Group cohomology and cohomological finiteness conditions (Lecture 3)

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Let $\mathcal{X}$ be a class of groups. We would like to build a bigger class of groups. One method is to define $H_1\mathcal{X}$ to be the class of all groups G such that there exists a finite dimensional contractible G-complex (G-CW complex in which cell stabilizers fix cells pointwise) X with isotropy subgroups in $\mathcal{X}$. Philip Hall developed some notation for different classes of groups and Peter Kropholler defined $H\mathcal{X}$.

$p\mathcal{X}$ "Poly $\mathcal{X}$" is the smallest extension closed class containing $\mathcal{X}$.

$Q\mathcal{X}$ All quotients of $\mathcal{X}$-groups.

$R\mathcal{X}$ Residually $\mathcal{X}$-groups.

$S\mathcal{X}$ Subgroups of $\mathcal{X}$-groups.

$H\mathcal{X}$ Smallest H_1-closed class containing $\mathcal{X}$.

Now we would like to look at the class of groups $\mathcal{F}$ which is the class of all finite groups. $LH\mathcal{F}$ is the class of all groups whose finitely generated subgroups are in $H\mathcal{F}$. One nice property that this class enjoys is that it is closed under L, H, P, and S.

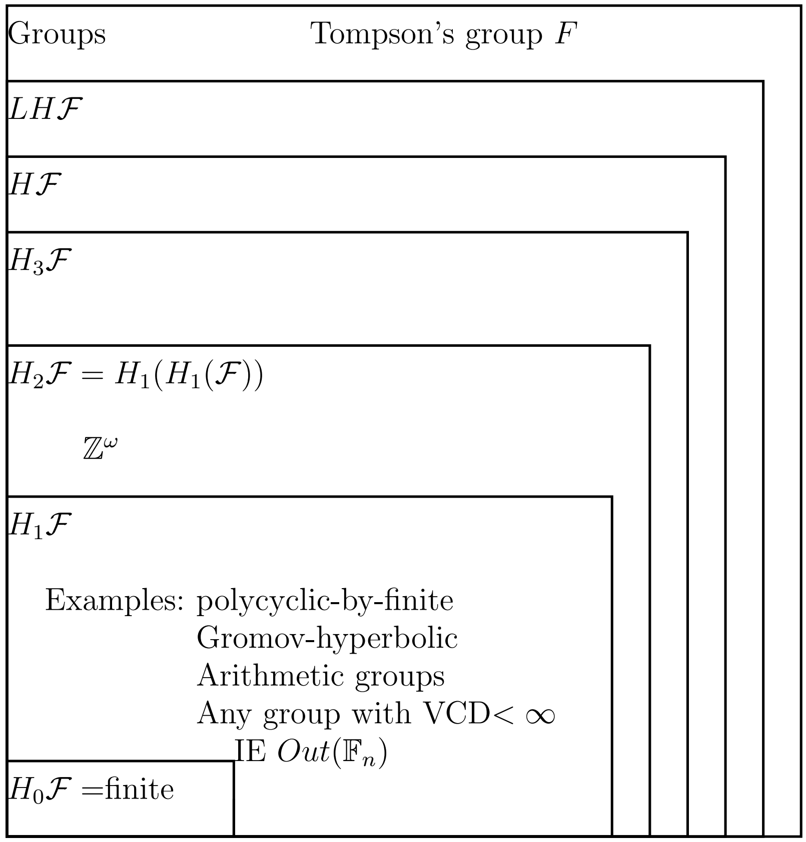
Definition 1. *For each ordinal α define H_α by*

$$H_\alpha\mathcal{X} = \begin{cases} H_1(H_{\alpha-1}\mathcal{X}) \text{ } if \text{ } \alpha \text{ } is \text{ } a \text{ } succesor \\ \cup_{\beta<\alpha} H_\beta\mathcal{X} \text{ } if \text{ } \alpha \text{ } is \text{ } a \text{ } limit \text{ } ordinal \end{cases}$$

Lemma 2. *If G is in $LH\mathcal{F}$ then $\mathcal{C}(\mathbb{Q}G) =$ "all $\mathbb{Q}G$-modules". Said another way this is the smallest class of $\mathbb{Q}G$-modules which contains all projectives and is closed under the 2-3 condtion and FCL.*

Proof. (Sketch) Fix some group G and consider the set $\mathcal{S}$ of subgroups of H such that $\mathbb{Q}G \otimes_{\mathbb{Q}H} M$ belongs to $\mathcal{C}(\mathbb{Q}G)$ for all $\mathbb{Q}H$-modules M. It suffices to prove that $G \in \mathcal{S}$. We will prove that

1. All subgroups of G in $H\mathcal{F}$ belong to $\mathcal{S}$.



2. All subgroups of G in $LH\mathcal{F}$ belong to $\mathcal{S}$.

Clearly $\mathcal{S}$ contains all finite subgroups.

Exercise 3. *Show that you can replact the 2-3 condition with a $(n-1)-n$ condition via simple induction.*

The above exercise implies that $\mathcal{S}$ is H_1-closed or more formally we will show that $\forall \alpha$ if $H \leq G$ and $H \in H_\alpha\mathcal{F}$ then $H \in \mathcal{S}$.

Assume that $\alpha = \beta + 1$. Then H acts on X (some finite dimensional contractible (ie exact) H-complex).

$$\begin{array}{ccccccccccccc} & & & & & & & & X & & & & \\ & & & & & & & & \downarrow & & & & \\ 0 & \to & C_n(X) & \to & \dots & \to & C_1(X) & \to & C_0(X) & \twoheadrightarrow & \mathbb{Q} & \to & 0 \end{array}$$

Apply $\mathbb{Q}G \otimes_{\mathbb{Q}H} \_$ to the cellular chain complex of X and then $M \otimes_{\mathbb{Q}} \_$. This gives you an exact sequence of finite length with $n-1$ modules belonging to $\mathcal{S}$ except one at the far right hand end. Step 2 then follows from [?].

□

Corollary 4. *$LH\mathcal{F}$ groups have type FP_∞ and finite cohomological dimension over $\mathbb{Q}$.*

One application of this involves $\mathbb{Q}$. Namely we can consider $\mathbb{Q}$ as a FP_∞-module over $\mathbb{Q}G$ and hence $projdim_{\mathbb{Q}G}(\mathbb{Q}) < \infty$.

Theorem 5. *(K-Mislin 1999) If $G \in LH\mathcal{F}$ and is of type FP_∞ over $\mathbb{Z}$ then there exists a finite dimensional model for $\underline{E}G$. In particular $G \in H_1\mathcal{F}$. ($\underline{E}G$ is a G-complex X with finite isotropy such that for all $H \in G$ X^H is contractible if H is finite and empty if H is infinite)*

It is important to see that not all groups belong to $LH\mathcal{F}$. Two classes of examples are:

1. Thompson's Group F which has the presentation $< x_0, x_1, x_2, \ldots | x_i^{-1} x_n x_i = x_{n+1} \; i < n >$ is FP_{infty} (Brown-Geoghegan) and $cd_{\mathbb{Z}} = cd_{\mathbb{Q}} = \infty$.
2. Every infinite $H\mathcal{F}$ group G admits a finite dimensional contractible G-complex without a global fixed point.

An important recent result in this area is due to (Arzhanteeva-Minasyan-Osin) involving SQ-universality of hyperbolic groups. One important line of reasearch being currently undertaken is to use this to construct groups that are $H\mathcal{F}$ but not $H_\alpha\mathcal{F}$ for all countable α ie $H_4\mathcal{F} > H_3\mathcal{F}$ and so on.