Group cohomology and cohomological finiteness conditions (Lecture 2)

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Question 1. *Where do cohomology theories come from?*

Answer 2. *They can either come from the cohomology of a space or we can define them purely from an algebraic point of view.*

From the algebraic point of view we start with a chain complex of R-modules

$$\cdots \to M_n \xrightarrow{d} M_{n-1} \to \dots$$

where the composition of any two consecutive maps is zero. Given any R-module N we get a cochain complex

$$\cdots \to Hom_R(M_{n-1}, N) \to Hom_R(M_n, N) \xrightarrow{d_*} Hom_R(M_{n+1}, N) \to \dots$$

and at the n^{th} position in the sequence we can compute Ker/Im to give us the n^{th} dimensional cohomology with coefficients in N.

If the long exact sequence axiom for cohomological functors is to hold then every short exact sequence of cochain complexes

$$A^* \rightarrowtail B^* \twoheadrightarrow C^*$$

should give us a long exact sequence in cohomology.

$$\begin{array}{ccccc}
A^* & \overset{\iota}{\rightarrowtail} & B^* & \overset{\pi}{\twoheadrightarrow} & C^* \\
\vdots & & \vdots & & \vdots \\
A^n & \rightarrowtail & B^n & \twoheadrightarrow & C^n \\
\downarrow & \circlearrowright & \downarrow & \circlearrowright & \downarrow \\
A^n & \rightarrowtail & B^n & \twoheadrightarrow & C^n \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

For example if we can create a diagram like the one above then a snake lemma argument gives us a long exact sequence. What we really want is for $Hom_R(M_*,)$

to produce short exact sequences when applied to short exact sequences. Thus we need $Hom_R(M_*,)$ to be an exact functor which implies that all the M_n should be projective. Summarizing the above, in order to get a cohomology theory we need a chain complex of projective modules.

Finiteness Conditions

Let R be a ring and M an R-module then

$Projdim_R(M) < \infty$ if and only if M has a projective (free) resolution of finite length.

M **is** $FP_\infty = FL_\infty$ if and only if M has a projective (free) resolution of finite type.

M **is** FP if and only if M has a finite projective resolution.

M **is** FL if and only if M has a finite free resolution.

If a group acts freely and cocompactly on [?] then G is of type FL. If M has $projdim < \infty$ and is of type FP_∞ then it is of type FP.

Let us recall the definition of $K_0(R)$.

Definition 3. *$K_0(R)$ is the free abelian group generated by $\{[P]$, where P is a finitely generated projective R-module and $[P]$ is the set of finitely generated projectives $\}$ modulo the relations*

$$[P \oplus Q] - [P] - [Q]$$

which allow us to do subtraction in this group.

Definition 4. *$K_0^{fp}(R)$ is defined by taking the free abelian group on $\{[M] | M$ is of type $FP\}$ modulo the relations $[M] - [M'] - [M'']$ whenever*

$$M'' \rightarrowtail M \twoheadrightarrow M'$$

is a short exact sequence.

For any ring R there is a isomorphism $\vee : K_0^{fp}(R) \to K_0(R)$ given by the following.

$$\begin{array}{ccc} P_* & \longrightarrow & M \\ & & \rightsquigarrow \vee \\ & & \sum_{i \geq 0} (-1)^i [P_i] \end{array}$$

In addition there is a map going the other direction $\wedge : K_0(R) \to K_0^{fp}(R)$ given by $[\hat{P}] = [P]$.

2-3 condition If $\mathcal{C}$ is a class of R-modules then $\mathcal{C}$ has the $2-3$ condition if and only if whenever $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in which at least 2 out of A, B, C are in $\mathcal{C}$ then they are all in $\mathcal{C}$.

FCL Say that $\mathcal{C}$ satisfies FCL if and only if it is closed under filtered colimits.

Definition 5. *(filtered colimits) Colimits go forward.*

Example 6.

$$\varinjlim(\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \stackrel{\times 3}{\to} \mathbb{Z} \stackrel{\times 4}{\to} \mathbb{Z} \stackrel{\times 5}{\to} \dots) = \mathbb{Q}$$

Example 7.

$$\varinjlim(\mathbb{Z} \stackrel{\times 0}{\to} \mathbb{Z} \stackrel{\times 0}{\to} \dots) = 0$$

Theorem 8. *(Lazard's Criterion 1960's) M is flat if and only if M is a filtered colimit of projective modules.*

Definition 9. *A left R-module M is flat if $\_ \otimes_R M$ is exact.*

Similarly to previous theorems that we have seen we can replace the projective assumption in Lazard's Criterion with free since

$$\varinjlim(P \oplus Q \stackrel{P_1}{\to} P \oplus Q \stackrel{P_2}{\to} \dots) = P$$

Lemma 10. *Let R be a ring and $\mathcal{C}$ the smallest $(2-3)$-closed and FCL-closed that contains the free rank one R-module R, then every module of type FP_∞ in $\mathcal{C}$ has finite projective dimension.*

One aside would be that if $G \in LHF$ then $\mathcal{C}(\mathbb{Q}G) = \{\text{all } \mathbb{Q}G\text{-modules}\}$.

Proof. (proof of 10) The proof uses complete cohomology $\widehat{Ext}^*_R(M, N)$. Fix M then $\widehat{Ext}^n_R(M, \_)$ for $n \in \mathbb{Z}$ is a cohomological functor similar to $Ext^n_R(M, \_)$ which is the classical cohomological functor. There are four nice properties that our new functor has namely:

- There is a long exact sequence axiom for $\widehat{Ext}^*_R(M, \_)$.
- $\widehat{Ext}^*_R(M, P) = 0$ when P is projective.
- If M is FP_∞ then $\widehat{Ext}^*_R(M, \_)$ commutes with filtered colimits.
- $\widehat{Ext}^0_R(M, N) = 0$ if and only if the projective dimension $Proj\ dim_R M < \infty$.

The first and the third bullets regular Ext satisfies as well. The second and the fourth properties regular Ext does not have.

To start the proof fix M of type FP_∞ and look at the class $\mathcal{X}$ of modules N such that $\widehat{Ext}^*_R(M, N) = 0$. We will take on trust that a complete cohomology theory exists that has this property. Then $\mathcal{X}$ contains all projective modules and specifically R. By the long exact sequence bullet $\mathcal{X}$ has the (2-3)-condition. The third bullet then says that $\mathcal{X}$ satisfies FCL and thus $\mathcal{C} \subset \mathcal{X}$. If $M \in \mathcal{C}$ then $M \in \mathcal{X}$ and $\widehat{Ext}^*_R(M, M) = 0$. By the fourth bullet M has finite projective dimension. □