9.2 – Arithmetic Sequences and Series

→ In the last section, we got a general overview of sequences and series. In this section we look at a specific type of sequence called an arithmetic sequence.

→ A sequence is arithmetic if the difference between consecutive terms is always the same. So

$a_2 - a_1 = a_3 - a_2 = a_4 - a_3$ etc. $= d$

→ d is called the common difference.

→ Arithmetic sequences are the easiest to use. They are analagous to lines.

Ex a) $1, 3, 5, 7, 9, \ldots$ → arithmetic with common difference $d = 2$

→ $a_n = 2n - 1$

b) $2, 6, 10, 14, 18, \ldots$ → arithmetic with $d = 4$

→ $a_n = 4n - 2$

c) $1, -4, -9, -14, -19, \ldots$ → arithmetic with $d = -5$

→ $a_n = -5n + 6$

d) $1, 4, 9, 16, 25, \ldots$ → not arithmetic

→ $a_n = n^2$

→ we can see from the example that the general form for the n^{th} term of an arithmetic sequence is

$$a_n = dn + c$$

→ d is common difference and c is the appropriate number to get our sequence started on the right number (when $n=1$).

Ex Find an arithmetic sequence with common difference 5 and first term $a_1 = 6$.

$$a_n = 5n + 1$$

$a_1 = 6,\ a_2 = 11,\ a_3 = 16,\ \ldots$

Ex Find an arithmetic sequence whose 4^{th} term is 22 and 13^{th} term is 67.

→ 13^{th} term is 9 terms more than 4^{th}, so we go through 9 common differences to get there.

so $a_{13} = a_4 + 9d \Rightarrow 67 = 22 + 9d \Rightarrow 45 = 9d \Rightarrow 5 = d$

Then $a_n = 5n + c$, → use one of the terms we know to find c.

$a_4 = 5(4) + c \Rightarrow 22 = 20 + c \Rightarrow 2 = c$

so $\boxed{a_n = 5n + 2}$

$7, 12, 17, 22, \ldots$

Ex Find a sequence whose 3rd term is 7 and 11th term is 39.

$a_{11} = a_3 + 8d \Rightarrow 39 = 7 + 8d \Rightarrow 32 = 8d \Rightarrow 4 = d$

so $a_n = 4n + C$ and $a_3 = 4(3) + C \Rightarrow 7 = 12 + C \Rightarrow -5 = C$

then $a_n = 4n - 5$

$-1, 3, 7, 11, 15, \ldots$

→ It turns out that there is a nice formula for finding the sum of a finite arithmetic series.

Ex Consider the series $\sum_{i=1}^{\infty} 3i - 1$

Find S_5, the fifth partial sum.

$S_5 = 2 + 5 + 8 + 11 + 14$

Instead of just adding, notice the following:

$2 \cdot S_5 = 2 + 5 + 8 + 11 + 14 + 2 + 5 + 8 + 11 + 14$

$= (2+14) + (5+11) + (8+8) + (11+5) + (14+2)$ → each adds to 16 and there are 5 copies

$= 5 \cdot 16$

$= 80$

So then $S_5 = \frac{80}{2} = 40$

→ In general, the sum of a finite arithmetic sequence with n terms is

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Ex Find the sum of the first 100 integers

$$S_{100} = \frac{100}{2}(1 + 100) = 50(101) = 5050$$

Ex Find the 36^{th} partial sum of the sequence

$5, 16, 27, 38, 49, \ldots$

$$S_{36} = \frac{36}{2}(a_1 + a_{36})$$

→ need a_{36}. Let's find a general form for the n^{th} term.

Common difference = 11 so $a_n = 11n + C$

$\Rightarrow a_1 = 11(1) + C$

$\Rightarrow 5 = 11 + C$

$\Rightarrow -6 = C$

so $a_n = 11n - 6$. Then $a_{36} = 11(36) - 6$

$= 396 - 6 = 390$

so $S_{36} = \frac{36}{2}(5 + 390) = 18(395) = 7110$

EX Determine the seating capacity of an auditorium if there are 36 rows with 15 seats in the first row, 18 seats in the second, 21 in the 3rd and so on.

→ common difference : $d = 3$

$a_n = 3n + c$

$a_1 = 3 + c \Rightarrow 15 = 3 + c \Rightarrow c = 12$

so $a_n = 3n + 12 \Rightarrow a_{36} = 3(36) + 12 = 108 + 12 = 120$

Then $S_{36} = \frac{36}{2}(15 + 120) = 18(135) = 2430$ seats