

# Quasi-Isometries

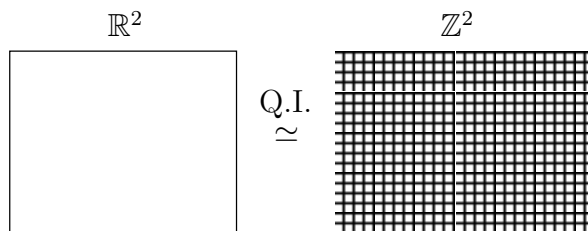
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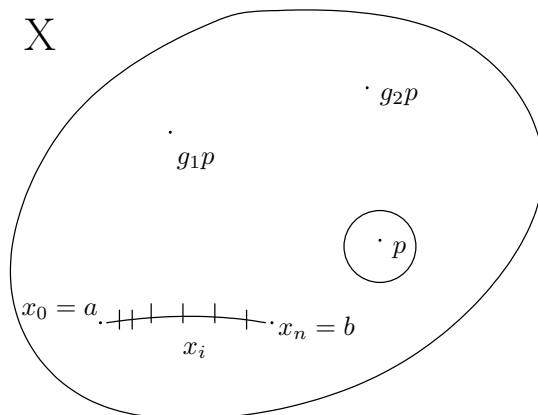
## Lecture 1

**Theorem 1.** *If  $G$  acts geometrically on  $X$  and  $Y$  (proper geodesic metric spaces) then  $X$  and  $Y$  are quasi-isometric.*

A geometric action is a group action that is cocompact, isometric, and properly discontinuous. The basic example of such an action is when  $K$  is compact,  $G = \pi_1(K)$  and  $X = \tilde{K}$  where  $\tilde{K}$  is the universal cover of  $K$ . The basic idea of a quasi-isometry is that if we hit a space with a quasi-isometry then the basic shape stays the same or if we stand far enough back the spaces look the same. One example is shown below with  $X = \mathbb{R}^2$ ,  $K = T^2$ ,  $G = \mathbb{Z}^2$ .



*Proof.* Fix  $G$ ,  $X$  and pick  $p \in X$ . Then since the action of  $G$  is cocompact there



exists an  $R$  such that the  $G$  translates of  $B_R(p)$  are the whole space  $X$ . Now if  $a$  and  $b$  are distinct points in  $X$  then subdivide the geodesic connecting  $a, b$  so that you get a sequence of points  $\{x_i\}$  such that  $a = x_0, x_1, \dots, x_n = b$ ,  $d(x_i, x_{i+1}) \leq 1$  and  $n \leq d(a, b) + 1$ .

We can find  $g_i \in G$  such that  $d(g_i p, x_i) \leq R$ . Since the action is isometric we know that

$$d(g_i p, g_{i+1} p) \leq 2R + 1 \iff d(p, g_i^{-1} g_{i+1} p) \leq 2R + 1$$

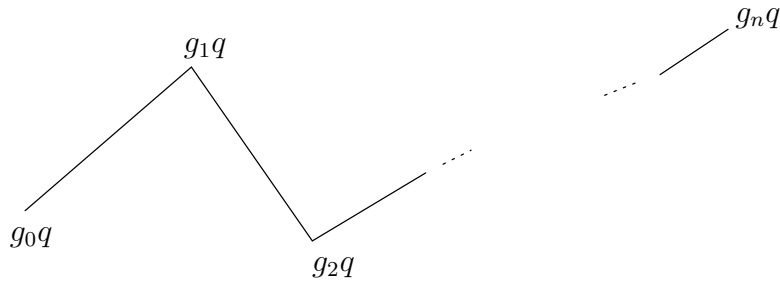
Since the action is properly discontinuous and the space is proper we see that

$$S = \{g \in G \mid d(p, gp) \leq 2R + 1\}$$

is finite. Look at  $Y$  with a base point  $q$ . Then

$$\text{Sup}_{g \in S} d(q, gq) = R' < \infty$$

since we are taking the  $\text{Sup}$  of a finite set. Consider the path in  $Y$  that connects  $(g_i q, g_{i+1} q)$



and notice that each segment has length less than  $R'$  and hence

$$d_y(g_0q, g_nq) \leq \text{total length} \leq R' n \leq R'(d(a, b) + 1).$$

□

**Definition 2** (Quasi-isometry version 1).  $X, Y$  are quasi-isometric if there exists an  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that for some  $(K, C)$

$$d_y(f(a), f(b)) \leq K d_x(a, b) + C \text{ and } d_x(g(a'), g(b')) \leq K d_y(a', b') + C$$

for all  $a, b \in X$  and  $a', b' \in Y$  and in addition

$$d(g \circ f(a), a) \leq C \quad \forall a \in X \quad \text{and} \quad d(f \circ g(b), b) \leq C \quad \forall b \in Y.$$

**Definition 3** (Quasi-isometry version 2). There exists an  $f : X \rightarrow Y$  so that

$$\frac{d(a, b) - C}{K} \leq d(f(a), f(b)) \leq K d(a, b) + C$$

and for all  $y \in Y$  there exists  $x \in X$  so that  $d(f(x), y) \leq C$ .

## Cayley Graphs

Let  $C(G, S)$  be the Cayley graph of the group  $G$  with respect to the generating set  $S$ . Then the vertex set corresponds to the elements of  $G$  and the edges are given by  $(g, g') \iff g^{-1}g' \in S$ . If a space  $X$  “looks like”  $C(G, S)$  for some choice of generating set then  $X$  is quasi-isometric to  $C(G, S)$ . This implies that for any two finite generating sets for the group  $G$  the corresponding Cayley graphs are quasi-isometric.

Now think about  $G = \pi_1(S)$  for a closed surface  $S$  and note that the universal cover is the same up to quasi-isometry. Thus using the previous theorem we see that

$$G \stackrel{Q.I.}{\simeq} \tilde{S} = S^2, \mathbb{R}^2, \text{ or } \mathbb{H}^2$$

and consequently there are only three quasi-isometry classes for  $\pi_1(S)$ .

**Question 4.** *What things share the same geometry as an arbitrary group  $G$ ?*

Some examples of quasi-isometries are

1.  $G$  is quasi-isometric to any finite index subgroup  $G' \leq G$ .
2.  $G$  is quasi-isometric to any quotient  $G/N$  if  $N$  is a finite and normal subgroup of  $G$ .

Sequences of the above two examples produce an equivalence relation on groups  $G$  called Weak Commensurability. These two examples put together with the theorem give the following results.

1. Any 2 groups that have isomorphic finite index subgroups are quasi-isomorphic.
2. The genus 2 surface is quasi-isometric to higher genus surfaces.

Now we have committed a lie by omission since we never showed that in fact the three spaces  $S^2, \mathbb{R}^2, \mathbb{H}^2$  are not quasi-isometric.

**Theorem 5** (Connect the dots). *Suppose  $G$  acts geometrically on  $X$  where  $X$  is contractible then if  $f : Y \rightarrow X$  ( $Y$  a geodesic metric space) such that*

$$d(f(y), f(y')) \leq Kd(y, y') + C$$

*then there exists an  $f'$  with*

$$d(f'(y), f(y)) \leq R < \infty$$

*for all  $y \in Y$  which is continuous and*

$$d(f'(y), f'(y')) \leq K'd(y, y') + C'.$$

**Corollary 6.** *If  $X, Y$  are both uniformly contractible and quasi-isometric to  $G$  then  $X, Y$  are proper homotopy equivalent.*

**Corollary 7.**  $\mathbb{Z}^n$  and  $\mathbb{Z}^m$  are not quasi-isometric for  $n \neq m$ .

## Lecture 2

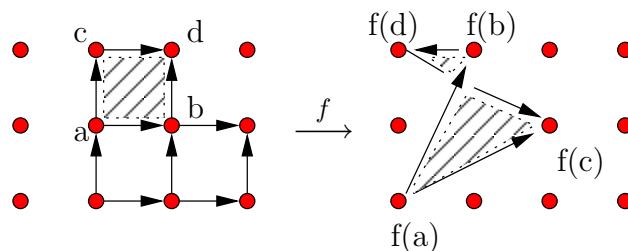
The general idea of the course is to develop quasi-isometry invariants and techniques and use them to tell different groups apart. In general there are four types of quasi-isometry invariants:

1. Topological
2. Geometric
3. Analytic
4. Asymptotic

We would like to use the connect the dots theorem to give a topological invariant, but first we will sketch a proof.

*Sketch of connect the dots in a special case.* The idea is to use the existence of a finite  $K(G, 1)$  ( $K$  is a finite complex with  $\pi_1(K) = G$  and  $\tilde{K}$  contractible) if it exists. The special case we will look at is when  $G = \mathbb{Z}^2$ ,  $K = T^2$ , and  $\tilde{K} = \mathbb{R}^2$ .

(claim) If  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is a quasi-isometry then it extends to a proper ( $\hat{f}^{-1}(K) = \text{compact}$  for all compact  $K \subset \mathbb{R}^2$ ) homotopy equivalence  $\hat{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  ( $\hat{f}$  is also a quasi-isometry).



What the picture describes is the following operation. First you map all the vertices over using  $f$ . Second you map all the edges connecting the vertices together across via  $f$ . This gives you a map of  $S^1 \rightarrow \mathbb{Z}^2$ . Thirdly you use the fact that  $K$  is a  $K(G, 1)$  and more specifically that  $\tilde{K}$  is contractible to show that this map actually extends to a map of the whole ball into the one-skeleton. Iterate this process for higher dimensional spheres and balls to prove the claim. □

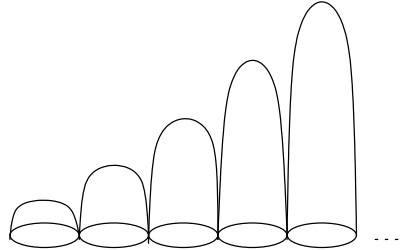
**Definition 8** (Uniformly Simply-Connected).  $X$  is uniformly simply connected if for all  $D$  there exists a  $D'$  so that any continuous map  $S^1 \rightarrow X$  with image of diameter  $\leq D$  bounds a disk of diameter  $\leq D'$ .

**Definition 9** (Uniformly Contractible).  $X$  is uniformly contractible if for all  $R$  there exists an  $R'$  such that for all  $x \in X$

$$B_R(x) \hookrightarrow B_{R'}(x)$$

is null-homotopic.

**Example 10** (Contractible but not uniformly contractible). *So each of the circles in the base has the same circumference  $C$  and then we are attaching a disk but going higher and higher up each time. This is clearly contractible, but is not uniformly contractible since for far out you will have to go arbitrarily high up to contract a loop of length  $C$ .*



In the case where  $G$  has a finite  $K(G, 1)$  then the Cayley graph of  $G$  is uniformly contractible and hence if  $G$  and  $H$  have finite  $K(\pi, 1)$ 's and are quasi-isometric then

$$H_c^i(\tilde{K}) \cong H^i(G, \mathbb{Z}G) \cong H^i(H, \mathbb{Z}H) \cong H_c^i(\tilde{L})$$

where  $K, L$  are the Cayley graphs of  $G, H$  respectively. The isomorphisms are a consequence of the previous argument since  $\tilde{K}, \tilde{L}$  are properly homotopy equivalent.

**Example 11.**

$$H_c^i(\mathbb{R}^n) = \begin{cases} 0 & i \neq n \\ \mathbb{Z} & i = n \end{cases}$$

*In particular this says that  $\mathbb{Z}^m, \mathbb{Z}^n$  are not quasi-isometric if  $m \neq n$ . Also note that*

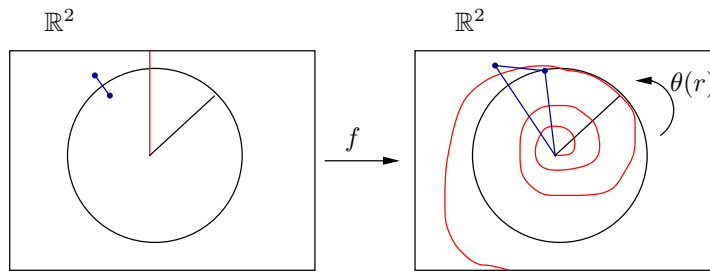
$$H_c^1(\text{Tree}) = \text{infinitely generated.}$$

The previous discussion shows that the homotopy type of the ends is a quasi-isometric invariant. In the case of a tree this shows that the natural boundary is a Cantor set.

**Exercise 12.**  *$G$  is finitely presented  $\iff G$  is uniformly simply connected (there exists a uniformly simply connected space on which  $G$  acts cocompactly and isometrically). This will show that being finitely presented is a quasi-isometric invariant.*

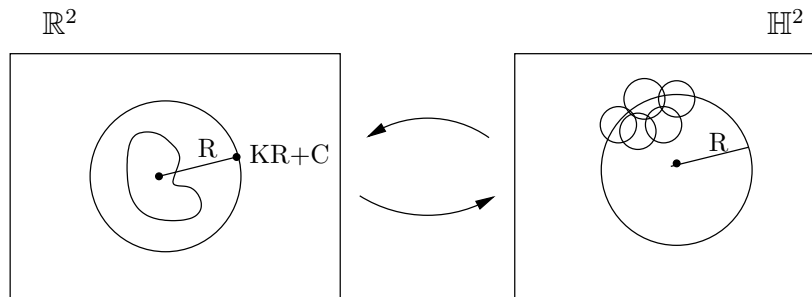
**Example 13.** *[Bad Example of a quasi-isometry from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .] If  $\theta(r) = \ln(1+r)$  then  $f$  is a quasi-isometry, however the red geodesic ray goes to a logarithmic spiral and hence  $f$  has a winding number of  $\infty$  as a map from the disk to the disk.*

An example of a property that is not a quasi-isometry invariant would be possessing a finite  $K(G, 1)$  ( $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$  but  $K(\text{trivial}, 1) = \text{point}$ ). It would be interesting if there was a torsion free counter-example. Having a finite  $k$ -skeleton for any  $k$  on the other hand is a quasi-isometric invariant.



**Question 14.** *Is  $\mathbb{R}^2$  quasi-isometric to  $\mathbb{H}^2$ ? Or equivalently is  $\mathbb{Z}^2$  quasi-isometric to  $\pi_1(S)$  where  $S$  is a surface of genus greater than two?*

These two spaces are homeomorphic so a topological invariant will not give us a negative answer. A geometric invariant namely growth will give us a negative answer. In  $\mathbb{R}^2$  a ball has area that is quadratic as a function of the radius  $R$ . In  $\mathbb{H}^2$  points are



widely separated (IE covered by balls of radius say 1) thus the area behaves like  $e^R$ . Let  $G$  be a group and  $1 \in S \subset G$  a finite generating set that is closed under inverses (makes things easier but not essential). Define

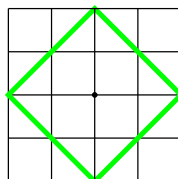
$$N_{(G,S)}(k) = |S^k| = \{g \in G | g = s_1 s_2 \dots s_j \text{ for } s_i \in S \text{ and } j \leq k\}$$

which is a count of the vertices in the ball of radius  $k$  in the Cayley graph.

**Example 15.** *Let  $G = \mathbb{Z}^2$  with*

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

*then  $N_{(G,S)}(k)$  is roughly  $k^2$ .*



If  $f : (G, S) \rightarrow (H, T)$  is a  $(K, C)$  quasi-isometry with  $f(id) = id$  then

$$f(B_k(G, S)) \subset B_{Kk+C}(H, T)$$

$$f^{-1}(B_k(H, T)) \subset B_{Kk+C}(G, S)$$

for all  $k$ . Now  $f$  is not quite injective and not quite surjective thus we have both overcounted and undercounted respectively. To make this precise for some  $A, B$  and for all  $k$  we have

$$N_{(G,S)}(k) \leq AN_{(H,T)}(Kk + C) + B$$

$$N_{(G,S)}(k) \geq \frac{N_{(H,T)}(\frac{k-C}{K}) - B}{A}.$$

This implies that  $N_{(G,S)} \approx A(k)$  ( $A(k)$  is a polynomial of degree  $d$ ) then so is  $N_{(H,T)}$  which gives us a quasi-isometric invariant. This gives another proof that

$$\mathbb{Z}^n \stackrel{QI}{\not\cong} \mathbb{Z}^m$$

for  $n \neq m$ . We can also see that

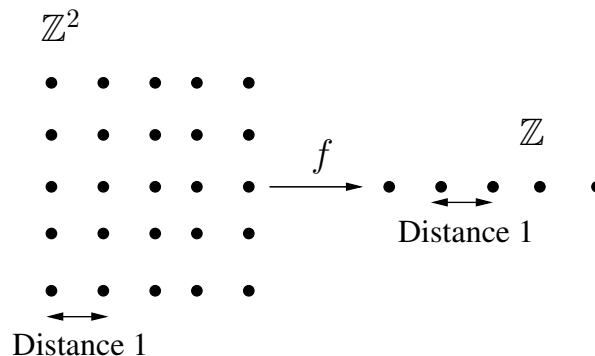
$$\mathbb{Z}^n \stackrel{QI}{\not\cong} \pi_1(S_g)$$

for  $g \geq 2$  since  $\pi_1(S_g)$  is hyperbolic and hence grows exponentially. It is important to note that these geometric methods can not distinguish  $\mathbb{H}^2$  and  $\mathbb{H}^{12}$  while topological invariants can. And all the methods that we have developed so far are no help in distinguishing  $\mathbb{H}^3$  and  $\mathbb{H}^2 \times \mathbb{R}$ . The hope is that some blending of these arguments will work, for example  $L^2$ -cohomology which is a quasi-isometric invariant.

## Lecture 3

We start with another proof that  $\mathbb{Z}^m \not\cong \mathbb{Z}^n$  if  $m \neq n$ .

*Proof.* Suppose  $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$  was a  $(K, C)$  quasi-isometry.



$$\begin{array}{c} \epsilon\mathbb{Z}^n \subset \mathbb{R}^n \\ \downarrow \text{"} f_\epsilon \text{"} \\ \epsilon\mathbb{Z}^m \subset \mathbb{R}^m \end{array}$$

For any  $\epsilon > 0$  let  $f_\epsilon : \epsilon\mathbb{Z}^n \rightarrow \epsilon\mathbb{Z}^m$  ( $\epsilon\mathbb{Z}^n$  is  $\mathbb{Z}^n$  with the normal metric rescaled by a factor of  $\epsilon$ ) and notice  $f = f_\epsilon$ .

$$d_{\epsilon\mathbb{Z}^n}(f_\epsilon(x), f_\epsilon(y)) = \epsilon d_{\mathbb{Z}^n}(f(x), f(y)) \leq \epsilon(Kd_{\mathbb{Z}^n}(x, y) + C) = Kd_{\epsilon\mathbb{Z}^n}^n(x, y) + \epsilon C$$

thus  $f_\epsilon$  is a  $(K, \epsilon C)$  quasi-isometry from  $\epsilon\mathbb{Z}^n \rightarrow \epsilon\mathbb{Z}^m$ . Taking a limit as  $\epsilon \rightarrow 0$  we see that  $\epsilon\mathbb{Z}^n, \epsilon\mathbb{Z}^m$  are sitting inside  $\mathbb{R}^n, \mathbb{R}^m$  more densely. Define  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  via the following: for  $x \in \mathbb{R}^n$  take any sequence  $x_i \in \epsilon_i\mathbb{Z}^n$  as  $\epsilon_i \rightarrow 0$  such that  $x_i \rightarrow x$  and let  $\hat{f} = \lim f_{\epsilon_i}(x_i)$ . This will be a  $(K, 0)$  quasi-isometry which implies that  $\hat{f}$  is  $K$ -bilipschitz and thus well-defined. Notice  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a  $K$ -bilipschitz homeomorphism. But  $\hat{f}$  is  $K$ -bilipschitz and is almost everywhere differentiable via Rademacher's Theorem with derivative a linear isomorphism of norm less than or equal to  $K$  which gives a contradiction.  $\square$

How does this construction generalize for more general groups  $\Gamma$ ? We would like to take a limit of the metric spaces  $\Gamma_\epsilon$  as  $\epsilon \rightarrow 0$ . IE take limits of sequences  $\{\gamma_{\epsilon_i} \in \Gamma_{\epsilon_i}\}$  as  $\epsilon_i \rightarrow 0$ . In general this construction yields something called an (could be lots of different ones) asymptotic cone of  $\Gamma$ . For example, the asymptotic cone of anything negatively curved is an  $\mathbb{R}$ -tree. A nice property that we might hope for is the space being locally compact (any ball (of radius 1) in the limit can be covered by finitely many  $(N_\delta)$   $\epsilon$ -balls for any  $\delta > 0$ ). Notice that

1. a ball of radius 1 in  $\Gamma_\epsilon$  equals a ball of radius  $\frac{1}{\epsilon} = R$  in  $\Gamma$
2. a ball of radius  $\delta$  in  $\Gamma_\epsilon$  equals a ball of radius  $\frac{\delta}{\epsilon} = \delta R$  in  $\Gamma$ .

Another way to describe  $N_\delta$  is the number of ball's needed to cover a bigger ball.

**Theorem 16** (Almost Claim). *If balls of radius one in the limit space can be covered by  $N_\delta$  balls of radius  $\delta$ , then in  $\Gamma$ , every  $R$ -ball can be covered by approximately  $N_\delta$  balls of radius  $\delta R$ .*

This implies that

$$\frac{Vol(B_R)}{Vol(B_{\delta R})} \leq N_\delta$$

thus  $R \rightarrow Vol(B_R)$  is bounded by a polynomial of degree  $\frac{\ln(N_\delta)}{\ln \delta}$ . In addition the degree and the fact that this is a polynomial are quasi-isometric invariants. It is important to note that the converse is true if the bound is a polynomial and thus taking nice limits requires polynomial growth.

**Theorem 17** (Gromov).  *$\Gamma$  has polynomial growth if and only if  $\Gamma$  is virtually nilpotent.*

A virtually nilpotent group is virtually a subgroup of  $n \times n$  upper triangular matrices with all ones on the diagonal for some  $n$ . IE

$$\left\{ \left( \begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{Z} \right\}$$

*Very fast outline.* 1. True for linear groups (IE matrices)

2. Build  $X = \lim_{\epsilon \rightarrow 0} \epsilon \Gamma$

3. Show that  $X$  is homogeneous (if you can do 2 then you can do 3)

4.  $Isom(X)$  is a Lie Group (technically very hard)

5.  $\Gamma \rightarrow Isom(X)$  is big enough to induct.

**Exercise 18.**  $\gamma \in \Gamma$  if  $d(\gamma x, x) \leq R$  for all  $x \in \Gamma$  then  $\gamma$  commutes with a finite index subgroup  $\Gamma' \subset \Gamma$ .

In particular if this is true for all  $\gamma \in \Gamma$  then  $\Gamma$  has a finite index abelian subgroup which takes care of the base case for the induction. □

**Example 19.** Let

$$\Gamma = \left\{ \left( \begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{Z} \right\}$$

where the multiplication is

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

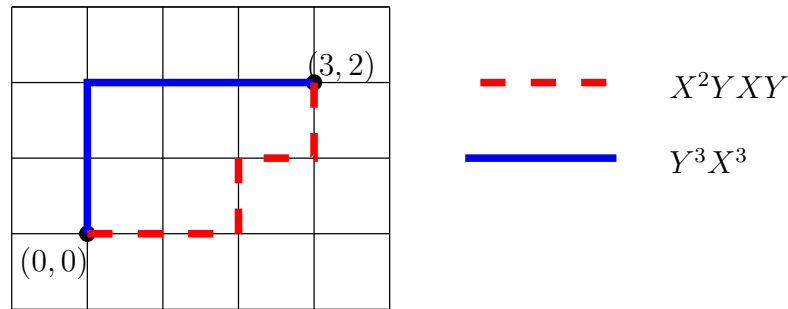
Note that the first two terms are abelian and the third term is abelian modulo some error term not depending on either  $z$  or  $z'$ . Let

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$XY = YXZ$  and  $Z$  commutes with everybody. This group is not  $\mathbb{Z}^3$  for two reasons. The first is that it is non-abelian and the second is that it is generated by two elements namely  $X, Y$  ( $Z^{-1} = [X, Y]$ ) not three. Let

$$G = \left\{ \left( \begin{array}{ccc} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) \mid x, y, z \in \mathbb{R} \right\} \stackrel{\text{homeo}}{\simeq} \mathbb{R}^3$$

then  $G/\Gamma$  is a compact 3-manifold with universal cover  $\mathbb{R}^3$  which has the Nil geometry of Thurston.



The Cayley graph for  $\Gamma$  maps to  $\mathbb{Z}^2$  by writing any word in  $X$  and  $Y$  and counting the number with signs of  $X$ 's and  $Y$ 's that appear. In the diagram both of the words trace out paths from the origin to  $(3, 2)$  but the distance between the actual endpoints of the words depends on the number of  $Z$ 's that you pick up when moving from one to another. This number of  $Z$ 's turns out to be exactly the area of the polygon traced out by the paths in the projection to  $\mathbb{Z}^2$  which in this case is 5. It is most efficient to do all the  $Y$ 's first then all the  $X$ 's.

$$X^n Y^n = Y^n X^n Z^{n^2} \Rightarrow Z^{n^2} = X^{-n} Y^{-n} X^n Y^n$$

this shows that the path from 0 to  $Z^{n^2}$  has length  $n^2$  when writing it as a product of  $Z$ 's but the actual word length is at most  $4n$ . This shows that when you are traveling vertically the word length is growing like  $\sqrt{n}$  and is not a geodesic. Thus  $\Gamma$  has polynomial growth like  $n^4$  which implies that it is not quasi-isometric to  $\mathbb{Z}^3$ . It is also not quasi-isometric to  $\mathbb{Z}^4$ .

In fact we are quite ignorant about nilpotent groups.

**Question 20.** Are there known quasi-isometric nilpotent groups that are not commensurable?

**Answer 21.** Yes. Cocompact lattices in the same nilpotent lie group.

The problem is that individual quasi-isometries can be awful and we don't know enough invariants or how to go the other direction. In fact the tools that we have developed can't really do more than what we have already shown.

## Lecture 4

One question that is both natural and interesting is

**Question 22.** Given a group  $G$ , which groups  $H$  are quasi-isometric to  $G$ ?

Unfortunately, the techniques we have developed can not help so we need something new. Suppose that  $G$  acts geometrically on a nice space  $X$  what works in a lot of cases is to get  $H$  to act on  $X$  or something similar. Two theorems of this type are:

**Theorem 23.** *If  $G$  is quasi-isometric to  $\mathbb{H}^n$  then  $G$  acts geometrically on  $\mathbb{H}^n$ .*

**Theorem 24.** *If  $G$  is quasi-isometric to  $\mathbb{F}_2$  (the free group on two letters) then  $G$  acts geometrically on some (locally finite) tree. Note: acting geometrically on a tree implies that  $G$  is virtually free.*

In order to approach the proofs additional information is needed.

**Definition 25** (Quasi-Action). *A  $(K, C)$  quasi-action of  $G$  on  $X$  is*

1.  $\forall g \in G$  associate a  $(K, C)$  quasi-isometry  $\phi_g : X \rightarrow X$ .
2.  $\forall g, h \in G \sup_x d(\phi_g \circ \phi_h(x)) \leq C$

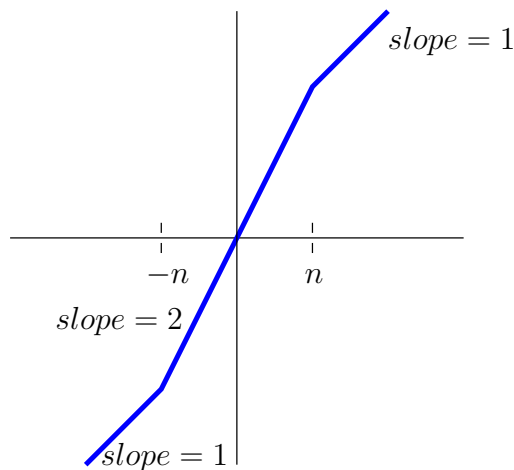
**Definition 26.** *Define  $QI(X) = \{f : X \rightarrow X \text{ a quasi-isometry}\} / f \sim g$  where  $f \sim g$  if and only if  $\sup_x d(f(x), g(x)) < \infty$ . It can be checked that  $QI(X)$  is a group.*

A quasi-action of  $G$  on  $X$  gives a uniform (image of  $\phi_g$  all have quasi-isometry constants  $(K, C)$ ) homomorphism  $G \rightarrow QI(X)$ , but the converse is not true whatsoever.

**Example 27.**  $\mathbb{Z}$  quasi-acts on  $\mathbb{R}$  by integer translation, but the induced homomorphism  $\mathbb{Z} \rightarrow QI(\mathbb{R})$  is trivial. It is completely mysterious what  $QI(\mathbb{R})$  really is, but some subgroups of  $QI(\mathbb{R})$  are known. The first is  $\mathbb{R}^\times$  via the map  $x \mapsto \lambda x$  for  $\lambda \neq 0$ . The second is the space of Lipschitz diffeomorphisms of  $\mathbb{R}$ . However there are bad things that happen in the topology. Let

$$f_n : \mathbb{R} \rightarrow \mathbb{R} \text{ given by } f_n(x) = \begin{cases} 2x & |x| \leq n \\ x + n & x > n \\ x - n & x < -n \end{cases}$$

$f_n \sim Id$  for every  $n$  but  $\lim_{n \rightarrow \infty} f_n = 2x \not\sim Id$  which implies that a constant sequence



of identity maps converge to a non-identity element of the group.

There are spaces  $X$  (even some nice ones) where  $Isom(X) \rightarrow QI(X)$  is an isomorphism (IE higher rank symmetric spaces, buildings, ...). There are even some groups  $G$  where  $QI(G) = G$  which is about as strong a rigidity result for quasi-isometries that is possible. It is suspected that most groups are of this form. One example would be [?].

**Claim 28.** *If  $G$  acts geometrically on  $X$  and  $H$  is quasi-isometric to  $G$  then  $H$  quasi-acts on  $X$ .*

*Proof.* Since  $G \stackrel{QI}{\cong} X$  and  $H \stackrel{QI}{\cong} G$  we see that  $H \stackrel{QI}{\cong} X$  and call this quasi-isometry  $f$ .  $H$  acts on itself isometrically by left multiplication so define the quasi-action to be  $\phi_H(x) = f(h \cdot f^{-1}(x))$ .  $\square$

We could have tried a more direct approach: let  $T : H \rightarrow G$  be the given quasi-isometry and define  $\phi_h(x) = h \cdot x = (Th) \cdot x$ . The problem is that  $T(h_1h_2)$  and  $T(h_1)T(h_2)$  as group elements in  $G$  might have nothing to do with one another which gives no control over the induced action of  $H$  on  $X$ . This shows that quasi-isometries do not behave like group homomorphisms.

*Proof.* (Outline for 23 and 24)

1. Get a quasi-action of  $H$  on  $X$ . (28)
2. Improve the quasi-action to an isometric action (techniques generally depend on  $X$ , for example different techniques will be utilized in 23 and 24).

$\square$

For  $\mathbb{H}^n$ , quasi-isometries of  $\mathbb{H}^n$  “=” quasi-conformal (mob, symmetric) homeomorphisms of  $S^{n-1}$ .

**Theorem 29.** *(Tukia for  $n > 3$  and Sullivan for  $n = 3$ ) A uniform group of quasi-conformal homeomorphisms of  $S^{n-1}$  is quasi-conformal conjugate to a group of conformal homeomorphisms.*

This is a hard theorem, but it completes step two in the proof of 23. Unfortunately the boundary of the space will not help once the space  $X$  no longer has negative curvature. For example, the quasi-isometry in 13 sends geodesic rays in  $\mathbb{R}^2$  to a logarithmic spiral which does not converge to anything nice on the boundary.

**Theorem 30.** *If  $G$  is quasi-isometric to  $\mathbb{Z}$  then  $G$  acts properly discontinuously and cocompactly by isometries on  $\mathbb{R}$ .*

*Proof.* (sketch 1) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a quasi-isometry the  $f$  either fixes or flips the ends  $\{\pm\infty\}$ . Let  $a_n = f(n)$  where  $n \in \mathbb{Z}$  since  $f$  is a quasi-isometry

$$|a_{n+1} - a_n| = K + C.$$

If  $|n - m| > K + C$  then  $|f(n) - f(m)| > 1$ . Take  $N > K + C$  such that  $a_N \neq a_0$  and without loss of generality assume that  $a_N > a_0$ . For all  $n \geq N$  from the previous fact  $|f(n) - f(0)| > 1$  and thus all the  $a_n$  must be on the same side of  $a_0$ . Thus  $a_{2N} \rightarrow \infty$  if  $N$  is large enough. If instead  $a_N < a_0$  then the sequence converges to  $-\infty$  instead.  $\square$

**Claim 31.** *If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a quasi-isometry then there exists  $D > 0$  such that one of the following holds*

1. (Coarsely Increasing) For all  $x, y$  with  $x \geq y + D$ ,  $f(x) \geq f(y) + 1$
2. (Coarsely Decreasing) For all  $x, y$  with  $x \leq y - D$ ,  $f(x) \geq f(y) + 1$

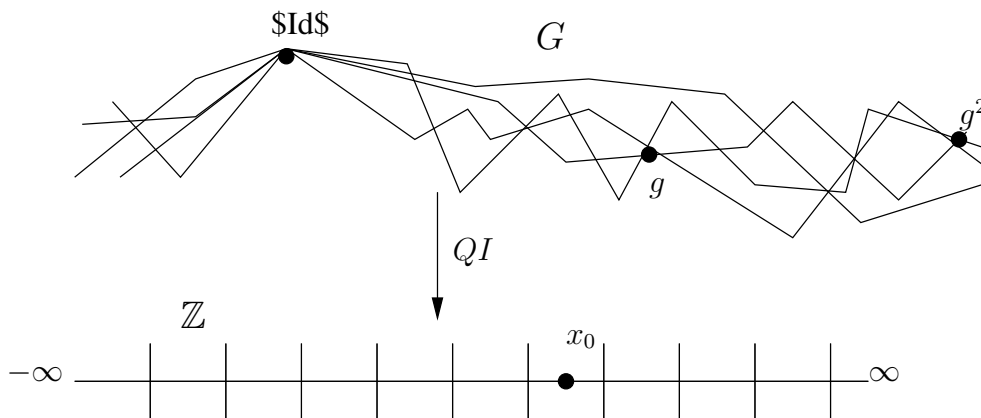
**Corollary 32.** *If  $G$  quasi-acts on  $\mathbb{R}$  preserving ends then there exists  $x \in \mathbb{R}$  such that  $d(f_g(x), x) \geq D$  which implies  $g$  has infinite order in  $G$ . This implies that if  $f(x) > x$  then  $f^2(x) > f(x)$  and the process can be iterated.*

The point is the following. Suppose  $f_g(x) \gg x$  then  $f_g(f_g(x)) \gg f_g(x)$  and  $f_g(f_g(x))$  is close to  $f_{g^2}(x)$ . Thus  $f_{g^2}(x) \gg f_g(x) \gg x$  and by induction  $f_{g^n}(x) \gg f_{g^{n-1}}(x)$ . In particular, the “orbits” of  $\{g^n\}$  are unbounded so:

- $$G \stackrel{QI}{\cong} \mathbb{R} \Rightarrow G \text{ quasi-acts on } \mathbb{R}.$$
- $\Rightarrow G' \subset G$  of index at most 2 quasi-acts preserving orientation.
  - $\Rightarrow$  There exists  $g \in G$  of infinite order  
(uses the fact that quasi-action is coarsely cocompact).
  - $\Rightarrow f|_{\langle g \rangle} \rightarrow \mathbb{R}$  is a quasi-isometry.
  - $\Rightarrow \langle g \rangle \subset G$  is finite index (coarsely properly discontinuous).

To summarize if  $G$  is quasi-isometric to  $\mathbb{Z}$  then there exists  $\mathbb{Z} \subset G$  of finite index.

## Lecture 5



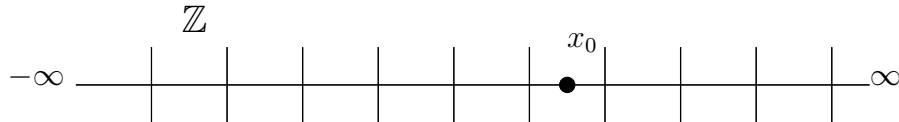
Last time we saw that if a group  $G$  was quasi-isometric to  $\mathbb{Z}$  then there exists  $g \in G$  of infinite order such that  $\dots < g^{n-1}x_0 < g^n x_0 < g^{n+1}x_0 < \dots$ . This gives a

quasi-action of  $G$  on  $\mathbb{R}$  that is uniformly proper ( $\forall R > 0$  there exists  $N$  so that  $\forall x \in X$   $|\{g \mid d(gx, x) < R\}| < N$ ) and cobounded ( $\exists R > 0$  so that  $\forall x$  and  $x' \in X \exists g \in G$  with  $d(gx, x') < R$ ) which is the quasi-action analogue of proper and cocompact. An example of a space and group that have these properties is given by the action of  $G$  on a Cayley graph.

**Exercise 33.** Check that the above properties are invariant under quasi-isometries. In fact, anything defined in the coarse uniform language will be quasi-isometric invariant.

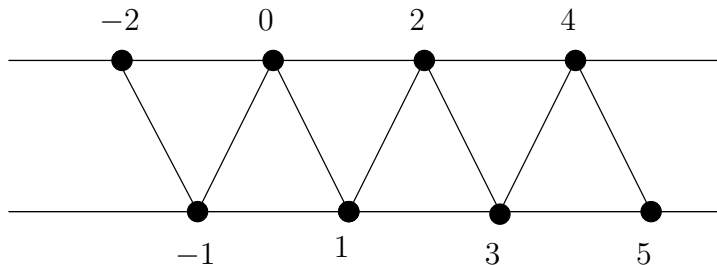
**Exercise 34.** Suppose  $G$  quasi-acts properly and coboundedly on  $X$  and  $G' \subset G$  also acts coboundedly then  $G'$  has finite index in  $G$ .

In the following picture we see that  $x_0$  disconnects  $\mathbb{Z}$  which gives a natural order on  $\mathbb{Z}$ . In the case of a general group  $G$  quasi-isomorphic to  $Z$  a point might not



disconnect the space.

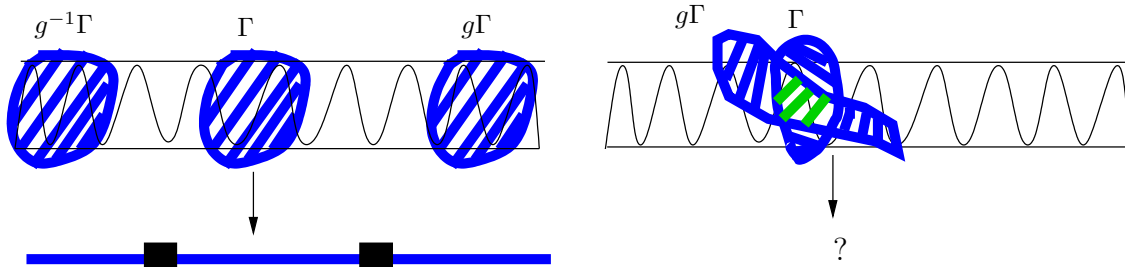
**Example 35.** Take  $G = \mathbb{Z}$  with the generating set  $\{\pm 1, \pm 2\}$  which has the following Cayley graph. In this example points do not disconnect the space.



This shows that the property of points disconnecting the space is not a quasi-isometric invariant. What can be said is that sets of some bounded size disconnect the space and thus talking about ordering of points that are far enough apart so that they were not removed in the disconnecting operation can be done.  $\mathbb{Z}$  is special as a large scale geometry since it has two ends. We can rephrase this as

**Theorem 36.** Any group with two ends has a cyclic subgroup of finite index.

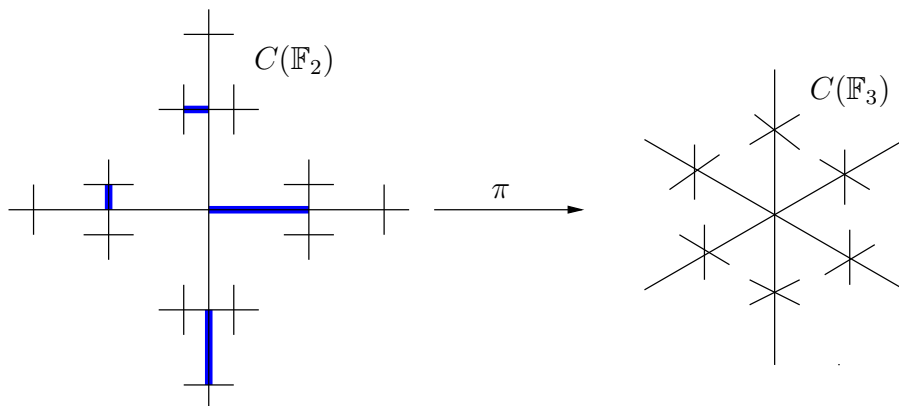
*Proof.* (Idea) Since it is two ended take a subset of the Cayley graph that separates it. If the translates of the chosen subset are disjoint then make each of the translates an edge and collapse everything in between to a vertex to get an action of  $G$  on  $\mathbb{R}$ . If the translates do intersect then the procedure described above does not yield an action on  $\mathbb{R}$ . To complete the proof we need to modify  $\Gamma$  into a new separating set  $\Gamma'$  in such a way that the translates of  $\Gamma'$  are disjoint.  $\square$



An extension of this technique can be used to prove the following.

**Theorem 37.** *If  $G$  is quasi-isometric to a finitely generated free group then  $G$  acts properly discontinuously and cocompactly on some (locally finite) tree.*

This is the same theorem as 24 since all finitely generated non-abelian free groups are quasi-isometric to  $\mathbb{F}_2$  (finite index subgroups). One way to see the quasi-isometry to  $\mathbb{F}_3$  is by collapsing one edge at every vertex in the Cayley graph of  $\mathbb{F}_2$ . The map  $\pi$

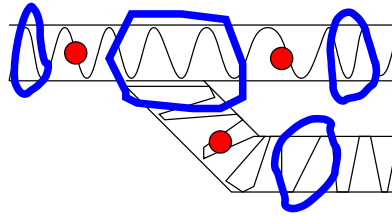


is a quasi-isometry in which we might have collapsed at most half the edges rounded up on any path thus

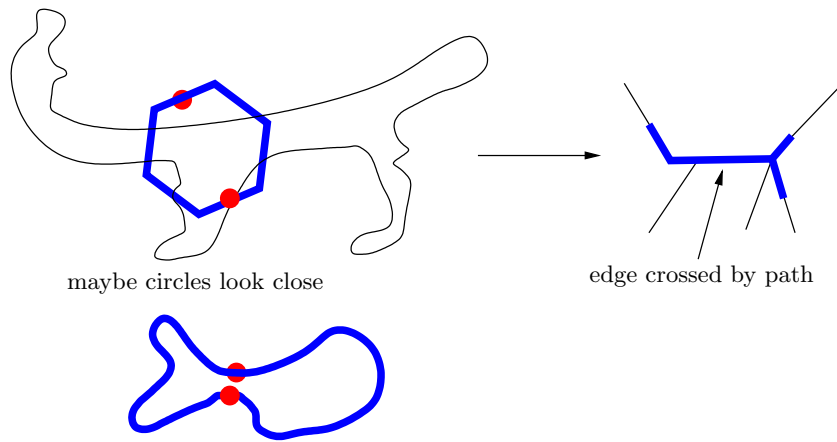
$$\frac{d(x, y) - 1}{2} \leq d(\pi(x), \pi(y)) \leq d(x, y).$$

At this point thinking that  $\mathbb{F}_2$  acts properly discontinuously and cocompactly on  $C(\mathbb{F}_3)$  might seem reasonable, but this is not the case. In general all groups quasi-isometric to a finitely generated free group act on some tree but we do not pick the tree, which is unlike when hyperbolic space was picked in the proof of 23. The trees that we are talking about have infinitely many ends (certainly true for bounded sets) since larger and larger bounded sets separate more and more.

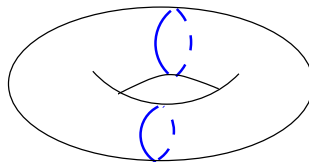
*Proof.* We would hope to build a dual tree like we did before. We still have the problem of overlapping translates of the disconnecting sets and that the vertices that we would like to define (circles) would all be adjacent in this picture giving a loop



and the resulting graph would not be a tree. Suppose  $G$  is quasi-isometric to  $\mathbb{F}_2$ , let  $X$  be a Cayley graph for  $G$ , then there exists  $D$  so that  $\pi_1(X)$  is generated by loops of diameter  $\leq D$ . Equivalently we can attach 2-cells of  $diam \leq D$  to  $X$  to get something simply connected. In the picture we are taking steps of bounded size and



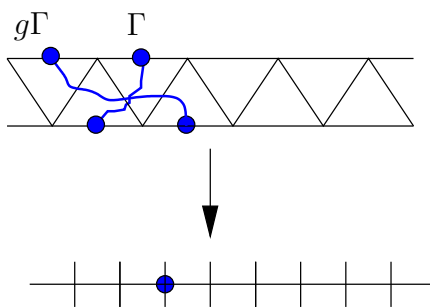
eventually the path closes. One operation that we can do is to write the loop as a union of smaller loops until the quasi-isometry constants take care of things. Why do we care about the separating set being simply connected? The answer is that we want to understand the sets that are doing the disconnecting, and hence we want them to be as simple as possible (why we require that the disconnecting set be simply connected). An example of a disconnecting set that we want to rule out is shown below. We have  $K$ , a  $G$  two complex which is quasi-isometric to our tree  $T$  (glued in



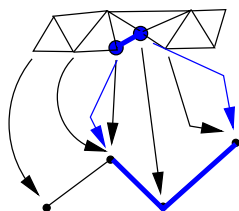
discs of bounded size), we may assume the map

$$K \rightarrow T$$

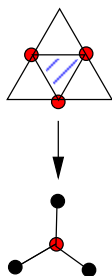
is continuous. Look at the preimage of a point and notice that with  $\mathbb{Z}$  we could cut



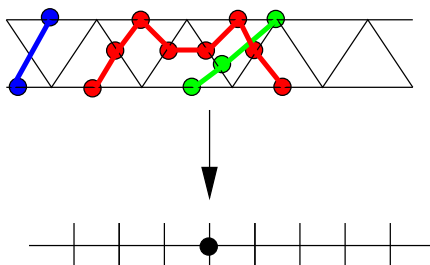
an edge in half to make a simpler disconnecting set. The same operation can be done in a tree. To do this go back to the connect the dots argument.



First map in points and then pick a geodesic to map to.



What you get is a triangle mapping onto a tripod. Then crush the triangle connecting the preimages of the center of the tripod to a point and map obviously with the boundary mapping in geodesically. Point inverses look like the picture below.



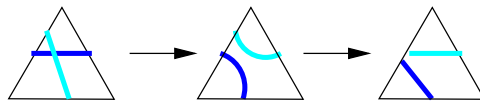
We call the disconnecting dual graphs tracks, and all we know about these tracks is that they are connected. We still have the problem that the tracks might not be disjoint. So find a graph that separates that are as short as possible. With luck we can do a cut and paste argument and then the minimal graphs will not intersect. Thus we get an action on some tree and then repeat the argument so that any unbounded chunks become manageable.

□

# 1 Lecture 6

The following is a summary of what happened last time. What can we know about some group  $G$  that is quasi-isometric to  $\mathbb{F}_k$  a non-abelian free group. To this end we did the following.

1. Build  $K$  a simply connected 2-complex with properly discontinuous and cocompact  $G$  action.
2. Find compact subsets  $C$  of  $K$  such that  $K/C$  has more than one unbounded component.
3. Use some definition to find “minimal” such  $C$ .
4. Hope that by minimality translates are disjoint. The picture to imagine is the following.



5. Conclude that  $G$  acts on the dual tree with finite edge stabilizers.

**Question 38.** *Why is the dual graph a tree?*

**Answer 39.** *Suppose not. Then there is some non-trivial loop in the dual graph and thus removing one of the translates of  $C$  does not disconnect the space which is a contradiction.*

**Question 40.** *Is the tree that you obtain necessarily locally finite?*

**Answer 41.** *No.*

**Theorem 42** (Stalling’s Ends Theorem). *Suppose that  $G$  is finitely presented and  $G$  has more than one end. Then  $G$  has a non-trivial splitting  $G = A *_C B$  or  $G = A *_C$  with  $C$  finite.*

*Proof.* If  $G$  is quasi-isometric to a free group then outline gives a variant of Dunwoody’s proof of Stalling’s Theorem where the finitely presented assumption is necessary to build the 2-complex and the assumption that  $G$  has more than one end enables us to find minimal compact sets that disconnect.  $\square$

**Question 43.** *What is the splitting that this theorem gives for  $\mathbb{Z}$ ?*

$$\mathbb{Z} = \begin{array}{c} \circlearrowleft \\ \downarrow \\ \mathbb{1} \end{array}$$

**Exercise 44.** ?

Why does the splitting process given by Stallings's Theorem terminate? The answer is that the group  $G$  is finitely presented. More precisely

1. If  $G$  is torsion free then  $G = A * B$
2. One can try to inductively split  $A$  and  $B$  until we get a graph of groups decomposition for  $G$  with  $\leq 1$  – ended vertex groups and finite edge groups (if  $G$  is torsion free this follows from Grushko's Theorem). But we need finitely presented to get this process to terminate in general.

In fact there are known examples where this process does not terminate. Dunwoody has an example of a finitely generated but not finitely presented group where the process does not stop.

What do we know so far?

1. If  $G$  is Gromov hyperbolic then we have  $\partial G$  which is quasi-isometry invariant.
2. If  $G$  splits you may be able to reduce your question to the pieces. IE analyze your mystery group in terms of quasi-isometry invariants of the pieces.
3. If  $G$  has polynomial growth (almost flat) then *Gromov* says that  $G$  is virtually nilpotent which reduces your question to nilpotent groups which is a very open area.
4. What if  $G$  has non-positive curvature? Unfortunately there is not a large scale (ie coarse) notion of non-positive curvature.

What about the specific case of three-manifold geometries? We can cut along tori to get an analogue of the graph of groups decompositions where the pieces have one of the following geometries:  $S^3$ ,  $S^2 \times S^1$ ,  $\mathbb{R}^3$ ,  $\mathbb{H}^3$ ,  $\mathbb{H}^2 \times \mathbb{R}$ , Nil, Sol, or  $\widetilde{SL_2\mathbb{R}}$ . There are certain geometries that we can “handle” at this point (handle means know all groups quasi-isometric to this geometry) namely  $S^3$  (QI to trivial group),  $S^2 \times S^1$  (compact),  $\mathbb{R}^3$ ,  $\mathbb{H}^3$ ,  $\mathbb{H}^2 \times \mathbb{R}$  (Kleiner spoke about this one), Nil. We are not going to worry about Sol.  $\widetilde{SL_2\mathbb{R}}$  turns out to be annoying but we will say some words about it.

An example of a manifold with  $\widetilde{SL_2\mathbb{R}}$  geometry is  $M = \{\text{unit tangent bundle of a closed surface } \Sigma \text{ of genus } \geq 2\}$ . In this case  $M$  is a non-trivial circle bundle over  $\Sigma$

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi_1(\Sigma) \rightarrow 1$$

which gives a non-trivial central extension. This almost looks like  $\pi_1(M) = \pi_1(\Sigma) \times \mathbb{Z}$  and so you might think that  $\widetilde{SL_2\mathbb{R}}$  is quasi-isometric to  $\mathbb{H}^2 \times \mathbb{R}$ , but  $\widetilde{SL_2\mathbb{R}}$  is not non-positively curved. There is a result in this vein which is the following exercise.

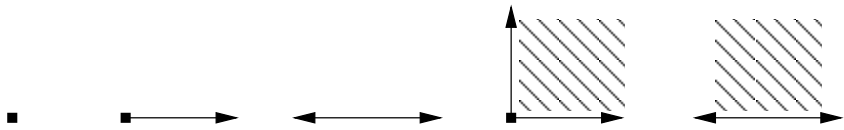
**Exercise 45.** *A central extension*

$$1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow H \rightarrow 1$$

*is quasi-isometric to  $H \times \mathbb{Z}$  if there is a Lipschitz section  $H \rightarrow G$ . This is not quite an if and only if statement but it is very close.*

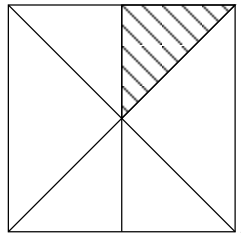
If  $X$  is  $CAT(0)$  one approach would be to try to control the quasi-flats of maximal rank (quasi-isometric embeddings of  $\mathbb{R}^n$  where  $n$  is as large as possible) in order to determine the question of what spaces  $X$  is quasi-isometric to.

**Example 46.** Examine  $\mathbb{H}^2 \times \mathbb{H}^2$ . Here the maximal flats are isometric copies of  $\mathbb{R}^2$  given by products of the form  $l_1 \times l_2$  where  $l_1$  and  $l_2$  are geodesics in different factors of  $\mathbb{H}^2 \times \mathbb{H}^2$ . A natural set of questions is where and how do two flats come close to each other. In each factor they can get close in three distinct ways namely they can intersect in a point (different endpoints on the boundary), can become close along a ray (can share one endpoint on the boundary), or they are the same geodesic (sharing 2 points on the boundary). This gives essentially 5 different behaviors:



The quarter plane is the basic building block of a two-dimensional flat. Build a graph to record the incidence data. Let the vertices be the vertical and horizontal rays up to bounded distance which equals  $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ . Connect the vertices by an edge if they bound a quadrant (this is the analogue to the visual boundary of hyperbolic space). This gives a complete bipartite graph which implies that a quasi-isometry of this space is a quasi-isometry of each of the factors because of the fact that switching sides is not allowed.

**Example 47.** Now let us look at  $SL_3\mathbb{R}/SO(3)$ . Here the graph defined in the same way is again bipartite with the basic unit being a sixth of a plane



The graph at “ $\infty$ ” is the incidence graph of points and lines in the real projective plane. It turns out that automorphisms of this graph are (essentially)  $SL_3\mathbb{R}$ . This relationship is analogous to the relationship of the curve complex to the mapping class group.

If you can control the flats then you have a handle on what is going on. Unfortunately, there are “bad” quasi-flats even in  $\mathbb{H}^2 \times \mathbb{H}^2$ . One example is that you can get a loop of length six in the bipartite graph. Thus you can glue six quadrants together to form a quasi-flat. A general way to build quasi-flats is to build them by stringing together a finite number of flats.