RESEARCH STATEMENT
TRAVIS MANDEL

INTRODUCTION

My research mostly focuses on the interplay between tropical geometry, Gromov-Witten theory, and mirror symmetry. I am especially interested in the theta function construction of Gross-Hacking-Keel-Siebert [GHKS], particularly the case of cluster varieties as in [GHKK14]. Much of my work as a postdoc has been on a series of papers [Man16, MR16, MRb, Man] which together show that these canonical theta function bases can be defined in terms of certain counts of holomorphic curves. I have also proved other interesting results on cluster algebras, including an indecomposability result on the theta function bases [Man17b] and a result on universal torsors of compactifications of cluster varieties [Man17a]. I am also very excited about new developments in [MRb] which connect multiplicities used in tropical curve counting to an $L_\infty$-algebra of polyvector fields on the mirror, evidently a key advancement towards a tropical explanation of mirror symmetry. Some of my ongoing work seeks to understand quantum and motivic refinements of the above constructions, e.g., relating quantum or Hall algebra theta functions to counts of real curves. All these ideas will be detailed below.

§1 will give an overview of my series of papers proving the Frobenius structure conjecture [GHK15, arXiv v1, §0.4] for cluster varieties. This conjecture states that the theta functions can be described in terms of certain counts of algebraic curves (precisely, descendant log Gromov-Witten invariants). The strategy is to first relate the theta functions (defined by [GHKS] in terms of “broken lines”) to tropical curve counts [Man16], then to relate these tropical curve counts to counts of holomorphic curves on toric varieties (using [MR16] combined with [MRb]), and then, using degeneration techniques, to relate these counts to the desired invariants of the mirror cluster variety [Man].

In §2, I will discuss some of my other work on cluster algebras. §2.1 focuses on my theorem from [Man17b] characterizing the theta functions as the atomic totally positive elements of the upper cluster algebra, essentially as conjectured by [FG09] (with slight modifications). §2.2 will discuss my paper [Man17a] on how to identify an upper cluster algebra with the Cox ring of a leaf of the corresponding partially compactified cluster Poisson variety $\overline{X}$. This allows one to realize theta function bases for all line bundles on $\overline{X}$, as opposed to just for the structure sheaf. I then discuss potential applications to homological mirror symmetry.

In §3, I will discuss [MRb], where H. Ruddat and I find that multiplicities of tropical curves can be understood using the Schouten-Nijenhuis bracket (and new higher brackets) on polyvector fields of the mirror. We are also developing another interpretation, a sort of “tropical field theory” which associates certain linear maps to tropical curves and then computes multiplicities by evaluating these linear maps on certain boundary conditions. From a mirror symmetry perspective, this is very significant because it relates the Hochschild cohomologies of the $A$- and $B$-models.

The tropical version of the Frobenius structure conjecture, as developed in [Man16] and discussed below in §1.1, is in fact much more general than needed for the applications discussed in §1. The theorem works in the same way over various different Lie algebras, in particular allowing quantum theta functions to be described in terms of $q$-deformed (Block-Göttsche style [BG16]) counts of tropical
curves, and also allowing the Hall algebra scattering diagrams of \textup{[Bri16]} to be expressed in terms of counts of tropical curves with multiplicities in the Hall algebra.

Motivated by this, upcoming joint work with H. Ruddat \textup{[MRa]} will define certain counts of real curves (alternatively, holomorphic disks with boundary on the real locus), and we will relate these to the $q$-deformed tropical curve counts (a la \textup{[Mik16]}). \S 4 will explain this result. I hope to someday use this to prove a quantum/real version of the Frobenius structure conjecture. Furthermore (cf. \S 4.3), I plan to eventually extend this beyond the case of cluster varieties to obtain a more general refinement of the Gross-Siebert program, in which I expect a real/open version of the A-model to be mirror to a quantum refinement of the B-model.

Finally, \S 5 will discuss ongoing joint work with M.-W. Cheung in which we investigate the Hall algebra tropical curves mentioned above. We have found that the multiplicities associated to the tropical curves here are spaces of composition series for various quiver representations.

0.1. A very brief introduction to cluster varieties. Before continuing, we quickly give some background on cluster varieties. Cluster varieties are constructed by gluing together a collection of algebraic tori $(\mathbb{k}^*)^n = \text{Spec} \mathbb{k}[N]$, called clusters, via certain birational maps called mutations, cf. \textup{[FG09]} and \textup{[GHK13]}. They have a large number of applications, with some examples of cluster varieties including double Bruhat cells of semisimple Lie groups \textup{[BFZ05]}, Grassmannians \textup{[Sco06]} and other partial flag varieties \textup{[GLS08]}, and various moduli spaces of local systems on punctured Riemann surfaces \textup{[FG06]}. They have quickly become a very active topic of research.

Happily, cluster varieties are also examples of spaces where the Gross-Siebert mirror symmetry program \textup{[GS11]} works very well, particularly the Gross-Hacking-Keel-Siebert theta function construction \textup{[GHKS]}. This was used in \textup{[GHKK14]} to construct canonical theta function bases $\{\vartheta_q\}_{q \in \mathbb{N}}$ for (roughly) the ring of global regular functions on the cluster variety (also known as the upper cluster algebra). Understanding certain canonical bases and positivity properties (as in \textup{[Lus90, Lus94]}) was one of the original motivations for defining cluster algebras in \textup{[FZ02]}, so this is indeed a major contribution to cluster theory. These theta bases and related constructions are the focus for much of the research I will discuss.

1. The Frobenius structure conjecture: theta functions in terms of curve counts

By a log Calabi-Yau with maximal boundary $(Y, D)$, we mean a smooth projective variety (or orbifold) $Y$, together with a divisor $D \in | - K_Y |$, such that $D$ contains a 0-stratum which in a local analytic neighborhood looks like a complete intersection of components of $D$. The most basic examples are smooth complete toric varieties, with $D$ being the toric boundary.

Cluster varieties form an important more general class of examples. According to \textup{[GHK13]}, these spaces $(Y, D)$ are roughly obtained by taking a smooth complete toric variety $(\overline{Y}, \overline{D})$ and then blowing up a collection of “hypertori” $\{(a_i + z^{e_i})| e_i | = 0 \} \cap D_{e_i}$. Here, the subscripts $i$ come from a finite index set $I$, $a_i \in \mathbb{k}^*$, each $e_i$ is in the cocharacter lattice $N$ for $\overline{Y}$ with $D_{e_i}$ denoting the corresponding boundary divisor, and the vectors $u_i$ are in the character lattice $M = N^*$ and satisfy a “skew-symmetrizability” condition with the vectors $e_i$. $| e_i |$ indicates the index of $e_i$.

For each $p \in N$, there is an associated “theta function” $\vartheta_p$ on the mirror to $(Y, D)$. These theta functions are constructed using combinatorial gadgets called scattering diagrams and broken lines (cf. \textup{[GHK15, GHKS, GHKK14]} for the surface, general, and cluster cases, respectively), but the Frobenius
structure conjecture [GHK15, arXiv v1, Conj. 0.8] predicts a geometric characterization in terms of certain counts of algebraic curves in \( Y \). I prove this for cluster varieties in [Man]:

**Theorem 1.1.** For \((Y,D)\) as in the cluster variety examples above, suppose \( D \) supports an ample divisor. Then there is a unique associative algebra structure on

\[
A := \bigoplus_{p \in \mathbb{N}} k[\text{NE}(Y)] \cdot \vartheta_p
\]

(\text{NE}(Y) denoting the Mori cone, or cone of curves, of \( Y \)) with \( \vartheta_0 = 1 \) such that the \( \vartheta_0 \)-coefficient of \( \vartheta_{p_1} \cdots \vartheta_{p_s} \) (each \( p_i \in \mathbb{N} \setminus \{0\} \)) is given by

\[
\sum_{\beta \in \text{NE}(\tilde{Y})} z^{\pi \cdot \beta} N_{\beta}(p_1, \ldots, p_s),
\]

where \( \pi : \tilde{Y} \to Y \) is a toric blowup such that each \( p_k \) corresponds to a boundary divisor \( D_{p_k} \) of \( \tilde{Y} \), and \( N_{\beta}(p_1, \ldots, p_s) \) is the (virtual) number of log curves of class \( \beta \) hitting \( D_{p_k} \) at a marked point with multiplicity \( |p_i| \) and also satisfying a generic point and \( \psi^{s-2} \)-condition (the \( \psi^{s-2} \)-condition specifies the underlying marked curve, without the data of the map to \( \tilde{Y} \)).

Similarly if \( D \) does not support an ample divisor, except that the multiplication might not be polynomial, so one must work formally.

**Remark 1.2 (Application to Fano’s).** As an application of Theorem 1.1, I note that it implies the version of mirror symmetry used in the Fano classification program described in [CCG+12]. In this setup, one considers a “quantum period” capturing the GW counts of degree \( d \) curves (for varying \( d \)) satisfying a point condition and a \( \psi^{d-2} \)-condition. One also considers a “classical period” determined by integrals \( \int_{\gamma} f^d \Omega \), where \( \gamma \) is the class of a certain (SYZ) torus, \( \Omega \) is a log holomorphic volume form, and \( f \) is a “superpotential” equal to \( \vartheta_{p_k} \), where the sum is over the primitive \( p_k \) such that \( D_{p_k} \) is an irreducible component of \( D \). The version of mirror symmetry used here states that these two periods should be equal, and indeed, this follows from Theorem 1.1.

As I will sketch, the proof of Theorem 1.1 combines several results from my time as a postdoc.

1.1. **The tropical Frobenius structure conjecture.** As a first step, in [Man16], I related the theta functions on the mirror to \((Y,D)\) as above to certain tropical curve counts, proving what I view as a tropical version of the Frobenius structure conjecture. Like in [GHKK14], I considered theta functions \( \{\vartheta_p\}_{p \in \mathbb{N}} \) constructed by using

\[
\mathcal{D}_m := \{(u^i, (1 + z^{e_i})^{|u^i|})| i \in I\}
\]

as the initial scattering diagram. We have an algebra \( A \) which, as a \( k \)-module, is given by

\[
A = \bigoplus_{p \in \mathbb{N}} k \cdot \vartheta_p.
\]

1Technically, I work with a certain lift of \( \mathcal{D}_m \) to a lattice \( \tilde{N} \) surjecting to \( N \) in which the vectors \( e_i \) and generators for the rays of \( \Sigma \) lift to a basis. The kernel of \( \tilde{N} \to N \) is related to \( \text{NE}(Y) \) and statements about the coefficients come from carefully understanding this. Relatively, the multiplication of theta functions is generally not polynomial, and so in general, defining the algebra \( A \) (even formally) may require working over something like a completion of \( k[\ker(\tilde{N} \to N)] \), rather than just over \( k \). For the sketch here though, I will ignore these important technicalities.
Theorem 1.3 (Man16). The $\vartheta_0$-coefficient of $\vartheta_{p_1} \cdots \vartheta_{p_s}$ is given by

$$\sum_{w \in \mathfrak{M}_p} N^{\text{trop}}_{w p} \frac{R_w}{|\text{Aut}(w)|},$$

where $\mathfrak{M}_p$ is the set of weight vectors $(w_i)_{i \in I}$, $w_i = (w_{i1} \leq \ldots \leq w_{il_i})$, $N^{\text{trop}}_{w p}$ is a count of tropical curves in $N \otimes \mathbb{R}$ with unbounded edges $E_k$ of weighted direction $p_k$ for each $k = 1, \ldots, s$ and $E_{ij}$ with weighted direction $w_{ij}e_i$ for each $i, j$, with the $E_{ij}$'s contained in generic translates of $u_i^+$, and with a marked $s$-valent vertex satisfying a generic point condition. $R_w := \prod_{i,j} \frac{(-1)^{w_{ij} - 1}}{w_{ij}^2}$ (which comes from coefficients of the dilogarithm and geometrically corresponds to multiple cover contributions).

Furthermore, these $\vartheta_0$-coefficients uniquely determine the associative multiplication rule.

Other coefficients of the theta function multiplication are expressed in terms of counts of tropical disks (tropical curves with one unbalanced vertex). It should be possible to express these counts in terms of punctured invariants [GS16], but for now I focus on the tropical curves.

I also give similar descriptions of the multiplication rule for quantum theta functions, and in an upcoming rewrite I will give a more general description that also works for Hall algebra scattering diagrams. These will be discussed in §4 and §5 respectively.

1.2. The descendant log correspondence theorem. The $s$-valence condition on the marked vertex for the tropical curves in Theorem 1.3 has long been expected to correspond to a $\psi^{s-2}$ condition on the corresponding algebraic invariants (cf. [Mik07, MR09, Gro10, Ove15, Gro15] for various results in this direction). The original driving motivation for my paper [MR16] with H. Ruddat was to prove this in the generality necessary for the invariants of Theorem 1.3. We achieved this and more, proving a correspondence theorem between tropical curve counts and descendant log Gromov-Witten numbers of toric varieties for any collections of incidence conditions and $\psi$-class conditions, even in higher genus cases (assuming non-superabundance).

The methods of our argument were mostly based on those of [NS06] with significant simplifications by our use of log Gromov-Witten theory [GS13, AC14]. A new observation key to our proof was that $\psi$-classes in our setting could be pulled back from the moduli space of curves.

Remark 1.4. The multiplicities with which the tropical curves are counted for Theorem 1.3 are quite different looking from the multiplicities used in [MR16] (defined as the index of certain a map of lattices as in [NS06]). Proving that these multiplicities are in fact equal was the main original motivation for [MR16], although this paper also achieves quite a bit more, cf. §3.

1.3. Completing the proof — degeneration and coefficients. Finally, in [Man], I combine the results discussed in §1.1 and 1.2 to obtain a description of the theta functions in terms of certain descendant log GW invariants of $(\overline{Y}, \overline{D})$. To prove the Frobenius structure conjecture, I then relate these to descendant log GW invariants of the blowup $(Y, D)$. I do this by constructing a degeneration of $(Y, D)$ into a union of $(\overline{Y}, \overline{D})$ with a bunch of “flaps” hanging off the boundary divisors $D_{c_i}$ ($i \in I$).

To apply the log degeneration formula of [KLR] or the relative one of [Li02], I prove that the log curves appearing in the counts are torically transverse. This is one of the trickier technical results of the paper, and the proof is based on a tropical interpretation of toric transversality. The proof of Theorem 1.1 then follows (see Footnote 1 for issues I have glossed over about how to relate the coefficients to NE($Y$)).
2. Further results on cluster varieties and theta functions

2.1. Total positivity. In the construction of the theta functions, each chamber of the scattering diagram $D$ corresponds to some local coordinate system—a (formal) Laurent series ring—on the mirror space where the theta functions are defined. [GHKK14] shows that the theta functions always have positive integer coefficients in these local coordinate systems. For “cluster $\mathcal{A}$-varieties,” the clusters (cf. §0.1) are a subset of these local coordinate systems, so it follows that the theta functions are “totally positive,” i.e., they have positive integer coefficients in each cluster.

The existence of totally positive bases like these was conjectured in [FG09]. This conjecture further predicted that the bases would exactly consist of the atomic totally positive functions, i.e., those which cannot be written as a sum of two other totally positive functions with integer coefficients. This characterization was shown to be false in [LLZ14]. However, I found that this is because [FG09] used total positivity with respect to the cluster atlas (i.e., in each cluster) rather than with respect to the then-undiscovered scattering atlas (i.e., with respect to each chamber of $D$):

**Theorem 2.1** ([Man17b, Thm. 1]). The theta functions are exactly the elements which are atomic with respect to the scattering atlas.

2.2. Cox rings and canonical bases for line bundles. [FG09] defines both cluster $\mathcal{A}$-varieties (whose ring of global sections is the upper cluster algebra) and cluster $\mathcal{X}$-varieties (or cluster Poisson varieties), along with a map $\mathcal{A} \to \mathcal{X}$ which realizes $\mathcal{A}$ as a torsor over a certain symplectic leaf $\mathcal{X}_\phi$ of $\mathcal{X}$, or using “generic coefficients,” a map $\mathcal{A}_t \to \mathcal{X}_\phi$. [GHK13, Thm. 4.4] showed that this map realizes $\mathcal{A}_t$ as the universal torsor over $\mathcal{X}_\phi$, roughly meaning that $\mathcal{A}_t$ can be viewed as the direct sum of all line bundles over $\mathcal{X}_\phi$, $\mathcal{A}_t = \bigoplus_{\mathcal{L} \in \operatorname{Pic}(\mathcal{X}_\phi)} \mathcal{L}$. However, the [GHK13] setup did not allow for (partial) compactifications $\overline{\mathcal{X}}_\phi$, and such compactifications are often what one is interested in. In [Man17a], I show how to realize (using “frozen variables”) a certain partial compactification $\overline{\mathcal{A}}_t$ of $\mathcal{A}_t$ as the universal torsor of a corresponding (partial) compactification $\overline{\mathcal{X}}_\phi$ of $\mathcal{X}_\phi$:

**Theorem 2.2** ([Man17a, Thm. 3.5]). $\overline{\mathcal{A}}_t$ is the universal torsor over the corresponding $\overline{\mathcal{X}}_\phi$.

Well-known special cases include Cox’s construction of homogeneous coordinates on toric varieties [Cox95] and the correspondence between weight spaces of basic affine space $G/U$ and line bundles on the corresponding flag variety $G/B$ ($G$ semisimple, $B \subset G$ Borel, $U$ the unipotent radical of $B$).

A nice application of this theorem is that it allows theta functions on $\overline{\mathcal{A}}_t$ to be viewed as sections of line bundles on $\overline{\mathcal{X}}_\phi$. Under certain assumptions, this yields theta function bases for every line bundle on $\overline{\mathcal{X}}_\phi$ [Man17a, Thm. 4.2]. In future work with M.-W. Cheung, we plan to use this to describe the derived category $D^b\operatorname{Coh}(\overline{\mathcal{X}}_\phi)$ in terms of theta functions (using the equivalence with $D^b(\operatorname{Mod} - A)$ for $A = \operatorname{End}(\bigoplus_i \mathcal{L}_i)$, $\{\mathcal{L}_i\}_i$ a full exceptional collection of line bundles, cf. [Bon90]). Since these theta functions can be expressed in terms of certain counts of holomorphic disks in the mirror (cf. [1] and [GS16]), we hope this will lead to a version of Homological Mirror Symmetry for cluster varieties.

3. Polyvector fields and tropical curve multiplicities

As mentioned in Remark 1.4, the multiplicities in [MR16] and [NS06] are given as the index of a certain map of lattices. In dimension 2 though, an expression of the multiplicity as a product of vertex multiplicities has been known since [Mik05]. The multiplicities of [Man16] are also products of...
vertex multiplicities, even in higher-dimensional cases (an update in progress will also express these multiplicities as iterated commutators of elements of a Lie algebra).

In [MR16], we investigate the multiplicities in [MR16] to obtain new multiplicity formulas in any dimension and with any collection of incidence and ψ-class conditions. One formula uses a product of vertex multiplicities divided by a product of edge multiplicities. Another puts a flow on the tropical curve by picking one vertex to be the sink. This flow is then used to express the multiplicity as a product of vertex multiplicities.

This latter formula leads to a very nice expression in terms of polyvector fields. Let Γ be a tropical curve in $N \otimes \mathbb{R}$, and let $V_\infty$ denote the sink. Incidence conditions are imposed on the non-compact edges. I.e., one may impose the condition that a non-compact edge $E$ is contained in a certain affine subspace $A_E$, say of codimension $d_E$. Let $L(A_E)$ denote the linear space parallel to $A_E$. Let $n_E \in L(A_E)$ denote the weighted unbounded direction of $E$ (0 if $E$ is contracted). Let $\omega_E$ denote a primitive element of $\Lambda^{d_E} M$ (where $M := N^*$) which is orthogonal to $L(A_E)$, i.e., such that $\iota_n \omega_E = 0$ for all $n \in L(A_E)$. We then associate to $E$ the element

$$z^{n_E} \otimes \omega_E \in P := \mathbb{Z}[N] \otimes \Lambda M,$$

where $\Lambda M := \bigoplus_{d=0}^{\dim N} \Lambda^d M$ is the exterior algebra of $M$. Equivalently, $P$ is the algebra of log polyvector fields on the dual algebraic torus $G_m(M)$. Note that this is an associative algebra under $(z^{n_1} \otimes \omega_1) \cdot (z^{n_2} \otimes \omega_2) := z^{n_1+n_2} \otimes (\omega_1 \wedge \omega_2)$.

We recursively associate an element $f_E = z^{n_E} \otimes \omega_E \in P$ to every edge $E$ of $\Gamma$ as follows. If $E_1, \ldots, E_s$ are the edges flowing into a vertex $V$, and $E_{\text{out}}$ is the edge flowing out of $V$, then $n_{\text{out}} := n_{E_1} + \cdots + n_{E_{\text{out}}}$ (so $n_{\text{out}}$ will be the weighted direction of $E_{\text{out}}$ going against the flow), and

$$f_{E_{\text{out}}} := l_s(f_{E_1}, \ldots, f_{E_s}) := z^{n_{\text{out}}} \otimes l_{n_{\text{out}}} (\omega_{E_1} \wedge \cdots \wedge \omega_{E_s}).$$

Note that $l_s(f_{E_1}, \ldots, f_{E_s}) = l_1(f_{E_1} \cdots f_{E_s})$. Finally, when $E_1, \ldots, E_s$ are the edges flowing into $V_\infty$, rigidity and balancing imply that $f_{E_1} \cdots f_{E_s} = 1 \otimes \omega_\infty$ for some $\omega_\infty \in \Lambda^{\dim N} M$, and we find

$$\text{Mult}(\Gamma) = |\omega_\infty|,$$

the index of $\omega_\infty$ in $\Lambda^{\dim N} M$.

We find that the brackets $l_k$ satisfy many nice properties. $l_1$ is, up to sign, the same as the BV-form obtained as the dual to $d \log$. Restricting to the closed elements $P_0 := \ker(l_1) \subset P$, one finds that $l_2 | P_0$ agrees with the well-known Schouten-Nijenhuis bracket. Furthermore, the brackets $l_k$ make $P_0$ into a strict $L$-infinity algebra. We observe that the usual Jacobi identity is related to the invariance of the tropical counts. This will be used in an update of [Man16] to prove invariance results for tropical curve counts over other Lie algebras, like the quantum torus algebra and the Hall algebra.

For $z^n \otimes \omega \in P$ with $\omega \in \Lambda^d M$, define $\text{deg}(z^n \otimes \omega) := d - 1$. The wall-crossing automorphisms in the Gross-Siebert program have the form $\exp ad f$ for $f$ in the degree 0 part of $P_0$, acting on the degree $(-1)$ part of $P$. We find that the (multi)derivative of this action, i.e., the pushforward action on polyvector fields, is obtained by extending the $\exp ad f$ action to all of $P$. Summarizing, the Gross-Siebert construction of mirror manifolds extends to give polyvector fields on the mirror using the Schouten-Nijenhuis bracket, and this bracket can also be used to compute multiplicities of tropical curves. We hope that this will lead to the discovery of “theta polyvector fields,” along with a description of these in terms of certain descendant log GW invariants, thus extending the Frobenius structure conjecture to higher-degree polyvector fields (and higher-codimension incidence conditions).
This seems to be a log version of the mirror symmetry relationship between quantum cohomology and polyvector fields, which correspond to the Hochschild cohomologies for the $A$- and $B$-models, respectively (cf. [Pas13 §1]).

We also describe a different, more general perspective on these multiplicity computations which works in higher genus and strongly resembles a TCFT like the $A$- and $B$-models of mirror symmetry which we are interested in studying (I am at least tentatively using the term “Tropical Field Theory”). Briefly, we observe that our log polyvector fields form a Frobenius algebra, with associative product being the usual exterior product and Frobenius trace coming from taking the dual pairing with a log holomorphically volume form (which we always have for log Calabi-Yau’s). This is known to give a 2D TQFT, which we can view as associating a map between spaces of polyvector fields to each of our tropical curves (really, to tropical ribbons). We modify this by applying the above-mentioned BV-form whenever we traverse an edge of the tropical curve. The result is a way to associate, to any tropical curve type, a linear map between spaces of log polyvector fields (one factor for each unbounded edge) in a way which is compatible with gluing. Furthermore, the desired tropical multiplicities are recovered by evaluating these linear maps on the log polyvector fields associated to the incidence conditions. I am optimistic that this tropical field theory viewpoint will lead to a clear understanding of the mirror symmetry predictions sketched in the previous paragraph.

4. Quantum refinements and real/open GW invariants

4.1. Quantum refinement of tropical counts and theta functions. There are well-known quantizations of cluster varieties, due to [BZ05] for certain cluster $A$-varieties and [FG09] for cluster $X$-varieties. Essentially, one replaces the clusters Spec $k[N]$ with the quantum torus algebra where $z_1^n z_2^m = q^{\omega(n_1,n_2)} z_1^{n_1+n_2}$ ($\omega$ a skew-symmetric form on $N$ coming from the cluster data) for $n_1, n_2 \in N$ and $q$ a new variable. Given $a \in \mathbb{Z}$, let $[a]_q := q^a - q^{-a}$. The quantized mutations are then automorphisms of the form $\exp a f$ for $f$ a quantum dilogarithm $\sum_{k \geq 1} \frac{(-1)^{k-1}}{k!} q^k z^k e^f$.

In [Man16], I worked with these quantum cluster varieties and constructed quantum theta functions. The quantum version of Theorem 1.3 is exactly the same as the classical version given above, except that $R^w_{\tau}$ is now $R^w_{\tau,j} := \prod_{i,j} \frac{(1-q)^{w_{ij}}}{-q^{w_{ij}}}$, and a vertex that previously had multiplicity $a$ will now have multiplicity $[a]_q$ (as in [BG10]), with the exception of the marked vertex $V_\infty$ whose multiplicity was previously 1 but now is a power of $q$ which depends on the ordering of the theta function multiplication.

4.2. Correspondence with real/open GW invariants. A natural question then is whether there is a Gromov-Witten theoretic description of these quantized versions of the descendant tropical counts $N^w_{\tau,p}$. Indeed, we find such a correspondence theorem in [MRa]. The construction, largely building off the ideas of [Mik16], is as follows:

Let $\text{Log} : N \otimes \mathbb{C}^* \to N \otimes \mathbb{R}$ be the map $n \otimes \lambda \mapsto n \otimes \log |\lambda|$. We obtain a 2-form $\tilde{\omega} := \text{Log}^* \omega$ on $N \otimes \mathbb{C}^*$. Recall from [1] that the tropical curves contributing to $N^w_{\tau,p}$ have weighted unbounded directions $\{p_k\}_{k=1,\ldots,s}$ and $\{w_{ij} e_i^j\}$ of degree $\Delta$. This data is called the degree $\Delta$ of the tropical curve. Let $Y$ denote a smooth toric compactification of $N \otimes \mathbb{C}^*$ which includes as boundary divisors all the $D_{p_k}$ and $D_e$. We are interested in oriented real curves of degree $2\Delta$, meaning that they have order $2|p_k|$ tangency with $D_{p_k}$ for each $k$ and order $2w_{ij}$ tangency with $D_e$ for each $i, j$. Note that the real locus of $C$ separates $C$ into two halves, with the orientation picking out one half which we call $C^+$.

A tropical condition that a non-compact (possibly contracted) edge $E$ be contained in an affine subspace $A_E$ will correspond to the condition that the corresponding marked point of $C$ maps to
Theorem 4.1. \( N_{w,p}^{\text{trop}} \) gives a signed count of the oriented real curves of class \( 2\Delta \) satisfying the conditions \( \mathbb{R} A_E \) for each non-compact edge \( E \) and satisfying a generic \( \psi^{s-2} \)-condition as described above. More precisely, viewing \( N_{w,p}^{\text{trop}} \) as a Laurent polynomial in \( q \), the coefficient of \( q^k \) is the signed count of such curves which satisfy the additional condition that \( \int_{C^+} \tilde{\omega} = \frac{1}{2} k \pi^2 \).

In other words, \( q \) serves as a sort of Novikov variable for \( \tilde{\omega} \). We note that a similar description applies to the counts relevant for scattering diagrams (with the two-dimensional scattering diagram case almost being what is covered by the result of [Mik16]), and also for the two-dimensional genus 0 Block-Göttsche invariants [BG16].

Our approach is similar to that of [NS06] and [MR16] (as opposed to the amoeba-based approach of [Mik16]). That is, we first prove the correspondence in the singular fiber of a toric degeneration, then use log deformation theory to pass to the nearby fiber (this step relies on the doubling of the degree), and then prove invariance as we move the the generic fiber (this is the trickiest step and requires working carefully with explicit homogeneous coordinates for the map). Invariance of the quantized tropical counts then implies the invariance of the real curve counts.

4.3. Future directions. It seems natural now to expect an analog of the Frobenius structure conjecture which relates quantum theta functions to real curves. I hope to develop this in the future.

I also hope to extend these ideas to quantize the Gross-Siebert program in other settings. Whenever the integral affine manifold \( B \) of their construction is equipped with a global integral two-form \( \omega \), I expect that the techniques for quantizing cluster algebras can be used to quantize the whole Gross-Siebert mirror construction, resulting in \( q \)-deformed mirrors equipped with quantum theta functions, hopefully related to counts of real curves in the original space. That is, I expect that a real version of the A-model will be mirror to a \( q \)-deformed version of the B-model.

5. Hall algebras and tropical curves

In [Bri16], Bridgeland defined the notion of a “Hall algebra scattering diagram” associated to a quiver \( Q \) with relations \( I \). In [MC], M.-W. Cheung and I will apply the ideas of [Man16] (sketched in §4.1) to this setting, thus obtaining a description of Hall algebra scattering diagrams in terms of tropical curves whose multiplicities are certain elements of the Hall algebra.

Roughly, elements of the Hall algebra \( H(Q, I) \) are linear combinations of spaces of representations of the quiver with relations \( (Q, I) \). \( H(Q, I) \) has a “convolution product” \( \ast \) making it into an associative \( \mathbb{C}(t) \)-algebra, and commutators with respect to this product (followed by a completion) yield the Lie algebra \( \mathfrak{g}_{\text{Hall}} \) over which the Hall algebra scattering diagram is defined.

Taking dimension vectors \( d \) gives a grading of \( \mathfrak{g}_{\text{Hall}} \) over the lattice \( N \). For each vertex \( i \in Q^{(0)} \), let \( e_i \) denote the corresponding generator of \( N \), and let \( S_i \) denote the corresponding simple representation of \( Q \). For each \( k \in \mathbb{Z}_{\geq 0} \), let \( \delta_k \) be the element of \( H(Q, I) \) corresponding to semisimple representation \( S_i^{\otimes k} \). We consider the initial scattering diagram \( \mathcal{D}_I = \{ (e_i^\perp, f_i) \}_{i \in Q^{(0)}} \), where \( e_i^\perp \subset M_\mathbb{R}, \) and \( f_i := \sum_{k \geq 0} \delta_k \) in \( \exp(\mathfrak{g}_{\text{Hall}}) \). [Bri16] Lemma 11.4 ensures that the resulting consistent scattering diagram \( \mathcal{D} \) will agree with the Hall algebra scattering diagram of [Bri16] Thm. 6.5 if the relations \( I \) come from a “genteel” potential. Theorem 4.3 applies to this setting, as does an analog that describes scattering...
diagrams in terms of tropical curves, or in terms of a refinement called tropical ribbons (which by the ideas in §4.2 will correspond to real curves). We find the following description for the multiplicities of these tropical ribbons:

Let $\Gamma$ be a tropical ribbon contributing to a wall of $\mathcal{D}$ with direction $d_{\text{out}}$, and let $\mathcal{A}_{d_{\text{out}}}$ denote the representations of $(Q, I)$ of dimension vector $d_{\text{out}}$. Let $\Gamma$ be a tropical curve whose unbounded directions are $\{w_{ij}e_i\}_{ij} \cup \{-d_{\text{out}}\}$. Then

$$\text{Mult}(\Gamma) = (-1)^{\sigma(\Gamma)} \sum_{E \in \mathcal{A}_{d_{\text{out}}}} |\mathcal{M}_{\Gamma,E}| \delta_E.$$  

Here, $(-1)^{\sigma(\Gamma)}$ is a certain sign depending on the ribbon structure, and $\mathcal{M}_{\Gamma,E}$ is roughly the space of all composition series $E = E_0 \supset E_1 \supset \cdots \supset E_d = \{0\}$ with the ordering of the simple objects $E_i/E_{i+1}$ determined by an ordering of the unbounded edges of $\Gamma$. $|\mathcal{M}_{\Gamma,E}|$ is then the generalized Poincaré polynomial in $t$ of the space $\mathcal{M}_{\Gamma,E}$. This yields a description of the Hall algebra scattering diagram in terms of tropical ribbons and these associated spaces of composition series.

References


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