

# Group cohomology and cohomological finiteness conditions (Lecture 1)

Peter Kropholler

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**Lemma 1.** (Definition/Lemma/Exercise) Let  $R$  be a ring and  $P$  a left  $R$ -module then the following are equivalent:

1. (Pragmatic viewpoint)  $P \oplus Q$  is free for some module  $Q$ .
2. (Category theory viewpoint) Given a surjection  $A \xrightarrow{\pi} B$  of modules then any map  $P \rightarrow B$  factors through  $\pi$ . This can also be restated in terms of a diagram

$$\begin{array}{ccccc}
 & & P & & \\
 & \nearrow \text{dotted} & \downarrow & & \\
 A & \longrightarrow & B & \longrightarrow & 0 \quad (\text{Exact Row})
 \end{array}$$

3. (Homological algebra viewpoint)  $\text{Hom}_R(P, \_)$  is an exact functor (if you plug in an exact sequence then what you get out is an exact sequence).

Then  $P$  is called projective when these properties hold.

It is worth noting that the third condition needs only to be checked for short exact sequences, and that for any module  $M$ ,  $\text{Hom}_R(M, \_)$  is always left exact so it is necessary to check exactness at only one place.

## Cohomological functors

Need to have a family of functors  $(U^n)_{n \in \mathbb{Z}}: R\text{-modules} \rightarrow \text{abelian groups}$  + connecting maps that satisfy two axioms.

1. (Long Exact Sequence) Given a short exact sequence  $A \rightarrow B \rightarrow C$  we get a long exact sequence

$$\dots \rightarrow U^{n-1} \xrightarrow{\delta} U^n A \rightarrow U^n B \rightarrow U^n C \xrightarrow{\delta} U^{n+1} A \rightarrow \dots$$

2. (optional axiom)  $U^n = 0$  for  $n < 0$  and  $U^n(I) = 0$  for  $n > 0$  and  $I$  injective.

Later we would like to compute the cohomological functor over spaces and  $A \twoheadrightarrow B \twoheadrightarrow C$  are going to be sheaves which will play the role of coefficients. For completeness here is the definition of an injective module.

**Definition 2.** *An injective module is a module  $I$  that results from the dualization of the second and third equivalent conditions in the definition of a projective module. More explicitly, let  $I$  be a left  $R$ -module then the following are equivalent:*

1. *Given an injection  $B \xrightarrow{\psi} A$  of modules then any map  $B \rightarrow I$  factors through  $\psi$ . Or equivalently you have the following diagram*

$$\begin{array}{ccccc}
 & & I & & \\
 & & \uparrow & & \\
 & \swarrow & & \searrow & \\
 A & \longleftarrow & B & \longleftarrow & 0 \quad (\text{Exact Row})
 \end{array}$$

2.  *$\text{Hom}(\_, I)$  is an exact functor.*

For example,

- $R$  is a division ring if and only if every module is free (assume  $1 \neq 0$ ).
- $R$  is a semi-simple Artinian ring (satisfies the descending chain condition on ideals) if and only if every module is projective.
- If  $R$  is an integral domain (commutative) then  $R$  is a PID if and only if every submodule of a free module is free.

**Example 3.** In  $\mathbb{Z}[\sqrt{-5}]$   $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$

- If  $R$  is a Dedekind domain if and only if every submodule of a free module is projective.

**Theorem 4.** (Quillen-Suslin 1976) *If  $R$  is a field or a PID then every finitely generated projective  $R[t_1, t_2, \dots, t_n]$ -module is free.*

**Example 5.** (Barridge-Dunwoody 1979) *There exists a 2-generator non-free projective module over  $\mathbb{Z}G$  where  $G = \text{trefoil knot group}$ . It would be nice if there were more examples of this sort, but they are sadly lacking.*

## K-theory

Define  $K_0(R)$  to be the Grothendieck group of finitely generated modules.

**Theorem 6.** (Eilenberg-Ganea, Wall 1965) *If  $G$  is a group with finite cohomological dimension then  $G$  has a finite dimensional  $K(G, 1)$ .*

The other direction of the theorem is as follows. Let  $X$  be a CW-complex associate to it the cellular chain complex  $C_*(X)$  by letting  $C_n(X) = H_n(X^n, X^{n-1}) \cong \{\text{free abelian group on the set of } n\text{-cells}\}$  where the isomorphism is a consequence of excision. If  $X$  is a  $G$ -CW complex (a CW-complex with a  $G$  action) then  $H_n(X^n, X^{n-1})$  are  $\mathbb{Z}G$ -modules. If the action is free then  $H_n(X^n, X^{n-1})$  are free  $\mathbb{Z}G$ -modules. If  $X$  is acyclic then  $C_*(X) \twoheadrightarrow \mathbb{Z}$  and you get free resolutions. If  $X$  is contractible then you get an exact chain. If  $X$  is finite dimensional then you get a resolution of finite length which implies that  $G$  has finite cohomological dimension.

What we would like is an easy way to reverse the process described above. Suppose that we resolve  $\mathbb{Z}$  as follows:

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0 \quad (\mathbb{Z}G)$$

where all the  $P_n$ 's are free or projective. This is easy to do just map a massive free module onto the kernel of the previous map. If we had another resolution

$$0 \rightarrow L \rightarrow Q_{n-1} \rightarrow Q_{n-2} \rightarrow \cdots \rightarrow Q_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where all the  $Q_i$ 's are free or projective then we would like to say something about  $L$  (which is not known to be either projective or free).

**Lemma 7** (Schanuel's Lemma). *Given the two resolutions above*

$$P_0 \oplus Q_1 \oplus P_2 \oplus \cdots \cong Q_0 \oplus P_1 \oplus Q_2 \cdots$$

This implies that  $L$  must be a projective module by the first definition since it is a direct summand on one side of the isomorphism given by Schanuel's Lemma and the other side is a free module. Another application of Schanuel's Lemma implies that any resolution of projectives must stop after  $n$ -steps. In fact, abstract resolutions will behave in roughly the same way as a projective resolution.

Let's try to build a CW-complex that imitates a free resolution. This normally starts by building a 2-complex to deal with glueing 1-cells to 0-cells, and then glue in additional higher cells to kill elements of homology. If  $L$  is free then building this complex called an Eilenberg-MacLane complex and it follows as above. If  $L$  is not free and is just projective then what happens?

**Theorem 8** (Eilenberg Swindle). *Know that  $L \oplus L'$  is free then define*

$$F = L \oplus (L' \oplus L) \oplus (L' \oplus L) \cdots \cong L \oplus F$$

*and note that  $F$  is free.*

This allows us to replace our projective module  $L$  with the module  $L \oplus F$  in the following way to get the last module to be free. IE replace

$$0 \rightarrow L \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \mathbb{Z} \rightarrow 0$$

with

$$0 \rightarrow L \oplus F \rightarrow Q_{n-1} \oplus F \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \mathbb{Z} \rightarrow 0$$

Starting with a group  $G$  that is finitely presented and of type  $FP_\infty$  (all modules are finitely generated, but the resolution might go on forever) we can build a  $K(G, 1)$  of finite type (finite number of cells in each dimension with possibly infinite dimension). If in addition  $G$  has  $cd < \infty$  then there exists a  $K(G, 1)$  which is finite dimensional (as a CW-complex the dimension of all the cells is bounded, but there could be infinitely many). A natural question to ask at this point is when can we build a  $K(G, 1)$  that is finite (finite dimensional and finite type)? We can not apply the Eilenberg swindle directly since the swindle exploited the fact that  $F$  was infinite dimensional. If one can perform a finitely generated version of the swindle then the group  $G$  is called stably free.

## Wall Obstruction

In  $K_0(R)$  look at

$$\sum_{i \geq 0} (-1)^i [P_i].$$

If this is equivalent in the group to a finite rank free module then we can do the swindle with a finitely generated group instead of  $F$ . Which is exactly the obstruction to being able to build a finite  $K(G, 1)$ .