Group cohomology and cohomological finiteness conditions

Peter Kropholler
Notes taken by William Malone

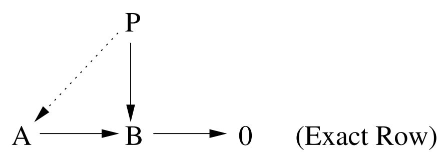
Berkeley Fall 2007

Lecture 1

Lemma 1.1. *(Definition/Lemma/Exercise) Let R be a ring and P a left R-module then the following are equivalent:*

1. *(Pragmatic viewpoint) $P \oplus Q$ is free for some module Q.*

2. *(Category theory viewpoint) Given a surjection $A \overset{\pi}{\twoheadrightarrow} B$ of modules then any map $P \to B$ factors through π. This can also be restated in terms of a diagram*



.

3. *(Homological algebra viewpoint) $Hom_R(P, \ )$ is an exact functor (if you plug in an exact sequence then what you get out is an exact sequence).*

Then P is called projective when these properties hold.

It is worth noting that the third condition needs only to be checked for short exact sequences, and that for any module M, $Hom_R(M, \ )$ is always left exact so it is necessary to check exactness at only one place.

Cohomological functors

Need to have a family of functors $(U^n)_{n\in\mathbb{Z}}$: $R - modules \to abelian\ groups$ + *connecting maps* that satisfy two axioms.

1. (Long Exact Sequence) Given a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ we get a long exact sequence

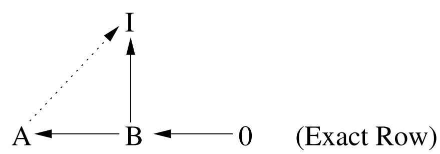
$$\cdots \to U^{n-1} \xrightarrow{\delta} U^nA \to U^nB \to U^nC \xrightarrow{\delta} U^{n+1}A \to \dots \ .$$

2. (optional axiom) $U^n = 0$ for $n < 0$ and $U^n(I) = 0$ for $n > 0$ and I injective.

Later we would like to compute the cohomological functor over spaces and $A \rightarrowtail B \twoheadrightarrow C$ are going to be sheaves which will play the role of coefficients. For completeness here is the definition of an injective module.

Definition 1.2. *An injective module is a module I that results from the dualization of the second and third equivalent conditions in the definition of a projective module. More explicitly, let I be a left R-module then the following are equivalent:*

1. *Given an injection $B \overset{\psi}{\rightarrowtail} A$ of modules then any map $B \to I$ factors through ψ. Or equivalently you have the following diagram*



2. *$Hom(\ ,I)$ is an exact functor.*

For example,

- R is a division ring if and only if every module is free (assume $1 \neq 0$).
- R is a semi-simple Artinian ring (satisfies the descending chain condition on ideals) if and only if every module is projective.
- If R is an integral domain (commutative) then R is a PID if and only if every submodule of a free module is free.

  Example 1.3. *In* $\mathbb{Z}[\sqrt{-5}]$ $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

- If R is a Dedekind domain if and only if every submodule of a free module is projective.

Theorem 1.4. *(Quitten-Suslin 1976) If R is a field or a PID then every finitely generated projective $R[t_1, t_2, \ldots, t_n]$-module is free.*

Example 1.5. *(Barridge-Dunwoody 1979) There exists a 2-generator non-free projective module over $\mathbb{Z}G$ where $G = trefoil$ knot group. It would be nice if there were more examples of this sort, but they are sadly lacking.*

K-theory

Define $K_0(R)$ to be the Grothendieck group of finitely generated modules.

Theorem 1.6. *(Eilenberg-Ganea, Wall 1965) If G is a group with finite cohomological dimension then G has a finite dimensional $K(G,1)$.*

The other direction of the theorem is as follows. Let X be a CW-complex associate to it the cellular chain complex $C_*(X)$ by letting $C_n(X) = H_n(X^n, X^{n-1}) \cong \{$free abelian group on the set of n-cells$\}$ where the isomorphism is a consequence of excision. If X is a $G-CW$ complex (a CW-complex with a G action) then $H_n(X^n, X^{n-1})$ are $\mathbb{Z}G$-modules. If the action is free then $H_n(X^n, X^{n-1})$ are free $\mathbb{Z}G$-modules. If

X is acyclic then $C_*(X) \twoheadrightarrow \mathbb{Z}$ and you get free resolutions. If X is contractible then you get an exact chain. If X is finite dimensional then you get a resolution of finite length which implies that G has finite cohomological dimension.

What we would like is an easy way to reverse the process described above. Suppose that we resolve $\mathbb{Z}$ as follows:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0 \ (\mathbb{Z}G)$$

where all the P_n's are free or projective. This is easy to do just map a massive free module onto the kernal of the previous map. If we had another resolution

$$0 \to L \to Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to \mathbb{Z} \to 0$$

where all the Q_i's are free or projective then we would like to say something about L (which is not known to be either projective or free).

Lemma 1.7 (Schanuel's Lemma)**.** *Given the two resolutions above*

$$P_0 \oplus Q_1 \oplus P_2 \oplus \ldots \cong Q_0 \oplus P_1 \oplus Q_2 \ldots .$$

This implies that L must be a projective module by the first definition since it is a direct summand on one side of the isomorphism given by Schanuel's Lemma and the other side is a free module. Another application of Schanuel's Lemma implies that any resolution of projectives must stop after n-steps. In fact, abstract resolutions will behave in roughly the same way as a projective resolution.

Let's try to build a CW-complex that imitates a free resolution. This normally starts by building a 2-complex to deal with glueing 1-cells to 0-cells, and then glue in additional higher cells to kill elements of homology. If L is free then building this complex called an Eilenberg-MacLane complex and it follows as above. If L is not free and is just projective then what happens?

Theorem 1.8 (Eilenberg Swindle)**.** *Know that $L \oplus L'$ is free then define*

$$F = L \oplus (L' \oplus L) \oplus (L' \oplus L) \ldots \cong L \oplus F$$

and note that F is free.

This allows us to replace our projective module L with the module $L \oplus F$ in the following way to get the last module to be free. IE replace

$$0 \to L \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0$$

with

$$0 \to L \oplus F \to Q_{n-1} \oplus F \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0$$

Starting with a group G that is finitely presented and of type FP_∞ (all modules are finitely generated, but the resolution might go on forever) we can build a $K(G,1)$ of finite type (finite number of cells in each dimension with possibly infinite dimension). If in addition G has $cd < \infty$ then there exists a $K(G,1)$ which is finite dimensional (as a CW-complex the dimensional of all the cells is bounded, but there could be infinitely many). A natural question to ask at this point is when can we build a $K(G,1)$ that is finite (finite dimensional and finite type)? We can not apply the Eilenberg swindle directly since the swindle exploited the fact that F was infinite dimensional. If one can perform a finitely generated version of the swindle then the group G is called stably free.

Wall Obstruction

In $K_0(R)$ look at

$$\sum_{i\geq 0}(-1)^n[P_i].$$

If this is equivalent in the group to a finite rank free module then we can do the swindle with a finitely generated group instead of F. Which is exactly the obstruction to being able to build a finite $K(G,1)$.

Lecture 2

Question 2.1. *Where do cohomology theories come from?*

Answer 2.2. *They can either come from the cohomology of a space or we can define them purely from an algebraic point of view.*

From the algebraic point of view we start with a chain complex of R-modules

$$\cdots \to M_n \xrightarrow{d} M_{n-1} \to \ldots$$

where the composition of any two consecutive maps is zero. Given any R-module N we get a cochain complex

$$\cdots \to Hom_R(M_{n-1}, N) \to Hom_R(M_n, N) \xrightarrow{d_*} Hom_R(M_{n+1}, N) \to \ldots$$

and at the n^{th} position in the sequence we can compute Ker/Im to give us the n^{th} dimensional cohomology with coefficients in N.

If the long exact sequence axiom for cohomological functors is to hold then every short exact sequence of cochain complexes

$$A^* \rightarrowtail B^* \twoheadrightarrow C^*$$

should give us a long exact sequence in cohomology.

$$\begin{array}{ccccc}
A^* & \overset{\iota}{\rightarrowtail} & B^* & \overset{\pi}{\twoheadrightarrow} & C^* \\
\vdots & & \vdots & & \vdots \\
A^n & \rightarrowtail & B^n & \twoheadrightarrow & C^n \\
\downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow \\
A^n & \rightarrowtail & B^n & \twoheadrightarrow & C^n \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

For example if we can create a diagram like the one above then a snake lemma argument gives us a long exact sequence. What we really want is for $Hom_R(M_*,)$ to produce short exact sequences when applied to short exact sequences. Thus we need $Hom_R(M_*,)$ to be an exact functor which implies that all the M_n should be projective. Summarizing the above, in order to get a cohomology theory we need a chain complex of projective modules.

Finiteness Conditions

Let R be a ring and M an R-module then

$Projdim_R(M) < \infty$ if and only if M has a projective (free) resolution of finite length.

M **is** $FP_\infty = FL_\infty$ if and only if M has a projective (free) resolution of finite type.

M **is** FP if and only if M has a finite projective resolution.

M **is** FL if and only if M has a finite free resolution.

If a group acts freely and cocompactly on [?] then G is of type FL. If M has $projdim < \infty$ and is of type FP_∞ then it is of type FP.

Let us recall the definition of $K_0(R)$.

Definition 2.3. *$K_0(R)$ is the free abelian group generated by $\{[P]$, where P is a finitely generated projective R-module and $[P]$ is the set of finitely generated projectives $\}$ modulo the relations*

$$[P \oplus Q] - [P] - [Q]$$

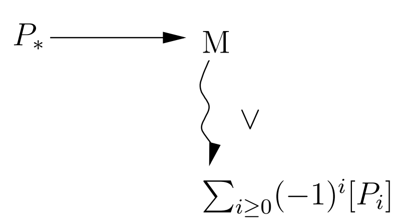
which allow us to do subtraction in this group.

Definition 2.4. *$K_0^{fp}(R)$ is defined by taking the free abelian group on $\{[M] | M$ is of type $FP\}$ modulo the relations $[M] - [M'] - [M'']$ whenever*

$$M'' \rightarrowtail M \twoheadrightarrow M'$$

is a short exact sequence.

For any ring R there is a isomorphism $\vee : K_0^{fp}(R) \to K_0(R)$ given by the following.



In addition there is a map going the other direction $\wedge : K_0(R) \to K_0^{fp}(R)$ given by $[\hat{P}] = [P]$.

2-3 condition If $\mathcal{C}$ is a class of R-modules then $\mathcal{C}$ has the $2-3$ condition if and only if whenever $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in which at least 2 out of A, B, C are in $\mathcal{C}$ then they are all in $\mathcal{C}$.

FCL Say that $\mathcal{C}$ satisfies FCL if and only if it is closed under filtered colimits.

Definition 2.5. *(filtered colimits) Colimits go forward.*

Example 2.6.

$$\varinjlim(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{\times 5} \dots) = \mathbb{Q}$$

Example 2.7.

$$\varinjlim(\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 0} \dots) = 0$$

Theorem 2.8. *(Lazard's Criterion 1960's) M is flat if and only if M is a filtered colimit of projective modules.*

Definition 2.9. *A left R-module M is flat if $\_ \otimes_R M$ is exact.*

Similarly to previous theorems that we have seen we can replace the projective assumption in Lazard's Criterion with free since

$$\varinjlim(P \oplus Q \xrightarrow{P_1} P \oplus Q \xrightarrow{P_2} \dots) = P$$

Lemma 2.10. *Let R be a ring and $\mathcal{C}$ the smallest $(2-3)$-closed and FCL-closed that contains the free rank one R-module R, then every module of type FP_∞ in $\mathcal{C}$ has finite projective dimension.*

One aside would be that if $G \in LHF$ then $\mathcal{C}(\mathbb{Q}G) = \{\text{all } \mathbb{Q}G\text{-modules}\}$.

Proof. (proof of 2.10) The proof uses complete cohomology $\widehat{Ext}^*_R(M, N)$. Fix M then $\widehat{Ext}^n_R(M, \_)$ for $n \in \mathbb{Z}$ is a cohomological functor similar to $Ext^n_R(M, \_)$ which is the classical cohomological functor. There are four nice properties that our new functor has namely:

- There is a long exact sequence axiom for $\widehat{Ext}^*_R(M, \_)$.
- $\widehat{Ext}^*_R(M, P) = 0$ when P is projective.
- If M is FP_∞ then $\widehat{Ext}^*_R(M, \_)$ commutes with filtered colimits.
- $\widehat{Ext}^0_R(M, N) = 0$ if and only if the projective dimension $Proj\ dim_R M < \infty$.

The first and the third bullets regular Ext satisfies as well. The second and the fourth properties regular Ext does not have.

To start the proof fix M of type FP_∞ and look at the class $\mathcal{X}$ of modules N such that $\widehat{Ext}^*_R(M, N) = 0$. We will take on trust that a complete cohomology theory exists that has this property. Then $\mathcal{X}$ contains all projective modules and specifically R. By the long exact sequence bullet $\mathcal{X}$ has the (2-3)-condition. The third bullet then says that $\mathcal{X}$ satisfies FCL and thus $\mathcal{C} \subset \mathcal{X}$. If $M \in \mathcal{C}$ then $M \in \mathcal{X}$ and $\widehat{Ext}^*_R(M, M) = 0$. By the fourth bullet M has finite projective dimension. □

Lecture 3

Let $\mathcal{X}$ be a class of groups. We would like to build a bigger class of groups. One method is to define $H_1\mathcal{X}$ to be the class of all groups G such that there exists a finite dimensional contractible G-complex (G-CW complex in which cell stabilizers fix cells pointwise) X with isotropy subgroups in $\mathcal{X}$. Philip Hall developed some notation for different classes of groups and Peter Kropholler defined $H\mathcal{X}$.

$p\mathcal{X}$ "Poly $\mathcal{X}$" is the smallest extension closed class containing $\mathcal{X}$.

$Q\mathcal{X}$ All quotients of $\mathcal{X}$-groups.

$R\mathcal{X}$ Residually $\mathcal{X}$-groups.

$S\mathcal{X}$ Subgroups of $\mathcal{X}$-groups.

$H\mathcal{X}$ Smallest H_1-closed class containing $\mathcal{X}$.

Now we would like to look at the class of groups $\mathcal{F}$ which is the class of all finite groups. $LH\mathcal{F}$ is the class of all groups whose finitely generated subgroups are in $H\mathcal{F}$. One nice property that this class enjoys is that it is closed under L, H, P, and S.

Definition 3.1. *For each ordinal α define H_α by*

$$H_\alpha\mathcal{X} = \begin{cases} H_1(H_{\alpha-1}\mathcal{X}) \; if \; \alpha \; is \; a \; succesor \\ \cup_{\beta<\alpha} H_\beta\mathcal{X} \; if \; \alpha \; is \; a \; limit \; ordinal \end{cases}$$

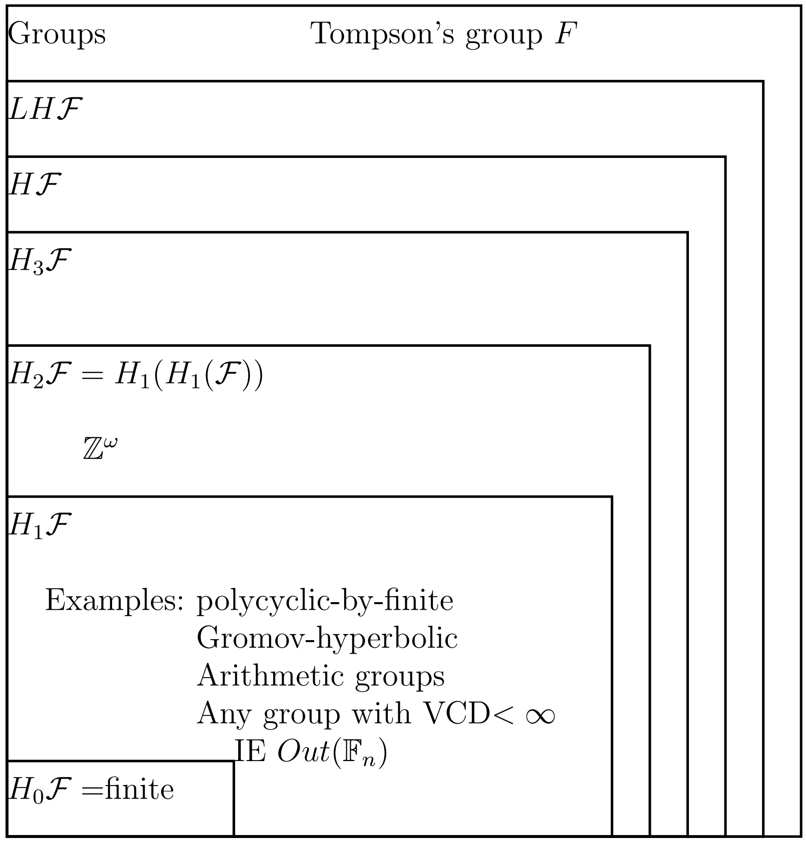
Lemma 3.2. *If G is in $LH\mathcal{F}$ then $\mathcal{C}(\mathbb{Q}G) =$"all $\mathbb{Q}G$-modules". Said another way this is the smallest class of $\mathbb{Q}G$-modules which contains all projectives and is closed under the 2-3 condtion and FCL.*

Proof. (Sketch) Fix some group G and consider the set $\mathcal{S}$ of subgroups of H such that $\mathbb{Q}G \otimes_{\mathbb{Q}H} M$ belongs to $\mathcal{C}(\mathbb{Q}G)$ for all $\mathbb{Q}H$-modules M. It suffices to prove that $G \in \mathcal{S}$. We will prove that

1. All subgroups of G in $H\mathcal{F}$ belong to $\mathcal{S}$.

2. All subgroups of G in $LH\mathcal{F}$ belong to $\mathcal{S}$.

Clearly $\mathcal{S}$ contains all finite subgroups.



Exercise 3.3. *Show that you can replact the 2-3 condition with a $(n-1)-n$ condition via simple induction.*

The above exercise implies that $\mathcal{S}$ is H_1-closed or more formally we will show that $\forall\alpha$ if $H \leq G$ and $H \in H_\alpha\mathcal{F}$ then $H \in \mathcal{S}$.

Assume that $\alpha = \beta + 1$. Then H acts on X (some finite dimensional contractible (ie exact) H-complex).

$$\begin{array}{c} \hspace{5em} X \\ \hspace{5em} \downarrow \\ 0 \longrightarrow C_n(X) \longrightarrow \dots \longrightarrow C_1(X) \longrightarrow C_0(X) \twoheadrightarrow \mathbb{Q} \longrightarrow 0 \end{array}$$

Apply $\mathbb{Q}G \otimes_{\mathbb{Q}H}$ _ to the cellular chain complex of X and then $M \otimes_{\mathbb{Q}}$ _. This gives you an exact sequence of finite length with $n-1$ modules belonging to $\mathcal{S}$ except one at the far right hand end. Step 2 then follows from [?].

□

Corollary 3.4. *$LH\mathcal{F}$ groups have type FP_∞ and finite cohomological dimension over $\mathbb{Q}$.*

One application of this involves $\mathbb{Q}$. Namely we can consider $\mathbb{Q}$ as a FP_∞-module over $\mathbb{Q}G$ and hence $projdim_{\mathbb{Q}G}(\mathbb{Q}) < \infty$.

Theorem 3.5. *(K-Mislin 1999) If $G \in LH\mathcal{F}$ and is of type FP_∞ over $\mathbb{Z}$ then there exists a finite dimensional model for $\underline{E}G$. In particular $G \in H_1\mathcal{F}$. ($\underline{E}G$ is a G-complex X with finite isotropy such that for all $H \in G$ X^H is contractible if H is finite and empty if H is infinite)*

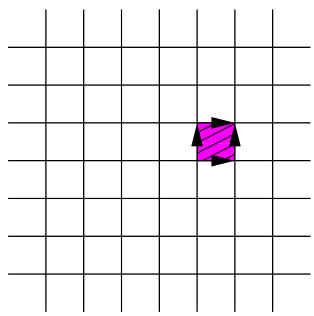
It is important to see that not all groups belong to $LH\mathcal{F}$. Two classes of examples are:

1. Thompson's Group F which has the presentation $< x_0, x_1, x_2, \dots | x_i^{-1} x_n x_i = x_{n+1}\ i < n >$ is FP_{infty} (Brown-Geoghegan) and $cd_{\mathbb{Z}} = cd_{\mathbb{Q}} = \infty$.
2. Every infinite $H\mathcal{F}$ group G admits a finite dimensional contractible G-complex without a global fixed point.

An important recent result in this area is due to (Arzhanteeva-Minasyan-Osin) involving SQ-universality of hyperbolic groups. One important line of reasearch being currently undertaken is to use this to construct groups that are $H\mathcal{F}$ but not $H_\alpha\mathcal{F}$ for all countable α ie $H_4\mathcal{F} > H_3\mathcal{F}$ and so on.

Lecture 4

$G = \mathbb{Z}^2$ acts on $\mathbb{R}^2$ to give a cell decomposition of the plane. Take the cellular chain



complex

$$0 \to C_2 \xrightarrow{d} C_1 \to C_0 \to \mathbb{Z} \to 0 (\textit{exact chain complex})$$

$$0 \to \mathbb{Z}G \to \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0 (\textit{chains of group elements})$$

which has one orbit of $0 - cells$ (are in bijection with G).

Want the space the group is acting on to be contractible and the actions to be free. If not then we need spectral sequences. If the group is not acting freely, then we get $H < G$ with $\mathbb{Z}[H \backslash G]$ and we have an exact sequence but maybe the modules are not free. The modules are permutation modules in general.

L-H-S spectral sequence

Given $K \trianglelefteq G$ and a free G/K-action on a contractible cell complex we get the spectral sequence

$$H^p(G/K, H^q(K, M)) = H^{p+q}(G, M)$$

Thompson's Group

K is free abelian of rank ∞. In $F_0 < F$ (of index 2) $K/(K \cap K^f)$ is abelian of finite rank for all $f \in F_0$.

Grothendieck spectral sequence

- $Hom_{\mathbb{Z}G}(P_*, \_)$
- $Hom_{\mathbb{Z}G}(\mathbb{Z}[G/K], \underline{M}) = M^k = H^0(K, M)$ where M^K picks out K fixed points and in particular it is a G/K-module.
- $H^0(G, M) = M^G$

Let M be any $\mathbb{Z}G$-module then

$$M \mapsto \cup_{H \in \mathcal{S}} M^H \xrightarrow{(\ )^G} .$$

What you get is a set that G acts on. Moreover $\varprojlim G/H$ inherits a monoid structure.

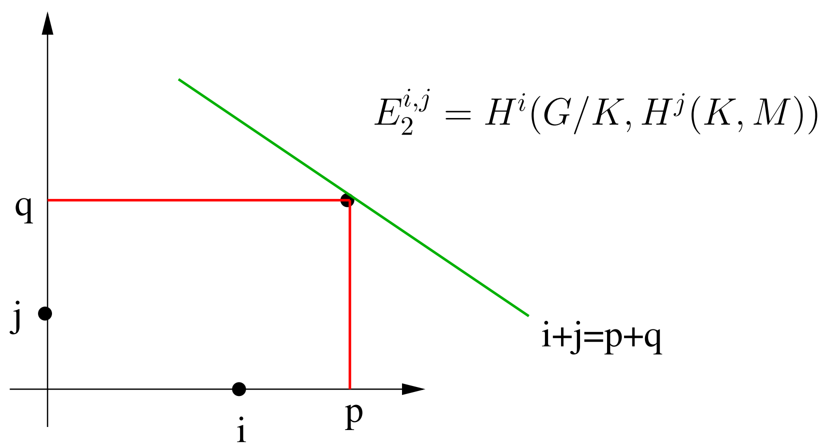
Lecture 5

Spectral Sequence Corner Argument

We are going to do the case of groups. Suppose we have a group extension

$$K \rightarrowtail G \twoheadrightarrow G/K$$

and M is a G-module. Let $cd(K) = q$ and $cd(G/K) = p$ then $cd(G) \leq p + q$ and $H^{p+q}(G, M) = H^p(G/K, H^q(K, M))$ which is the $L - H - S$ spectral sequence. Here



we are assuming that $p, q \geq 1$. The cohomological dimension conditions say that you have 0's outside the box and by the degree of the pages the (p, q) entry in never changing so $E_2^{p,q} = E_\infty^{p,q}$. Choose an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \dots$$

and pass to the cochain complex I^* of injectives (exact except at I^0)

$$0 \to 0 \to 0 \to I^0 \to I^1 \to I^2 \to \dots \ .$$

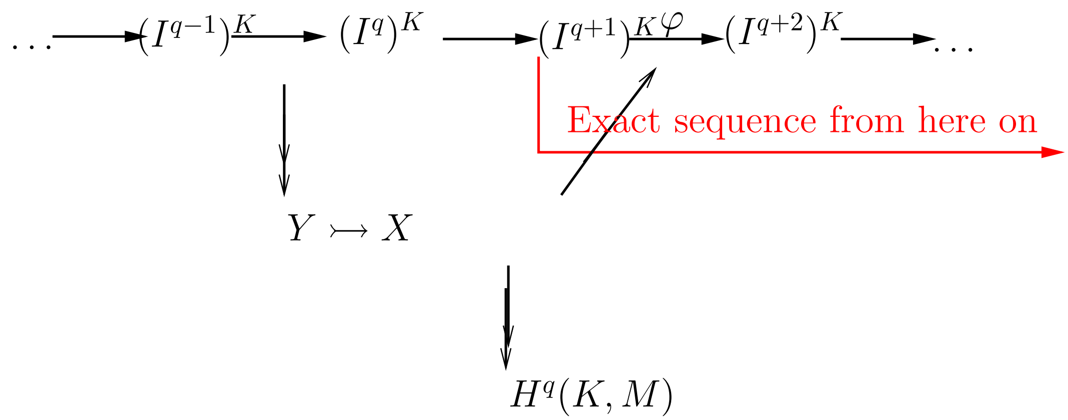
Now apply $H^0(G, \_) = (\ \ )^G = (G\text{fixed points in resolution})$ as a composite. $H^0(G/K, H^0(K, \ \ ))$
N.B. I^j is injective as a $K - module$

$$\dots \to (I^0)^K \to (I^1)^K \to (I^2)^K \to \dots$$

and

$$H^j(K,M) = \frac{Ker((I^j)^K \to (I^{j+1})^K)}{Im((I^{j-1})^K \to (I^j)^K)}.$$

We are working with injectives for theoretical reasons. Working out $Ext^*_R(A,B)$ we could use a projective resolution of A or an injective resolution of B. $H^*(G,M) = Ext^*_{\mathbb{Z}G}(\mathbb{Z},M)$ so apply H^0 to get



We would like to prove

$$Hom_{G/K}(\ \ , I^K) = Hom_G(\ \ , I)$$

on G/K modules (the equal sign is an adjunction and $Hom_G(\ \ ,I)$ is exact if I is injective).

$$\dots \to 0 \to (I^q)^K \to (I^{q+1})^K \to (I^{q+2})^K \to \dots$$

then apply $H^0(G/K,\ \ )$ to get

$$\dots \to (I^q)^G \to (I^{q+1})^G \to \dots \to (I^{q+j})^G \to \dots$$

where

$$(I^{q+j})^G \rightsquigarrow H^j(G/K,X) = H^{q+j}(G,M) = 0 \; if \; j > p.$$

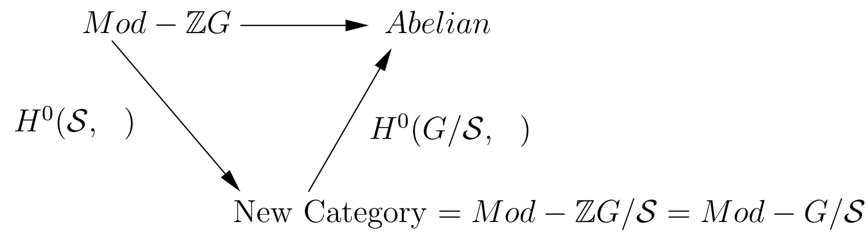
This implies $H^p(G/K,X) \cong H^p(G/K,H^q(K,M))$.

Recall that $\mathcal{S}$ is an admissable family of subgroups of G. In particular, $\mathcal{S}$ is closed under conjugation and finite intersections. Assume that $\mathcal{S} \neq \emptyset$. This new category is the full subcategory of $\mathbb{Z}G$ modules consisting of objects M such that

$$M = \cup_{H \in \mathcal{S}} M^H.$$

We need to know the following three things.

1. $Mod - G/\mathcal{S}$ has enough injectives



2. What is $H^0(\mathcal{S},\ )$? Answer is $H^0(\mathcal{S}, M) = \cup_{H\in\mathcal{S}} M^H$
3. If I is G-injective then $H^0(\mathcal{S}, I)$ is $G/\mathcal{S}$-injective

The three items above are crucial things to check, but once we know that property one holds we get property 3 basically for free.

$G, \mathcal{S}$ are given so define

$$\widehat{G}_{\mathcal{S}} = \varprojlim_{H\in\mathcal{S}} H\backslash G$$

in spirit where $H\backslash G = \{Hg | g \in G\}$. Define $\widehat{G}_{\mathcal{S}} = \{$set of functions $f : \mathcal{S} \to \mathcal{P}(G)$ such that $\forall H \in \mathcal{S}$, $f(H) \in H\backslash G$ *and* $f(K) \subset f(H)$ *whenever* $K \subset H\}$. Given $f \in \widehat{G}_{\mathcal{S}}$ and $x \in G$ such that $f(H) = Hx$ then define $H^f = H^x = x^{-1}Hx$.

Exercise 5.1. *Check that the above definition is well defined.*

Then we can make $\widehat{G}_{\mathcal{S}}$ into a monoid with multiplication given by

$$f \cdot f'(H) = f(H)f'(H^f).$$

If $f(H) = Hx$ then $f \cdot f'(H) = HxH^x_y$ for some y which is then equal to H_{xy}. Thus

$$\begin{aligned}(f\cdot f')\cdot f''(H) &= f\cdot f'(H)f''(H^{f\cdot f'})\\ &= f(H)f'(H^f)f''(H^{f\cdot f'}) \text{ and}\\ f\cdot(f'\cdot f'')(H) &= f(H)f'(H^f)f''((H^f)^{f'})\end{aligned}$$

Now it is not hard to prove the lemma to check that associativity holds.

The next natural question is when is this monoid a group and as a partial answer we have the following lemma.

Lemma 5.2. *If for all $K \subset H$ both in $\mathcal{S}$ there exists an $L \trianglelefteq H$, $L \subset K$, and $L \in \mathcal{S}$ then $\widehat{G}_{\mathcal{S}}$ is a group.*

Lecture 6

Examples of admissable families

1. (Residually finite groups G) Then $\mathcal{S}$ is the set of subgroups of finite index, $Mod{-}G/\mathcal{S}$ is the category of discrete $\widehat{G}$-modules and we have (Peter's Notation)

   $$H^*(G/\mathcal{S}, M) = \underrightarrow{lim}_{U \trianglelefteq_f G}(H^*(G/U, M^U))$$

   which is classical Galois cohomology. There is an inflation map

   $$\underrightarrow{lim}_{U \trianglelefteq_f G}(H^*(G/U, M^U)) \to H^*(G, M)$$

   which is forgetful, by which we mean that it maps continuous cocycles to the corresponding cocycles.

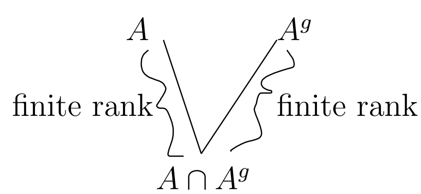
2. (Commensurated Case) E.g. $\mathcal{S}$ is the set of subgroups commensurable with $H \leq G$ is admissable provided $Comm_G(H) = G$ (i.e. H is commensurated).

   Example 6.1. *$SL_n(\mathbb{Z}) \subset SL_n(\mathbb{Q})$ where $< x > \leq < x, y | (x^p)^y = x^q >$ (i.e. H is commensurated).*

3. (Thompson's Group F) In this group there are many admissable sets of subgroups

   $$E.g. \ < x_1x_0^{-1}, x_3x_2^{-1}, x_5x_4^{-1}, \cdots >= A.$$

   For all $g \in F_0 < F$ ($F_0 < F$ of index 2) we have that $A \cap A^g$ is free abelian of infinite rank. We can combine this with a spectral sequence to deduce



$$H^*(F, M) = 0 \ if \ H^*(B, M = 0) \ where \ B \in \mathcal{S}$$

.

4. (An example where you do not get a group) Let $G = Perm(\mathbb{N})$ and $\mathcal{S} = \{Fix(S) | S\ finite \subset \mathbb{N}\}$ then $\{fix\{1\}, fix\{1,2\}, fix\{1,2,3\}, \dots\} \subset \mathcal{S}$. Now

$$Fix1,2,\dots,n\backslash Perm(\mathbb{N}) \hookrightarrow \mathbb{N} \times \dots \mathbb{N}$$

given by the map $\sigma \mapsto ((1)\sigma, (2)\sigma, \dots, (n)\sigma)$ and this is well defined. In this case

$$\varprojlim Fix(S)\backslash Perm(\mathbb{N}) \cong Monom(\mathbb{N})$$

where $Monom(\mathbb{N})$ are the monomorphisms of $\mathbb{N}$. Here we see that $n \mapsto n+1$ has no inverse and thus there is no group structure.

Groups and Modules of type FP_∞

Theorem 6.2. *(Mislin-K) $H\mathcal{F}$-groups of type FP_∞ belong to $H_1(\mathcal{F})$*

A stronger theorem is in fact true namely:

Theorem 6.3. *(Mislin-K) $H\mathcal{F}$-groups of type FP_∞ have a finite dimensional $\underline{E}$ where $\underline{E}G$ is a G-complex X such that*

1. *X^H is contractible for finite $H \leq G$.*
2. *$X^H = \emptyset$ is contractible for infinite $H \leq G$.*

Open Problem 6.4. *Does every $H_1\mathcal{F}$-group have a finite dimensional $\underline{E}G$?*

Question 6.5. *If G has a finite dimensional $\underline{E}$ then is the poset of non-trivial finite subgroups G-homotopy equivalent to a finite dimensional complex?*

There are theorems about modules of type FP_∞ over $H\mathcal{F}$ groups which is closely related to permuation modules with finite isotropy.