9.1 – Sequences and Series

→ In this chapter, we start to talk about things that are a bit different than the things we've discussed previously. Our first new ~~topic~~ topic is sequences & series.

→ A sequence is a function whose domain is the set of positive integers. The functional values
(sometimes nonnegative)

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

are called the terms in the sequence.

→ So really, a sequence is just a list of numbers.

Ex Suppose we have a sequence $a_n = 2n+1$. The first 4 terms are

$a_1 = 2(1)+1 = 3$

$a_2 = 2(2)+1 = 5$

$a_3 = 2(3)+1 = 7$

$a_4 = 2(4)+1 = 9$

Ex the first ~~4~~ 5 terms of $a_n = 3n + (-1)^n$ are

$a_1 = 3(1) + (-1)^1 = 3-1 = 2$

$a_2 = 3(2) + (-1)^2 = 6+1 = 7$

$a_3 = 3(3) + (-1)^3 = 9-1 = 8$

$a_4 = 3(4) + (-1)^4 = 13$

$a_5 = 3(5) + (-1)^5 = 15-1 = 14$

Ex Find the 1st 4 terms of $a_n = \frac{(-1)^n}{n^2+1}$

$a_1 = \frac{(-1)^1}{1^2+1} = \frac{-1}{2}$

$a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$

$a_3 = \frac{(-1)^3}{3^2+1} = \frac{-1}{10}$

$a_4 = \frac{(-1)^4}{4^2+1} = \frac{1}{17}$

→ any time you have $(-1)^n$ multiplying the terms in a sequence, the sign will alternate.

→ Now instead of being told the n^{th} term in a sequence, lets try to guess it.

Ex Find the n^{th} term of

a) 3, 7, 11, 15, 19, ...

$a_n = 4n-1$

b) −2, +5, −10, +17, −26

$a_n = (-1)^n(n^2+1)$

→ Sometimes a sequence is defined in terms of previous terms. A sequence like this is called a recursive sequence.

→ The Fibonacci Sequence is a famous recursive sequence.

Fibonacci Sequence:

$a_0 = 1,\ a_1 = 1,\quad a_k = a_{k-2} + a_{k-1}$ for $k \geq 2$

$a_0 = 1$

$a_1 = 1$

$a_2 = a_0 + a_1 = 1 + 1 = 2$

$a_3 = a_1 + a_2 = 1 + 2 = 3$

$a_4 = a_2 + a_3 = 2 + 3 = 5$

$a_5 = a_3 + a_4 = 3 + 5 = 8$

$\vdots$

→ Some other sequences are defined using things called factorial

↳ If n is a positive integer then n factorial is

$$n! = n(n-1)(n-2)\cdot \ldots \cdot (4)(3)(2)(1)$$

↳ product of all the integers up to n.

→ we define $0! = 1$

→ factorials follow the order of operation like exponents.

$$2n! = 2(n!) = 2\left(n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (3) \cdot 2 \cdot 1\right)$$

$$(2n)! = 2n \cdot (2n-1)(2n-2) \cdot \ldots\ 3 \cdot 2 \cdot 1$$

Ex first 4 terms of $a_n = \dfrac{n^2}{(n-1)!}$

$$a_1 = \frac{1^2}{(1-1)!} = \frac{1}{0!} = \frac{1}{1} = 1$$

$$a_2 = \frac{2^2}{(2-1)!} = \frac{4}{1!} = 4$$

$$a_3 = \frac{3^2}{(3-1)!} = \frac{9}{2!} = \frac{9}{2\cdot 1} = \frac{9}{2}$$

$$a_4 = \frac{4^2}{(4-1)!} = \frac{4^2}{3!} = \frac{16}{3\cdot 2\cdot 1} = \frac{16}{6} = \frac{8}{3}$$

→ Factorials are often easy to evaluate in expressions because lots of things cancel.

Ex Evaluate

a) $\dfrac{6!}{2!\,3!} = \dfrac{6\cdot 5\cdot 4\cdot 3!}{2!\,3!} = \dfrac{6\cdot 5\cdot 4}{2\cdot 1} = 60$

b) $\dfrac{4!\,3!}{5!} = \dfrac{4!\,3!}{5\cdot 4!} = \dfrac{3\cdot 2\cdot 1}{5} = \dfrac{6}{5}$

c) $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)\,n!}{n!} = (n+2)(n+1) = n^2+3n+2$

→ well end up using factorials quite a bit in later sections.

→ Sometimes we may want to add up the terms in a sequence. We use summation or Sigma notation for this.

→ If a_n is a sequence, ~~the sum of th~~

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

i → index of summation. n → upper limit. 1 → lower limit

Ex a) $\sum_{i=1}^{3} 2i = 2(1) + 2(2) + 2(3) = 2+4+6 = 12$

↳ or $\underbrace{2(1+2+3)}_{2\sum_{i=1}^{3} i} = 2(6) = 12$

b) $\sum_{i=1}^{3} i+2 = (1+2) + (2+2) + (3+2)$

$= 1+2+2+2+3+2$

$= (1+2+3) + (2+2+2)$

$= \underbrace{(1+2+3)}_{\sum_{i=1}^{3} i} + \underbrace{2(1+1+1)}_{2\sum_{i=1}^{3} 1}$

$= 6+6$

$= 12$

Properties of Sums

1) $\sum_{i=1}^{n} c = cn$

2) $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

3) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4) $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Note: $\sum_{i=1}^{n} a_i b_i \neq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$

→ When we add up terms in a sequence, we call the sum a Series. If we add up n terms in the sequence, we call it an n^{th} partial sum or a finite series

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n \quad \rightarrow n^{th} \text{ partial sum}$$

→ If we add up an infinite number of terms, it's called an infinite series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$$

Ex for the series $\sum_{i=1}^{\infty} \frac{6}{10^i}$

a) find the 3rd partial sum

$$\sum_{i=1}^{3} \frac{6}{10^i} = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} = .6 + .06 + .006 = 0.666$$

b) Find the sum of the series

$$\sum_{i=1}^{3} \frac{6}{10^i} = \frac{6}{10^1} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \ldots$$

$$= 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$$

$$= 0.66666\ldots = 2/3$$

→ We can add up an infinite number of things and still get a finite result!