A SURGICAL PROOF OF THE CONTRACTIBILITY OF THE COMPLEX OF MINIMIZING CYCLES

We give a short proof of the contractibility of the complex of minimizing cycles. Our proof is inspired by Hatcher's proof of the contractibility of the arc complex for a surface [2]. This theorem was used by the authors to compute the cohomological dimension of the Torelli group for a surface [1]. We start by briefly recalling the definition of the complex of minimizing cycles.

Curves and cycles. Let $S = S_g$ be a closed connected orientable surface of genus g. A 1-cycle in S is a finite formal sum

$$\sum k_i c_i$$

where $k_i \in \mathbb{R}$, and each c_i is an oriented simple closed curve in S. The set $\{c_i : k_i \neq 0\}$ is called the *support* of the 1-cycle. We say that the 1-cycle is *simple* if the curves of the support are pairwise disjoint, and we say that it is *positive* if each k_i is positive.

A *basic cycle* is a simple positive 1-cycle $\sum k_i c_i$ with the property that its support forms a linearly independent subset of $H_1(S, \mathbb{R})$.

Let $\mathcal{S}$ denote the set of isotopy classes of oriented simple closed curves in S. There is a natural set map from the set of 1-cycles in S to $\mathbb{R}^{\mathcal{S}}$.

A *multicurve* in the surface S is a nonempty collection of disjoint simple closed curves in S that are isotopically nontrivial and isotopically distinct.

Construction of the complex. The first step is to fix an arbitrary nontrivial element x of $H_1(S, \mathbb{Z})$. Next, let $\mathcal{M}$ be the set of isotopy classes of oriented multicurves M in S with the property that each oriented curve of M appears in the support of some basic cycle for x supported in M. The set $\mathcal{M}$ has a natural partial ordering under inclusion (the orientation is important here). As such, we can think of $\mathcal{M}$ as a category where the morphisms are inclusions.

Given $M \in \mathcal{M}$, let P_M be the polytope in $\mathbb{R}^{\mathcal{S}}$ given by the convex hull of the images of the basic cycles for x supported (with orientation) in M. The collection $\{P_M\}$ is a category under inclusion, which we denote $\mathcal{F}$. The functor given by

$$M \mapsto P_M$$

is an isomorphism from $\mathcal{M}$ to $\mathcal{F}$. We define the *complex of minimizing cycles* $\mathcal{B}(S)$ as the geometric realization of the colimit of this functor:

$$\mathcal{B}(S) = \left| \operatorname*{colim}_{M \in \mathcal{M}} \{P_M\} \right|.$$

An auxiliary complex. Let $\tilde{\mathcal{M}}$ be the set of isotopy classes of oriented multicurves M in S with the property that each oriented curve of M lies in the support of some 1-cycle that represents x and is supported in M. If in the definition of $\mathcal{B}(S)$ we replace $\mathcal{M}$ with $\tilde{\mathcal{M}}$, we get a space, which we denote $\tilde{\mathcal{B}}(S)$, that contains $\mathcal{B}(S)$. The space $\tilde{\mathcal{B}}(S)$ has the structure of a cell complex, except that some of the "cells" are not compact.

A characterization of $\mathcal{B}(S)$. Below, we give two alternate characterizations of points of $\mathcal{B}(S)$. For a positive 1-cycle $c = \sum k_i c_i$, an equation of the form $[c_{i_1}] + \cdots + [c_{i_n}] = 0$ ($n > 0$) is called a *one-sided relation*, and we may say that c *satisfies* a one-sided relation. Also, we say that c is a *minimizing cycle* for x if there is a hyperbolic metric on S for which c realizes the minimum length over all 1-cycles in S representing x (the length of c is the sum $\sum k_i \ell(c_i)$, where $\ell(c_i)$ is the length of the unique geodesic homotopic to c_i).

Lemma 1. *Let c be a 1-cycle in S. The following are equivalent.*

1. *c represents a point of $\mathcal{B}(S)$.*
2. *c is a minimizing cycle for x.*
3. *c satisfies no one-sided relations.*

Proof. The equivalence of (1) and (2) is the content of Lemma 3.12 in our original paper [1].

That (2) implies (3) is straightforward: if c satisfies a one-sided relation, say, $[c_1] + \cdots [c_n] = 0$, then we may shorten c by subtracting ϵ from the coefficients of $c_1, \ldots, c_n$, where ϵ is any positive number smaller than the minimum of $\{k_1, \ldots, k_n\}$.

To show that (3) implies (1), we suppose that c satisfies no one-sided relations. The goal is to show that there is a basic cycle for x supported in $\{c_i\}$ with some fixed c_{i_0} in the support. If c is not already basic, that means $S - \cup c_i$ is disconnected. If S' is one component, then the c_i that form the boundary of S' satisfy a relation of the form

$$[c_1] + \cdots + [c_p] = [c_{p+1}] + \cdots + [c_{p+q}]$$

with $p, q > 0$. Suppose without loss of generality that $i_0 > p$ and that k_1 is the minimum of $\{k_1, \ldots, k_p\}$. The following is a new simple positive 1-cycle that represents x:

$$c' = ((k_2 - k_1)c_2 + \cdots (k_p - k_1)c_p) + ((k_{p+1} + k_1)c_{p+1} + \cdots (k_{p+q} + k_1)c_{p+q}) + \sum_{i > p+q} k_i c_i.$$

The 1-cycle c' is a simple positive 1-cycle that contains c_{i_0} in its support. What is more, $S - c'$ has one fewer connected component than $S - c$. By induction, we can find a 1-cycle representing x that is supported in $\{c_i\}$ and that is nonseparating in S. Such a 1-cycle is basic. $\square$

Surgery on 1-cycles. Suppose that d and d' are two 1-cycles in S representing points of $\mathcal{B}(S)$, assume that d and d' have the minimal number of intersections in their isotopy classes, and let c be the 1-cycle $td + (1-t)d'$, where $t \in [0,1]$; note that c represents the class x. In general, c is not simple. We now explain how to do surgery to convert c into a simple 1-cycle $\mathrm{Surger}(c)$, which is canonical up to isotopy.

By Poincaré Duality, the homology class x corresponds to a cohomology class, and hence to a homotopy class of based maps from S to the circle S^1. We will use the cycle c to construct an explicit representative φ of this homotopy class.

As above, say that $c = \sum k_i c_i$. We thicken each c_i to a band (annulus) $A_i = S^1 \times [0, k_i]$. We choose the A_i and coordinates (θ_i, t_i) so that the following conditions hold (cf. the left hand side of Figure 1).

(1) c_i is oriented in the positive θ direction.
(2) The intersection of c_j with A_i $(i \neq j)$ is a coordinate arc.
(3) The intersection of two annuli is a coordinate rectangle in both annuli.
(4) The rectangles of intersection are disjoint.
(5) In each rectangle of intersection, an arc is a coordinate arc in one annulus if and only if it is a coordinate arc in the other.

We can now define a measure μ_i on the set of arcs in A_i. For an arc $\alpha : [0,1] \longrightarrow A_i$, we define $\mu_i(\alpha) = \int_\alpha dt_i$. We get a measure μ_i on arcs in S by adding the measures of the components contained in A_i. Finally, we obtain a measure μ on arcs in S given by

$$\mu(\alpha) = \sum \mu_i(\alpha).$$

For a closed loop α, we see that $\mu(\alpha)$ is the same as the algebraic intersection of α with c.

We are now ready to define the map $\varphi : S \longrightarrow S^1$. Let p be an arbitrarily chosen basepoint of S, let q be an arbitrary point of S, and let $\alpha_q : [0,1] \longrightarrow S$ be an arbitrary path from p to q. Thinking of S^1 as $\mathbb{R}/\mathbb{Z}$ we define $\varphi(q)$ as the fractional part of $\mu(\alpha_q)$. Since x is an integral homology class, this map is well-defined.

We now use the map $\varphi : S \longrightarrow S^1$ to define the simple 1-cycle $\mathrm{Surger}(c)$. The preimage of each regular point of φ in S^1 is a 1-manifold, that is, a collection of pairwise disjoint curves in S. If we consider the union of the preimages of regular points in S^1 all at once, we see some number of parallel families of curves. Each family has a well-defined thickness, coming from the measure on the circle (we use the measure coming from $\mathbb{R}$). As such, we replace each family of curves with a single weighted curve. Also, we discard any curves that are homotopic to a point. The resulting 1-chain is the desired simple 1-chain $\mathrm{Surger}(c)$. From the construction, we see that $\mathrm{Surger}(c)$ corresponds to the same cohomology class as c (integrating against $\mathrm{Surger}(c)$ gives a map homotopic to φ), and so $\mathrm{Surger}(c)$ represents the homology class x.

All of the choices made were unique up to isotopy, and so the 1-cycle $\mathrm{Surger}(c)$ is a well-defined point of $\tilde{\mathcal{B}}(S)$.

In what follows, we will use the fact that, for any fixed 1-cycle c coming from a point of $\mathcal{B}(S)$, the function

$$d \mapsto \mathrm{Surger}(tc + (1-t)d)$$

is a continuous map from $\mathcal{B}(S) \times [0,1]$ to $\tilde{\mathcal{B}}(S)$. For continuity, the key point is that the condition that some isotopy class is in the support of $\mathrm{Surger}(tc+(1-t)d)$ is an open condition.

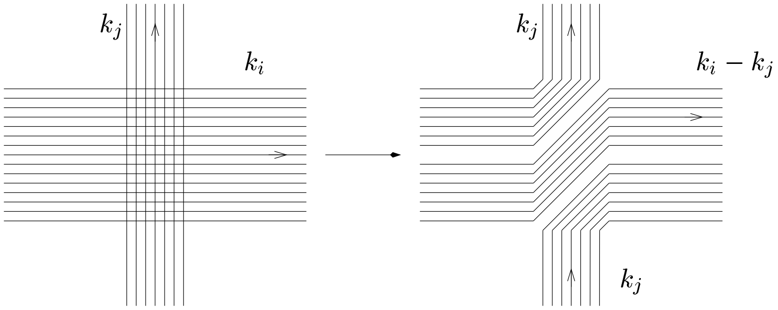

Figure 1. Surgery on 1-cycles

The construction of $\mathrm{Surger}(c)$ is more easily explained by a single picture; see Figure 1. The well-definedness is easy to see in this figure. What is not easy to see in the figure is that $\mathrm{Surger}(c)$ is a 1-cycle. For instance, consider on the torus two curves α and β that intersect once, and let d be the 1-cycle $\epsilon\alpha + (1-\epsilon)\beta$ where ϵ is irrational. If we perform the surgery indicated in Figure 1 at the intersection, the result is not a well-defined 1-cycle. The point is that d does not represent a rational homology class. In our construction of $\mathrm{Surger}(c)$ it was essential that x was an integral class (a slight modification is needed in the rational case).

Draining 1-cycles. The result of the surgery defined above is always a simple positive 1-cycle representing x. However, it is not necessarily true that this 1-cycle represents a point of $\mathcal{B}(S)$; rather, this 1-cycle is in general a point in $\tilde{\mathcal{B}}(S)$. To address this issue, we now define a strong deformation retraction $\mathrm{Drain} : \tilde{\mathcal{B}}(S) \to \mathcal{B}(S)$.

As a basic example, suppose that c is the formal sum of a 1-cycle representing a point of $\mathcal{B}(S)$ with a weighted oriented separating curve d. The cycle $\mathrm{Drain}(c)$, in this case, is obtained by "draining" the coefficient of d to 0 (we think of the subsurface bounded by d as the drain).

We now return to the general case; let c be a 1-cycle representing a point of $\tilde{\mathcal{B}}(S)$. Let $\{S_i\}$ be the set of embedded subsurfaces of S that have boundary in c. Say that $\{S_1, \ldots, S_k\}$ is the subset of $\{S_i\}$ consisting of subsurfaces that give rise to 1-sided relations. Suppose that the 1-sided relation corresponding to S_1 is

$$[c_1] + \cdots + [c_p] = 0.$$

For small enough ϵ, we obtain a new 1-cycle representing x by starting with c and decreasing each of the coefficients of $c_1, \ldots, c_p$ by ϵ.

We may simultaneously consider all of the k one-sided relations at once: again, we start with c, and we decrease the coefficient of c_i by $n\epsilon$, where $0 \leq n \leq k$ is the number of one-sided relations in which c_i appears.

We adjust weights in this way, until some coefficient becomes 0. At this point, we repeat the process, starting with the new 1-cycle. When there are no more 1-sided relations, we stop. The resulting 1-cycle is called Drain(c). By Lemma 1, Drain(c) represents a point of $\mathcal{B}(S)$. Note in particular, that, since no arbitrary choices were made, the 1-cycle Drain(c) is well-defined.

The way we defined the Drain map, it is a strong deformation retraction of $\tilde{\mathcal{B}}(S)$ onto $\mathcal{B}(S)$. Below, it will suffice to think of Drain as a continuous map $\tilde{\mathcal{B}}(S) \to \mathcal{B}(S)$.

The proof of contractibility. We choose an arbitrary basepoint v of $\mathcal{B}(S)$. We will now define a deformation retraction of $\mathcal{B}(S)$ to the point v.

Denote by c a 1-cycle representing v. The retraction $H : \mathcal{B}(S) \times [0,1] \longrightarrow \mathcal{B}(S)$ is given by

$$H(w,t) = \mathrm{Drain}(\mathrm{Surger}(tc + (1-t)d))$$

where w is an arbitrary point of $\mathcal{B}(S)$, and d is a representative 1-cycle.

The map H is a strong deformation retraction of $\mathcal{B}(S)$ onto the point v, and so $\mathcal{B}(S)$ is contractible.

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