the quotient complex is contractible (a tree). Since vertex stabilizers are the Dehn twists about the corresponding separating curves in S_2 , and edge stabilizers are trivial (Theorem 5.10, plus the fact that any two nonisotopic separating curves in S_2 must intersect), the lemma will then follow from the classical Bass–Serre theory of group actions on trees [36, Theorem 13].

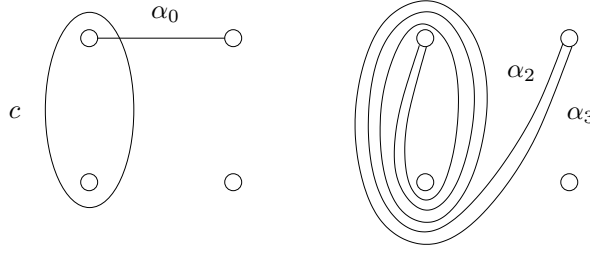
Let $([a], [a'], [b], [b'])$ be a symplectic basis for $H_1(S_2, \mathbb{Z})$, where a and b are the curves in the statement of the lemma (take the algebraic intersection of $[a]$ and $[a']$, say, to be 1). It follows from elementary linear algebra that the splittings of $H_1(S_2, \mathbb{Z})$ compatible with M are in bijection with the cosets $([a'] + k[b]) + \langle [a] \rangle$, and hence are indexed by \mathbb{Z} .

We now claim that two vertices form an edge in the quotient complex if and only if they are adjacent in \mathbb{Z} . It will follow that the quotient complex is isomorphic to the standard triangulation of \mathbb{R} .

If two vertices are adjacent in \mathbb{Z} , then one can draw disjoint arcs in S' corresponding to those vertices. This is proven by drawing the picture: if α_0 is an arc in S' corresponding to the integer 0, then $T_c^k(\alpha_0)$ is an arc corresponding to k , where T_c is the Dehn twist about any fixed curve in S' that comes from a curve in S_2 , represents the class $[a] + [b]$, and intersects α_0 in one point; see Figure 6 for an illustration. One can check that $T_c^k(\alpha_0)$ and $T_c^{k+1}(\alpha_0)$ are disjoint, and so the vertices corresponding to k and $k + 1$ are connected by an edge in the quotient.

If two vertices labelled by $k, k' \in \mathbb{Z}$ are connected in the quotient, this means that there are corresponding arcs in S' that are disjoint, and hence corresponding curves in S_2 that are disjoint. As above, these curves represent primitive homology classes $[a'] + k[b] + j[a]$ and $[a'] + k'[b] + j'[a]$. Since the curves are disjoint, we have $j = j'$ (otherwise, the algebraic intersection is nonzero). The curves in S_2 span a maximal isotropic subspace (i.e. a maximal subspace where the intersection form is trivial); in particular, they span the subspace spanned by $[a']$ and $[b]$. Hence, the determinant of the pair of vectors $[a'] + k[b]$ and $[a'] + k'[b]$ must be ± 1 , and we see $k = k' \pm 1$, which is what we wanted to show. \square

Genus 1. We now need to understand the stabilizer in $\mathcal{I}(S_2)$ of a vertex of $\mathcal{B}(S_2)$ corresponding to a single nonseparating curve c in the class x . This is the genus 1 version of Theorem D. We say that a homology splitting is *compatible with x* if x lies in one component of the splitting.

FIGURE 6. The arcs α_0 , α_2 , and α_3 from the proof of Lemma 6.1.

Lemma 6.2. *The group $\text{Stab}_{\mathcal{I}(S_2)}(c)$ is an infinitely generated free group, with one Dehn twist generator for each symplectic splitting compatible with x .*

Proof. We cut S_2 along c and obtain a twice-punctured torus S' . To find a generating set for $\text{Stab}_{\mathcal{I}(S_2)}(c)$, we consider the action on the complex of basic cycles for S' . To this end, we fix a homology class $y \in H_1(S_2, \mathbb{Z})$ coming from a nonseparating curve d in S_2 that is disjoint (and isotopically distinct) from c . The classes x and y correspond to classes in $H_1(S', \mathbb{Z})$, which we also call x and y .

The choice of the class y gives rise to a complex of cycles $\mathcal{B}(S')$, which is contractible (the argument is the same as the closed case, with only cosmetic changes).

Given any two disjoint (isotopically distinct) nonseparating curves in S' , their homology classes necessarily differ by $\pm x$. It follows that an integral basic cycle for y consists of a single curve in the homology class $y + kx$ (along with $-kc$) and that $\mathcal{B}(S')$ is a tree. We also deduce that the vertices of the quotient are indexed by \mathbb{Z} , and that if two vertices in the quotient are connected by an edge, then they are adjacent in \mathbb{Z} . Also, if two vertices are adjacent in \mathbb{Z} , then we can realize them by disjoint curves. Thus the quotient of $\mathcal{B}(S')$ by $\text{Stab}_{\mathcal{I}(S_2)}(c)$ is isomorphic to the standard triangulation of \mathbb{R} .

As in Lemma 6.1, $\text{Stab}_{\mathcal{I}(S_2)}(c)$ is freely generated by the vertex stabilizers (we take one copy for each vertex of the quotient). By Lemma 6.1, each vertex stabilizer is an infinitely generated free group, with one Dehn twist generator for each homology splitting compatible with x and $y + kx$.

It remains to establish the bijection between generators of $\text{Stab}_{\mathcal{I}(S_2)}(c)$ and homology splittings compatible with x . By Lemma 6.1, a generator corresponds to a unique homology splitting compatible with x . On the other hand, given a homology splitting compatible with x , we consider the induced splitting of y , and we get a unique generator. It is clear that these identifications are inverses of each other, so we are done. \square

Genus 2. We finally use Lemmas 6.1 and 6.2 to obtain the full version of the Mess theorem.

Proof of Theorem D. By Theorem E, the complex of cycles $\mathcal{B}(S_2)$ is contractible. We also have that it is one dimensional by Lemma 2.1, and so it is in fact a tree. As in the previous lemmas, we need to argue that the quotient $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ is

contractible, and then do some bookkeeping to check that there is exactly one generator for each homology splitting.

The vertices of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ are in one to one correspondence with the homology classes of basic cycles for x ; that is, if two nonseparating curves, or two pairs of nonseparating curves, in S_2 represent the same homology classes, then there is an element of $\mathcal{I}(S_2)$ taking one to the other [34, Lemma A.3]. There is a single distinguished vertex corresponding to the homology class x , and infinitely many vertices corresponding to pairs of homology classes (a basic cycle in S_2 has one or two curves).

In order to show that $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ is contractible, we will assign a “weight” to each vertex, and show that each vertex is connected by an edge to exactly one vertex of smaller weight. We declare the weight of the distinguished vertex corresponding to a single curve in the class x to be 1. For a vertex given by a basic cycle $pa + qb$ with $p, q > 0$, we define the weight to be $p + q$.

Let v be the vertex corresponding to the positive cycle $pa + qb$. There are exactly two classes that can be represented disjointly from the multicurve $\{a, b\}$, namely, $[a] + [b]$ and $[a] - [b]$. Hence, there are four potentially adjacent vertices, given by the pairings of $[a]$ and $[b]$ with these two new classes.

The class $[a] - [b]$ cannot pair with either $[a]$ or $[b]$ to give a vertex of smaller weight. Indeed, we have $(p + q)[a] + q([b] - [a]) = x$ and $(p + q)[b] + p([a] - [b]) = x$, and so in either case the weight is larger than $p + q$.

The weights of the other two nearby vertices are both smaller than $p + q$, since we have $(q - p)[b] + p([a] + [b]) = x$ and $(p - q)[a] + q([a] + [b]) = x$. As $q - p$ is negative, it follows from Lemma 3.9 that v is not connected by an edge to the vertex corresponding to the first equation. Since $p - q$ is positive, it follows from Lemma 3.13 that v is connected to the other vertex. This completes the proof of contractibility of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$.

Now for the bookkeeping. The stabilizer of any lift of the distinguished vertex is an infinitely generated free group generated by one Dehn twist generator for each homology splitting compatible with x by Lemma 6.2. The stabilizer of a lift of a vertex corresponding to two homology classes is an infinitely generated free group generated by one Dehn twist for each splitting compatible with that pair by Lemma 6.1. The theorem follows. \square

Remark. It is more illuminating to draw a diagram of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$. It is naturally subdivided into pieces, corresponding to different 2-dimensional isotropic subspaces of $H_1(S, \mathbb{Z})$ that contain x . Each such subspace is group isomorphic to \mathbb{Z}^2 . The set of bases for \mathbb{Z}^2 are depicted via the Farey graph: vertices correspond to (unsigned) primitive vectors, and edges correspond to bases. There is a distinguished vertex x , and the rest of the component of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ corresponding to this subspace is the tree shown in Figure 7. The entire complex $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ is obtained by gluing infinitely many of these trees along their distinguished vertices.

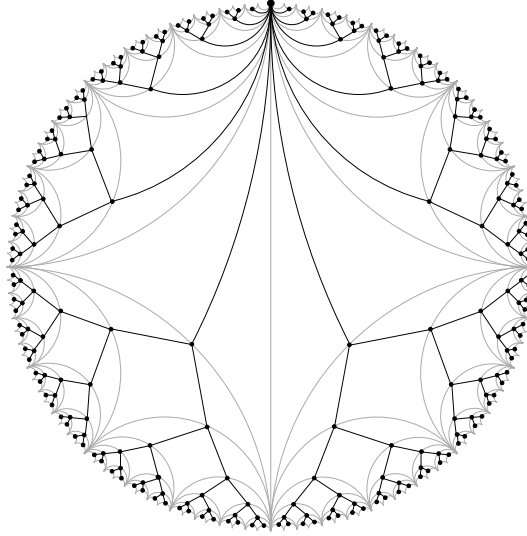


FIGURE 7. A piece of $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ corresponding to a particular Lagrangian subspace of $H_1(S_2, \mathbb{Z})$. The Farey graph is drawn lightly for reference.

7. INFINITE GENERATION OF HOMOLOGY

The goal of this section is to prove Theorem C, which states that, for $g \geq 2$, the group $H_{3g-5}(\mathcal{I}(S_g), \mathbb{Z})$ is infinitely generated. The basic idea is to use induction (the base case is Theorem D) and the spectral sequences associated to the appropriate Birman exact sequences.

Throughout this section, we will be forced to consider homology with various local and global coefficients. When the coefficient ring is \mathbb{Z} , it will be convenient to omit this from the notation.

7.1. Spectral sequences. We recall some facts about spectral sequences; for an introduction to the theory, see [18], [15], or [10].

Hochschild–Serre spectral sequence. Given a short exact sequence of groups

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

there is an associated Hochschild–Serre spectral sequence. The second page of this spectral sequence is the two-dimensional array of homology groups

$$E_{p,q}^2 = \begin{cases} H_p(Q, H_q(K)) & p, q \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the coefficients $H_q(K)$ are considered to be local: Q acts on K by conjugation. The main difficulties in calculation come from this action.

We remark that Fact 5.4 is an easy consequence of the Hochschild–Serre spectral sequence.

Fact 7.1. *Consider the Hochschild–Serre spectral sequence associated to a short exact sequence*

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

and assume $\text{cd}(Q)$ and $\text{cd}(K)$ are finite. For $p = \text{cd}(Q)$ and $q = \text{cd}(K)$, we have

$$H_{p+q}(G) \cong E_{p,q}^2 = H_p(Q, H_q(K)).$$

Cartan–Leray spectral sequence. For a group G acting cellularly and without rotations on a contractible complex X , we can associate the Cartan–Leray spectral sequence (to act *without rotations* means that, if a cell is fixed by some element, then the cell is fixed pointwise). The first page of this spectral sequence is given by

$$E_{p,q}^1 = \begin{cases} \bigoplus_{\sigma \in X_p} H_q(G_\sigma) & p, q \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where X_p is the set of p -cells of X/G , and G_σ is the stabilizer in G of a lift of the cell σ .

We only need the following basic fact; see [10, Section VII.7] and [15, Section 18].

Fact 7.2. *Consider the Cartan–Leray spectral sequence associated to the action of the group G on the contractible space X . Suppose that $E_{p,q}^1 = 0$ for $p+q > k$. Then $E_{0,k}^1 = \bigoplus H_k(G_v)$ injects as a subgroup of $H_k(G)$ (the sum is over all vertices v of X/G).*

We remark that Quillen’s upper bound on the cohomological dimension of a group given in the introduction is an immediate consequence of the Cartan–Leray spectral sequence; this is Exercise 4 in Section VIII.2 of [10].

7.2. Torelli Birman exact sequences. In this section, we give two versions of the Birman exact sequence that are particular to the Torelli group: the first, due to Putman, allows us to pass from two punctures to one puncture; the second, due to Johnson, enables us to go from there to the closed case. For each sequence, we analyze the action of the quotient on the kernel.

Let v be a vertex of $\mathcal{B}(S_g)$ corresponding to a single curve c (the image of v in $\mathcal{B}(S_g)/\mathcal{I}(S_g)$ is the distinguished vertex). We can think of the stabilizer in $\mathcal{I}(S_g)$ of v (equivalently, of c) as a subgroup of $\text{Mod}(S_{g-1,2})$, and so, as in the Birman exact sequence (Theorem 5.3), there is a forgetful map

$$\text{Stab}_{\mathcal{I}(S_g)}(v) \rightarrow \text{Mod}(S_{g-1,1})$$

whose kernel is a subgroup of $\pi_1(S_{g-1,1})$.

Lemma 7.3. *The image of $\text{Stab}_{\mathcal{I}(S_g)}(v)$ in $\text{Mod}(S_{g-1,1})$ is a subgroup of $\mathcal{I}(S_{g-1,1})$. Moreover, the sequence*

$$1 \rightarrow K \rightarrow \text{Stab}_{\mathcal{I}(S_g)}(v) \rightarrow \mathcal{I}(S_{g-1,1}) \rightarrow 1$$

where K is the commutator subgroup of $\pi_1(S_{g-1,1})$, is exact.

Proof. The first statement of the lemma is proven in the same way as the observation in the proof of Lemma 5.11 (page 23). The surjectivity of the map $\text{Stab}_{\mathcal{I}(S_g)}(v) \rightarrow \mathcal{I}(S_{g-1,1})$ can be seen as follows: embed $S_{g-1,1}$ in $S_{g-1,2}$ so that $S_{g-1,2} - S_{g-1,1}$ is a twice-punctured disk; given an element of $\mathcal{I}(S_{g-1,1})$ we extend it by the identity to get an element of $\text{Stab}_{\mathcal{I}(S_g)}(v)$. The description of K is due to Putman [34, Theorem 4.1] (Putman uses surfaces with boundary instead of with punctures, but only inconsequential changes to his proof are required). \square

We also have the following short exact sequence, due to Johnson [25].

Lemma 7.4. *For $g \geq 2$, we have*

$$1 \rightarrow \pi_1(S_g) \rightarrow \mathcal{I}(S_{g,1}) \rightarrow \mathcal{I}(S_g) \rightarrow 1.$$

As indicated in Section 7.1, we can apply the Hochschild–Serre spectral sequence to understand the groups in Lemmas 7.3 and 7.4. Since coefficients are local, we will need to understand the actions of the quotient groups on the kernels of the two sequences.

In the next lemma, we consider the action (by conjugation) of $\mathcal{I}(S_g)$ on $H_i(\pi_1(S_g)) \cong H_i(S_g)$ coming from the sequence in Lemma 7.4. This is the restriction of the usual action of $\text{Mod}(S_g)$ on $H_1(S_g)$.

Lemma 7.5. *The action of $\mathcal{I}(S_g)$ on $H_i(S_g)$ is trivial for $i = 0, 1, 2$.*

Proof. The case $i = 0$ is trivial. When $i = 1$, the statement of the lemma is the definition of $\mathcal{I}(S_g)$. For $i = 2$, the lemma is equivalent to the fact that elements of $\mathcal{I}(S_g)$ preserve the orientation of S_g . \square

We now consider the action of $\mathcal{I}(S_{g-1,1})$ on $H_1(K)$ coming from Lemma 7.3. Unlike the previous case, the action here is nontrivial (for instance, take an element of K that is the commutator of two simple loops that intersect only at the basepoint, and consider the action by a Dehn twist about a separating curve that is disjoint from the first loop and intersects the other one twice). However, we will be able to find a subgroup of $H_1(K)$ on which $\mathcal{I}(S_{g-1,1})$ acts trivially (Lemma 7.7 below). In the proof of Theorem C, this will allow us to use global coefficients.

First, we have a technical lemma about commutators.

Lemma 7.6. *Let $F = \langle x_1, \dots, x_{2n} \rangle$ be the free group generated by the x_i , let K be the commutator subgroup of F , and let $h \in K$ be the product of commutators $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$. The class of h in $H_1(K)$ is nontrivial.*

Proof. The group $H_1(K)$ can be thought of as the first homology of the 1-skeleton of the standard cubing of \mathbb{R}^{2n} . It is clear now that h represents a nontrivial class. \square

As above, we consider the action by conjugation of $\mathcal{I}(S_{g-1,1})$ on $H_1(K)$ coming from Lemma 7.3. We now give an invariant submodule of $H_1(K)$.

Lemma 7.7. *There is a nontrivial subgroup of $H_1(K)$ on which $\mathcal{I}(S_{g-1,1})$ acts trivially.*

Proof. Let $\alpha_1, \beta_1, \dots, \alpha_{g-1}, \beta_{g-1}$ be the usual generators for $\pi_1(S_{g-1,1})$, and let δ be the following element of $\pi_1(S_{g-1,1})$:

$$\delta = [\alpha_1, \beta_1] \cdots [\alpha_{g-1}, \beta_{g-1}].$$

Clearly, $\delta \in K$. We will show that the class of δ in $H_1(K)$ is fixed by $\mathcal{I}(S_{g-1,1})$. By Lemma 7.6, this is a nontrivial element of $H_1(K)$, and so the module generated by this element is the desired submodule of $H_1(K)$.

Thinking of $\text{Stab}_{\mathcal{I}(S_g)}(v)$ as a subgroup of $\text{Mod}(S_{g-1,2})$, the image of δ under the map $K \rightarrow \text{Stab}_{\mathcal{I}(S_g)}(v)$ in Lemma 7.3 is the Dehn twist T_d about a curve d that bounds a twice-punctured disk in $S_{g-1,2}$ (we think of δ as the simple loop in this disk that is based at one puncture and surrounds the other).

To prove the lemma, it suffices to show that, given an arbitrary element f of $\mathcal{I}(S_{g-1,1})$, there is a lift F of f to $\text{Stab}_{\mathcal{I}(S_g)}(v)$ (thought of as a subgroup of $\text{Mod}(S_{g-1,2})$) so that $FT_dF^{-1} = T_{F(d)}$ is conjugate in K to T_d . In fact, we will be able to construct the lift F so that $F(d) = d$, and so $T_{F(d)}$ is actually equal to T_d .

Let $S_{g-1,1} \rightarrow S_{g-1,2}$ be a fixed embedding with the property that $S_{g-1,2} - S_{g-1,1}$ is the twice-punctured disk bounded by d . Via this embedding, any homeomorphism of $S_{g-1,1}$ can be regarded as a homeomorphism of $S_{g-1,2}$ (extend by the identity). Given $f \in \mathcal{I}(S_{g-1,1})$, we take any representative homeomorphism, think of it as a homeomorphism of $S_{g-1,2}$, and call the resulting mapping class F . By construction, $F(d) = d$. It is clear that F maps to f under the map $\text{Mod}(S_{g-1,2}) \rightarrow \text{Mod}(S_{g-1,1})$ that forgets one puncture of $S_{g-1,2}$. It remains to check that F is an element of $\text{Stab}_{\mathcal{I}(S_g)}(v)$.

We embed $S_{g-1,2}$ into S_g so that the punctures correspond to homotopic nonseparating curves (that is, “uncut”). If the isotopy class corresponding to the punctures is c , we can extend F to an element of $\text{Mod}(S_g)$ that is well-defined up to Dehn twists about c . Since the support of F is on a surface of genus $g-1$ and one puncture, and F acts trivially on the homology of this subsurface, we can choose a lift of F to $\text{Mod}(S_g)$ that lies in $\mathcal{I}(S_g)$. This completes the proof. \square

7.3. Proof of infinite generation of top homology. This section contains the proof of Theorem C, which states that $H_{3g-5}(\mathcal{I}(S_g), \mathbb{Z})$ is infinitely generated.

Lemma 7.8. *If $H_{3g-5}(\mathcal{I}(S_g))$ is infinitely generated, then $H_{3g-3}(\mathcal{I}(S_{g,1}))$ is infinitely generated.*

Proof. We know that $\text{cd}(\pi_1(S_g)) = 2$. By Theorem A, $\text{cd}(\mathcal{I}(S_g)) = 3g-5$. Applying Fact 7.1 to the short exact sequence of Lemma 7.4, and using the identification $H_2(\pi_1(S_g)) = H_2(S_g) \cong \mathbb{Z}$, we find

$$H_{3g-3}(\mathcal{I}(S_{g,1})) \cong H_{3g-5}(\mathcal{I}(S_g), \mathbb{Z}).$$

By Lemma 7.5, the coefficients in the last homology group are global. Therefore, by Theorem C, $H_{3g-3}(\mathcal{I}(S_{g,1}))$ is infinitely generated. \square

We record the following corollary of Lemma 7.8 (use Lemma 7.4 and Fact 5.4).

Corollary 7.9. *For $g \geq 2$, we have $\text{cd}(\mathcal{I}(S_{g,1})) = 3g-3$.*

Lemma 7.10. *If v is any vertex of $\mathcal{B}(S_g)$, then $H_{3g-5}(\text{Stab}_{\mathcal{I}(S_g)}(v))$ injects into $H_{3g-5}(\mathcal{I}(S_g))$.*

Proof. We consider the Cartan–Leray spectral sequence associated to the action of $\mathcal{I}(S_g)$ on $\mathcal{B}(S_g)$. By Proposition 5.1, we know that $E_{p,q}^1 = 0$ whenever $p+q > 3g-5$. By Fact 7.2, the group $E_{0,3g-5}^1 = \bigoplus H_{3g-5}(\text{Stab}_{\mathcal{I}(S_g)}(v))$, where the sum is over a set of representatives of the vertices of $\mathcal{B}(S_g)/\mathcal{I}(S_g)$, injects into $H_{3g-5}(\mathcal{I}(S_g))$. The lemma follows. \square

We are now ready to prove the theorem.

Proof of Theorem C. We proceed by induction on the genus g . Theorem D tells us that $\mathcal{I}(S_2)$ is an infinitely generated free group, and so $H_1(\mathcal{I}(S_2))$ is infinitely generated. Let $g \geq 3$, and assume the theorem is true for $g-1$, that is, $H_{3g-8}(\mathcal{I}(S_{g-1}))$ is infinitely generated.

Let v be a vertex of $\mathcal{B}(S_g)$ represented by a single curve in the class x . By Lemma 7.10, it suffices to show that $H_{3g-5}(\text{Stab}_{\mathcal{I}(S_g)}(v))$ is infinitely generated.

Consider the short exact sequence in Lemma 7.3. Since K is free, we have $\text{cd}(K) = 1$, and Corollary 7.9 gives $\text{cd}(\mathcal{I}(S_{g-1,1})) = 3g-6$. Applying Fact 7.1 to the sequence in Lemma 7.3, we obtain

$$H_{3g-5}(\text{Stab}_{\mathcal{I}(S_g)}(v)) \cong H_{3g-6}(\mathcal{I}(S_{g-1,1}), H_1(K)).$$

Thus, the theorem is reduced to showing that $H_{3g-6}(\mathcal{I}(S_{g-1,1}), H_1(K))$ is infinitely generated. By Lemma 7.7, there is a nontrivial submodule M of $H_1(K)$ on which $\mathcal{I}(S_{g-1,1})$ acts trivially. Since M is torsion free (it is a subgroup of the first homology of a free group), the universal coefficient theorem gives

$$H_{3g-6}(\mathcal{I}(S_{g-1,1}), M) \cong H_{3g-6}(\mathcal{I}(S_{g-1,1})) \otimes M.$$

By the inductive hypothesis and Lemma 7.8, the latter is infinitely generated.

Now, the short exact sequence of modules

$$1 \rightarrow M \rightarrow H_1(K) \rightarrow H_1(K)/M \rightarrow 1$$

induces a long exact sequence of homology groups, in which we find the following:

$$\begin{aligned} \cdots \rightarrow H_{3g-5}(\mathcal{I}(S_{g-1,1}), H_1(K)/M) &\rightarrow H_{3g-6}(\mathcal{I}(S_{g-1,1}), M) \\ &\rightarrow H_{3g-6}(\mathcal{I}(S_{g-1,1}), H_1(K)) \rightarrow \cdots \end{aligned}$$

By Corollary 7.9, the first term shown is trivial. Thus, the infinitely generated group $H_{3g-6}(\mathcal{I}(S_{g-1,1}), M)$ injects into $H_{3g-6}(\mathcal{I}(S_{g-1,1}), H_1(K))$, and the theorem follows. \square

In general, the vertices of $\mathcal{B}(S_g)$ correspond to basic cycles for x supported on k curves, with $1 \leq k \leq g$. When $k < g$, the cohomological dimension of the vertex stabilizer is $3g-4-k$. By an argument similar to that of Theorem C, the top dimensional integral homology of a vertex stabilizer ($k < g$) is infinitely generated. For $k > 1$, there are infinitely many vertices of that type. Therefore, to prove that $H_{3g-4-k}(\mathcal{I}(S_g), \mathbb{Z})$ is infinitely generated, one would only have to show that the quotient of each vertex group by its incoming edge groups is nontrivial.

Question 7.11. *Is it true that $H_{3g-4-k}(\mathcal{I}(S_g), \mathbb{Z})$ is infinitely generated for $2 \leq k \leq g-1$?*

8. TORELLI MESS SUBGROUPS

We now briefly describe Mess's proof of Theorem 1.1. We do this for completeness, and also because Mess's argument is not published in the exact form that gives Theorem 1.1. His original paper is a preprint, available from Institut des Hautes Études Scientifiques [30].

The basic idea is to inductively define subgroups Γ_g of $\mathcal{I}(S_g)$ with the property that $\text{cd}(\Gamma_g) = 3g - 5$. The theorem then follows from Fact 5.5 (monotonicity).

We employ the theory of Poincaré duality groups. A group Γ is a Poincaré duality group if the trivial $\mathbb{Z}\Gamma$ -module \mathbb{Z} admits a finite length projective resolution by finitely generated $\mathbb{Z}\Gamma$ -modules, and $H^i(\Gamma, \mathbb{Z}\Gamma)$ is trivial in all but one dimension, where it is group isomorphic to \mathbb{Z} ; see [27] and [5] for background. Fundamental groups of closed aspherical manifolds are Poincaré duality groups, for example.

We will need the following fact [27, Theorem 3].

Theorem 8.1. *Given a short exact sequence of groups*

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

where K and Q are Poincaré duality groups, it follows that G is a Poincaré duality group and $\text{cd}(G) = \text{cd}(K) + \text{cd}(Q)$.

To start the inductive process, we define Γ_2 to be the group generated by a single Dehn twist about a separating curve in S_2 . Since $\Gamma_2 = \pi_1(S^1) \cong \mathbb{Z}$, we have that Γ_2 is a Poincaré duality group. Of course, $\text{cd}(\Gamma_2) = 1$.

We now assume that Γ_g is given, and that it is a Poincaré duality group with cohomological dimension $3g - 5$. We will define Γ_{g+1} in two steps.

Let S'_g be the surface obtained from S_g by removing an open disk. We consider the *relative mapping class group* $\text{Mod}(S'_g, \partial S'_g)$, which is defined as $\pi_0(\text{Homeo}^+(S'_g, \partial S'_g))$, where $\text{Homeo}^+(S'_g, \partial S'_g)$ is the subgroup of elements of $\text{Homeo}^+(S'_g)$ that fix $\partial S'_g$ pointwise. Let $UT(S_g)$ denote the unit tangent bundle for S_g .

We have the following generalization of the classical Birman exact sequence due to Johnson [25].

Theorem 8.2 (Relative Birman exact sequence). *Let $g \geq 2$. We have*

$$1 \rightarrow \pi_1(UT(S_g)) \rightarrow \text{Mod}(S'_g, \partial S'_g) \rightarrow \text{Mod}(S_g) \rightarrow 1.$$

It was noticed by Johnson that the entire image of $\pi_1(UT(S))$ lies in the Torelli subgroup of $\text{Mod}(S'_g, \partial S'_g)$ [25]. Therefore, the preimage of Γ_g in $\text{Mod}(S'_g, \partial S'_g)$ is a subgroup of the Torelli group $\mathcal{I}(S'_g, \partial S'_g)$. We call this group Γ'_{g+1} . By Theorem 8.1, Γ'_{g+1} is a Poincaré duality group of cohomological dimension $3(g+1) - 5$ (since $UT(S_g)$ is a closed aspherical 3-manifold, $\pi_1(UT(S_g))$ is a Poincaré duality group of dimension 3).

The inclusion of S'_g into S_{g+1} induces an injective homomorphism $\text{Mod}(S'_g, \partial S'_g) \rightarrow \text{Mod}(S_{g+1})$ (see, e.g., [32, Corollary 4.2]) which restricts to an inclusion on the level of Torelli groups. We define Γ_{g+1} to be the image of Γ'_{g+1} . Since Γ_{g+1} is isomorphic to Γ'_{g+1} , we have that Γ_{g+1} is a Poincaré duality group of dimension $3(g+1) - 5$, and this completes the proof of Theorem 1.1.

Remark. While the above argument certainly proves Theorem 1.1, we would like to point out that more is true. Not only are the Γ_g Poincaré duality groups, but they also are fundamental groups of closed aspherical manifolds, with base Γ_{g-1} and fiber $UT(S_{g-1})$. The proof of this stronger statement is given by Ivanov [22, Section 6.3].

Remark. Often the easiest way to get a lower bound on the cohomological dimension of a group is to find a large abelian subgroup (cf. Fact 5.5). It is a theorem of Vautaw, however, that the largest free abelian subgroup of $\mathcal{I}(S_g)$ has rank $2g - 3$ [40]. In Figure 8, we exhibit such a subgroup in genus 5 (the example generalizes to higher genus). In that example, each generator is a Dehn twist about a separating curve and so we see that we have \mathbb{Z}^{2g-3} inside $\mathcal{K}(S_g)$. This gives the lower bound for $\text{cd}(\mathcal{K}(S_g))$ given in the introduction.

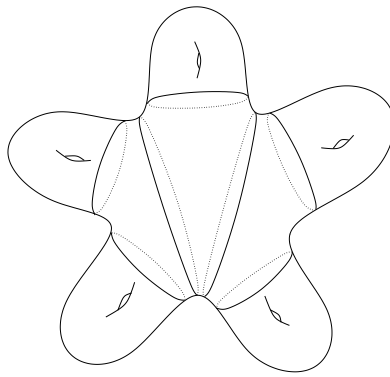


FIGURE 8. The Dehn twists about these curves generate a free abelian subgroup of maximal rank in $\mathcal{I}(S_5)$.

Remark. The construction of the Mess subgroups clearly illustrates the discrepancy of g between $\text{cd}(\mathcal{I}(S_g))$ and $\text{vcd}(\text{Mod}(S_g))$. To get a group of cohomological dimension $4g - 5$ in $\text{Mod}(S_g)$, Mess simply augments the inductive construction of the Γ_g by adding the Dehn twist about a nonseparating curve in the new handle at each stage. This new Dehn twist generates a direct factor of the Mess subgroup for $\text{Mod}(S_g)$. By only adding these Dehn twists on $g - k$ of the new handles, the construction gives the correct lower bound for Conjecture 1.2.

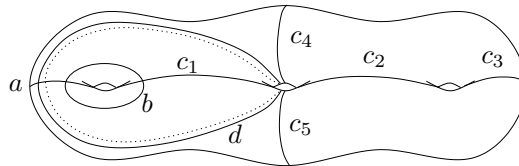


FIGURE 9

On the other hand, we note that the conjecture cannot be proven by applying the Quillen condition to the action of $\mathcal{I}^k(S_g)$ on $\mathcal{B}(S_g)$. To see this, we consider the group $\mathcal{I}^1(S_3)$, and the configuration shown in Figure 9; the handle being fixed

homologically is the one spanned by a and b (say $x = [a]$). There is a 2-cell of $\mathcal{B}(S_3)$ corresponding to the c_i : $[c_1] + [c_2] + [c_3] = x$, and each of c_4 and c_5 adds a “borrowing dimension”. The conjecture says that $\text{cd}(\mathcal{T}^k(S_g)) = 6$, so we would need that the stabilizer of $\{c_i\}$ in $\mathcal{T}^1(S_3)$ has cohomological dimension at most 4. On the contrary, we can find a free abelian group of rank 5 in this stabilizer: the group generated by $T_{c_2}, T_{c_3}, T_{c_4}, T_{c_5}$, and T_d .

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