A.2 - Exponents and Radicals

→ To multiply a number (or algebraic expression) by itself, we use an exponent.

Ex

a) $\underset{\text{base}}{2}^{\overset{\text{exponent}}{4}} = 2\cdot 2\cdot 2\cdot 2 = 16$ → "2 to the 4th power"

b) $\underbrace{(-3)}_{\text{base}}{}^{\overset{\text{exponent}}{3}} = (-3)(-3)(-3) = -27$ → "-3 to the 3rd power"

Properties

1) $a^m a^n = a^{m+n}$ $\qquad 3^2\cdot 3^3 = (3\cdot 3)(3\cdot 3\cdot 3) = 3^5$

2) $\dfrac{a^m}{a^n} = a^{m-n}$ $\qquad \dfrac{2^3}{2^2} = \dfrac{2\cdot 2\cdot 2}{2\cdot 2} = 2^1$

3) $a^{-m} = \dfrac{1}{a^m}$

4) $a^0 = 1$; $a \neq 0$ → an number raised to the zeroth power is 1

5) $(ab)^m = a^m b^m$ $\qquad (2\cdot 3)^3 = (2\cdot 3)(2\cdot 3)(2\cdot 3) = 2^3 3^3$

6) $(a^m)^n = a^{mn}$ $\qquad (2^2)^3 = (2\cdot 2)(2\cdot 2)(2\cdot 2) = 2^6$

) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\qquad \left(\dfrac{2}{3}\right)^3 = \left(\dfrac{2}{3}\right)\left(\dfrac{2}{3}\right)\left(\dfrac{2}{3}\right) = \dfrac{2^3}{3^3}$

Ex: Simplify

a) $(2a^2b^{-3})(-3a^{-1}b^4) = -6a^{2-1}b^{-3+4} = -6ab$

b) $(3a^2b)^3 = 3^3(a^2)^3b^3 = 27a^6b^3$

c) $\left(\frac{2x^2}{y}\right)^3 = \frac{2^3(x^2)^3}{y^3} = \frac{8x^6}{y^3}$

EX write with positive exponents

a) $\left(\frac{x}{y^3}\right)^{-1} = \frac{x^{-1}}{y^{-3}} = \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{y^3}\right)} = \frac{1}{x} \cdot \frac{y^3}{1} = \frac{y^3}{x}$

b) $\frac{2a^{-2}b^3}{4a^{-3}b^{-4}} = \frac{a^{-2-(-3)}\,b^{3-(-4)}}{2} = \frac{ab^7}{2}$

c) $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}x^{-4}}{y^{-2}} = \frac{\frac{1}{3^2}\,\frac{1}{x^4}}{\frac{1}{y^2}} = \frac{\frac{1}{9x^4}}{\frac{1}{y^2}} = \frac{1}{9x^4} \cdot \frac{y^2}{1} = \frac{y^2}{9x^4}$

Radicals

→ If $a = b^n$, we say b is the n^{th} root of a.

$b = \sqrt[n]{a}$ or $b = a^{1/n}$ → a fractional exponent is a radical

EX a) $\sqrt{36} = 6$ because $6^2 = 36$

b) $27^{1/3} = 3$ because $3^3 = 27$

c) $\sqrt[4]{16} = 2$ because $2^4 = 16$

d) $\sqrt{-9}$ not a real number because you can't square a number & get a negative result

Properties of Radicals

1) $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ $\qquad$ $\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

2) $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ $\qquad$ $\sqrt[3]{3} \cdot \sqrt[3]{5} = \sqrt[3]{15}$

3) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ $\qquad$ $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}} = \sqrt[3]{\dfrac{32}{4}} = \sqrt[3]{8} = 2$

4) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ $\qquad$ $\sqrt{\sqrt[3]{6}} = \sqrt[6]{6}$

$\left(\sqrt[n]{a}\right)^n = a$ $\qquad$ $\left(\sqrt{7}\right)^2 = 7$

Ex Simplify

1) $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

2) $\sqrt[3]{32x^4} = \sqrt[3]{8 \cdot 4 \cdot x^3 \cdot x} = \sqrt[3]{8}\,\sqrt[3]{x^3}\,\sqrt[3]{4x} = 2x\sqrt[3]{4x}$

3) $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3} = 2\sqrt{16}\sqrt{3} - 3\sqrt{9}\sqrt{3}$

$= 8\sqrt{3} - 9\sqrt{3}$

$= (8-9)\sqrt{3} = -\sqrt{3}$

Rationalizing the Denominator

↳ Technically, an expression is not in simplest form when it has a radical in the denominator.

Ex Rationalize the denominator (remove radicals)

i) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2\sqrt{3}}{3}$

ii) $\frac{3}{\sqrt[3]{4}} = \frac{3}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{4^2}}{\sqrt[3]{4^2}} = \frac{3\sqrt[3]{16}}{\sqrt[3]{4^3}} = \frac{3\sqrt[3]{16}}{4}$

↳ what if the denominator is more complicated? Ex $\frac{4}{2-\sqrt{3}}$

↳ notice: $\underbrace{(2-\sqrt{3})(2+\sqrt{3})}_{\text{FOIL}} = 4+2\sqrt{3}-2\sqrt{3}\;\sqrt{9} = 4-3=1$

Then

$$\frac{4}{2-\sqrt{3}} = \frac{4(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{4(2+\sqrt{3})}{1} = 4(2+\sqrt{3})$$

→ these are called conjugates

↳ multiplying conjugates gives a result that has no radicals.

Ex Rationalize the denominator

$\frac{2}{3+\sqrt{5}}$ → multiply both numerator and denominator by the conjugate, $3-\sqrt{5}$

$$= \frac{2(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{2(3-\sqrt{5})}{9-3\sqrt{5}+3\sqrt{5}-\sqrt{25}} = \frac{2(3-\sqrt{5})}{9-5} = \frac{2(3-\sqrt{5})}{4}$$

$$= \frac{3-\sqrt{5}}{4}$$

Rational Exponents

Another way we can write a radical is using a rational exponent.

Ex a) $\sqrt{3} = 3^{1/2}$

b) $\sqrt[4]{12} = 12^{1/4}$

c) $2^{3/2} = \sqrt{2^3} = \left(\sqrt{2}\right)^3$

d) $4^{-2/3} = \dfrac{1}{4^{2/3}} = \dfrac{1}{\sqrt[3]{4^2}} = \dfrac{1}{\left(\sqrt[3]{4}\right)^2}$

→ The numerator in the exponent is the power + the denominator is the root

→ In WebWork, you can write $\sqrt{x}$ as sqrt(x), or $x^{1/2}$, but every other root must be written using a rational exponent. For example $\sqrt[3]{x}$ is $x^{1/3}$ written x^(1/3) or x**(1/3).

Ex Simplify

a) $3a\sqrt[3]{a^2} = 3aa^{2/3} = 3a^{1+2/3} = 3a^{\frac{3}{3}+\frac{2}{3}} = 3a^{5/3}$

b) $\dfrac{\left(-2x^{3/2}\right)}{\sqrt[3]{x}} = \dfrac{-2x^{3/2}}{x^{1/3}} = -2x^{\frac{3}{2}-\frac{1}{3}} = -2x^{\frac{9}{6}-\frac{2}{6}} = -2x^{\frac{7}{6}}$