8.5 - Applications of Matrices and Determinants

→ There are many applications of determinants. In this section, we'll look at just a few of them

→ The first application is called Cramer's rule.

Consider the system

$$\begin{aligned} a_1x+b_1y &= c_1 \\ a_2x+b_2y &= c_2 \end{aligned} \longrightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This has the solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

We can write these as determinants

$$x = \frac{\overbrace{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}^{D_x}}{\underbrace{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}_{D}} \qquad y = \frac{\overbrace{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}^{D_y}}{\underbrace{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}_{D}}$$

→ In the numerator, you replace the column corresponding to the variable you're solving with the answer column and then take the determinant. In the denominator, you take the determinant of the coefficient matrix.

Ex Solve

$2x - 5y = 3$

$-4x + 3y = 8$

$D = \begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix} = 6 - 20 = -14$, $D_x = \begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix} = 9 + 40 = 49$

$D_y = \begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix} = 16 + 12 = 28$

so $x = \dfrac{49}{-14} = \dfrac{-7}{2}$, $y = \dfrac{28}{-14} = -2$

This is generalizable to larger systems

Ex Solve

$-x + 2y - 3z = 1$

$2x + z = 0$

$3x - 4y + 4z = 2$

$D = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + 0 - 1\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} = -2(8-12) - 1(4-6)$
$= 8 + 2 = 10$

→ replace 1st column w/ answer

$D_x = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = -0 + 0 - 1\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -1(-4-4) = 8$

→ replace 2nd column with answer

$D_y = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} + 0 - 1\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -2(4+6) - 1(-2-3)$
$= -20 + 5 = -15$

→ replace 3rd column w/ answer

$D_z = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ -4 & 2 \end{vmatrix} + 0 - 0 = -2(4+4) = -16$

so $x = \frac{D_x}{D} = \frac{8}{10} = \frac{4}{5}$

$y = \frac{D_y}{D} = \frac{-15}{10} = \frac{-3}{2}$

$z = \frac{D_z}{D} = \frac{-16}{10} = \frac{-8}{5}$

→ Our second application is finding the area of a triangle.

→ The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

→ the $\pm$ is just to mean the area should be positive

<u>Ex</u> Find the area of the triangle with vertices $(1,0)$, $(2,2)$, and $(4,3)$

$$\text{Area} = \pm \frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \pm \frac{1}{2}\left(1 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 0 + 1 \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right)$$

$$= \pm \frac{1}{2}\left((2-3) + (6-8)\right)$$

$$= \pm \frac{1}{2}(-1-2)$$

$$= \pm \frac{-3}{2} \rightarrow \boxed{\text{Area} = \frac{3}{2}}$$

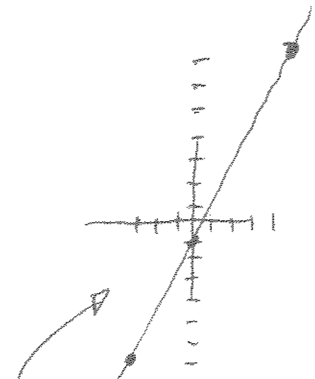
Ex Find the area of a triangle with vertices $(-3,-7)$, $(0,1)$ and $(4,7)$

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} -3 & -7 & 1 \\ 0 & -1 & 1 \\ 4 & 7 & 1 \end{vmatrix} = \pm\frac{1}{2}\left(-3\begin{vmatrix} -1 & 1 \\ 7 & 1 \end{vmatrix} - 0 + 4\begin{vmatrix} -7 & 1 \\ -1 & 1 \end{vmatrix}\right)$$

$$= \pm\frac{1}{2}\left(-3(-1-7)+4(-7+1)\right)$$

$$= \pm\frac{1}{2}(24-24)$$

$$= 0$$



→ An area of zero means the 3 points lie on the same line

→ we can use determinants to test to see if 3 points are collinear.

→ our last example is a fun one. we can use matrices to encode a message. well set up our system as follows:

0 = ___ → space

1 = A

2 = B

3 = C

etc.

26 = Z

we could write the word HELLO as

$$\begin{matrix}[8 & 5 & 12] & [12 & 15 & 0] \\ H & E & L & L & O & \end{matrix}$$

→ we use an invertible 3x3 matrix to encode the word

→ So the HEL part, $[8 \quad 5 \quad 12]$ after multiplication with the invertible matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & 4 \end{bmatrix}$ becomes

uncoded   encoding matrix   coded

$$[8 \quad 5 \quad 12]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & 4 \end{bmatrix} = [8-5+12 \quad 8(-2)+5-12 \quad 8(2)+5(3)+12(4)]$$

$$= [15 \quad -23 \quad 79]$$

→ The reason we use an invertible matrix is that then we have a way to decode the words.

Suppose B is the uncoded 1X3 matrix and C is the coded one. Then

$$BA = C \Rightarrow BAA^{-1} = CA^{-1} \Rightarrow BI = CA^{-1} \Rightarrow B = CA^{-1}$$

So from C and the inverse of A, we can get back the uncoded matrix B.