9.6 - Counting Principles

Ex A college needs 2 faculty: chemist and a mathematician.

There are 3 applicants for chemistry and 5 applicants for math.

How many possible ways are there to fill the positions

→ you can pair each chemistry applicant with one of 5 math applicants, giving a total of

$3 \cdot 5 = 15$ ways to fill the positions

→ This is an application of the Fundamental Counting Principle which says if E_1 and E_2 are 2 events ~~st~~ E_1 can occur in m_1 ways and then E_2 can occur in m_2 ways. the 2 events can occur in

$m_1 \cdot m_2$ ways

Ex License plate consists of 2 letters followed by a 4 digit number. Letters "O" and "I" are omitted and no letters or numbers are repeated. How many distinct license plates are there?

choices

$\underline{24}$ $\underline{23}$ $\underline{10}$ $\underline{9}$ $\underline{8}$ $\underline{7}$ → we see how many ways we can fill each slot.

Total of $24 \cdot 23 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 2,782,080$ license plates

Ex 4 boys, Jim, John, Bill, + Bob walk through a doorway one at a time. How many orderings are there for how they can walk through?

$\underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 4! = 24$ orderings

→ The last example was an example of something called a permutation

→ A permutation is an ordering of elements where one element is first, one is second, one is third, etc. (order matters)

→ There are $n!$ permutations of n elements.

Ex How many ways can 6 people sit in a 6 passenger car?

→ it's an ordering (permutation) of 6 people.

$6! = 720$ ways

Ex 7 racers run the 100-meter dash. How many ways can sprinters come in 1st, second, and third? (no ties)

$$\underset{1st}{\underline{7}} \quad \underset{2nd}{\underline{6}} \quad \underset{3rd}{\underline{5}} \qquad 7 \cdot 6 \cdot 5 = \frac{7!}{4!} = \frac{7!}{(7-3)!} = 210 \text{ ways}$$

→ The number of   of n elements taken r at a time is $_nP_r = \frac{n!}{(n-r)!}$

<u>Ex</u> How many distinguishable permutations are there of the letters MISSISSIPPI

→ 11! permutations of the letters, but they're not all distinguishable.

4! permutations of the I's → (we can't tell them apart so we don't want to count all of them)

4! permutations of the S's.

2! Permutations of the P's.

So $\dfrac{11!}{4!\,4!\,2!} = \dfrac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4!}{4!\,4!\,2!} = \dfrac{11\cdot10\cdot9\cdot\cancel{8}\cdot7\cdot\cancel{6}\cdot5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot2\cdot1}$

$= 11\cdot10\cdot9\cdot7\cdot5 = 34{,}650$

distinguishable permutations

The denominator keeps us from overcounting the indistinguishable permutations

<u>Ex</u> 5 students are selected from a pool of 20 for an experiment. How many different groups of 5 are possible.

$\underline{20}\ \ \underline{19}\ \ \underline{18}\ \ \underline{17}\ \ \underline{16}$ → $20\cdot19\cdot18\cdot17\cdot16$

→ if you have a group of 5, you can order it in 5! ways. but you still have the same group. So really there are

$\dfrac{20\cdot19\cdot18\cdot17\cdot16}{5!} = \dfrac{20!}{15!\,5!} = {}_{20}C_5 = 15{,}504$ different groups.

→ In the last example, the members in the group mattered, but not the order we chose those members.

→ When order doesn't matter, we call it a combination.

→ The number of combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n-r)!\,r!}$$

→ also written $\binom{n}{r}$

→ we usually say "n choose r"

Ex There are 8 applicants for 4 job openings. 3 of the applicants are women. How many ways are there to fill the 4 positions?

→ $_8C_4 = \binom{8}{4} = \frac{8!}{4!4!} = \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1} = 70$ ways

→ How many ways if exactly 2 selections are women?

→ Choose 2 women from 3: $\binom{3}{2}$ ways → Event 1

→ Choose 2 men from 5: $\binom{5}{2}$ ways → Event 2

⇒ Total # of ways $= \binom{3}{2}\cdot\binom{5}{2} = \frac{3!}{2!1!}\cdot\frac{5!}{3!2!} = 3\cdot\frac{5\cdot 4}{2} = 30$ ways

→ Let's look at some of the ways you can be dealt different hands when playing 5 card stud with a single deck.

→ There are 52 cards in a deck. 4 suits with 13 cards in each suit.

1) # of ways to be dealt a hand: $\binom{52}{5} = \frac{52!}{47!\,5!} = 2{,}598{,}960$ 5

2) # of ways to be dealt 2 pair:

Choose 2 of 13 ranks: $\binom{13}{2}$

Choose 2 of 4 suits: $\binom{4}{2}$

Choose 2 of 4 suits: $\binom{4}{2}$

Choose 1 of 11 remaining ranks: $\binom{11}{1}$

Choose 1 of 4 suits: $\binom{4}{1}$

$$\text{Total} = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1} = \frac{13\cdot 12}{2}\cdot\frac{4\cdot 3}{2}\cdot\frac{4\cdot 3}{2}\cdot 11\cdot 4 = 123552$$

3) # of ways to be dealt full house

choose 1 of 13 ranks: $\binom{13}{1}$

Choose 3 of 4 suits: $\binom{4}{3}$

Choose 1 of 12 ranks: $\binom{12}{1}$

Choose 2 of 4 suits: $\binom{4}{2}$

$$\text{Total} = \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13\cdot 4\cdot 12\cdot\frac{4\cdot 3}{2} = 3744$$

4) # of ways to be dealt straight flush (including royal flush)

choose 1 of 10 starting numbers: $\binom{10}{1}$

choose 1 of 4 suits $\binom{4}{1}$

$\left.\right\}$ $\binom{10}{1}\binom{4}{1} = 10\cdot 4 = 40$

5) # of ways to be dealt royal flush

Choose 1 of 4 suits: $\binom{4}{1}$

Choose 1 of 1 starting numbers: $\binom{1}{1}$

$\left.\right\}$ $\binom{4}{1}\binom{1}{1} = 4\cdot 1 = 4$