9.5 The Binomial Theorem

→ Consider the following

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4x3y+y^3$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

→ notice: - x exponent steps down by 1 + y up by 1 in each term
- sum of x and y exponents is always the same as the original exponent.
- There is symmetry in the coefficients.

Binomial Theorem

In the expansion of $(x+y)^n$

$$(x+y)^n = x^n + nx^{n-1}y + \dots + {}_nC_r x^{n-r}y^r + \dots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$$nCr = \frac{n!}{(n-r)!\,r!}$$

→ Sometimes $\binom{n}{r}$ is written instead of nCr

→ these are called binomial coefficients and give us the coefficients in the binomial expansion.

Ex Evaluate

a) $_8C_2 = \dfrac{8!}{(8-2)!\,2!} = \dfrac{8!}{6!\,2!} = \dfrac{8\cdot 7\cdot 6!}{6!\,2!} = \dfrac{8\cdot 7}{2\cdot 1} = 28$

b) $\dbinom{7}{3} = \dfrac{7!}{(7-3)!\,3!} = \dfrac{7!}{4!\,3!} = \dfrac{7\cdot 6\cdot 5\cdot 4!}{4!\,3!} = \dfrac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$

c) $_8C_6 = \dfrac{8!}{(8-6)!\,6!} = \dfrac{8!}{2!\,6!} = \dfrac{8\cdot 7\cdot 6!}{2!\,6!} = \dfrac{8\cdot 7}{2\cdot 1} = 28$

d) $\dbinom{8}{8} = \dfrac{8!}{(8-8)!\,8!} = \dfrac{8!}{0!\,8!} = 1 \quad \to 0! = 1$

↳ notice the symmetry between a) and c)

→ There is another way we can find binomial coefficients.
It is called Pascal's Triangle

$$
\begin{array}{ccccccccccccccccl}
 & & & & & & & & 1 & & & & & & & & \leftarrow \text{row zero} \\
 & & & & & & & 1 & & 1 & & & & & & & \leftarrow \text{row 1} \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \leftarrow \text{row 6} \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \leftarrow \text{row 7}
\end{array}
$$

$$
\begin{array}{cccccccc}
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
{}_7C_0 & {}_7C_1 & {}_7C_2 & {}_7C_3 & {}_7C_4 & {}_7C_5 & {}_7C_6 & {}_7C_7
\end{array}
$$

→ The entries in the rows of pascal's Triangle are binomial coefficients.

Ex Write the expansion of $(x+y)^4$ using Pascal's Triangle.

→ Look in row 4 to get the binomial coefficients, the coefficients in the expansion.

coefficients: 1, 4, 6, 4, 1

so $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Ex Expand $(x+2)^3$ → Binomial Coefficients: 1, 3, 3, 1

↳ takes place of y

$$(x+2)^3 = x^3 + 3x^2(2) + 3x(2)^2 + 2^3$$
$$= x^3 + 6x^2 + 27x + 8$$

Ex Expand $(x-2y)^4$ → $(x+(-2y))^4$

↳ takes place of y

Binomial coefficients: 1, 4, 6, 4, 1

$$(x-2y)^4 = x^4 + 4x^3(-2y) + 6x^2(-2y)^2 + 4x(-2y)^3 + (-2y)^4$$
$$= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$$

↳ whenever we expand a binomial with subtraction in it, the signs alternate in the expansion

Ex Expand $(2x+3)^3$ $\to$ binomial coeff: 1, 3, 3, 1

$$(2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)(3^2) + 3^3$$
$$= 8x^3 + 36x^2 + 54x + 27$$

Ex Find the 4th Term in the expansion of $(x-10z)^7$

Binomial coefficients: ${}_7C_0$, ${}_7C_1$, ${}_7C_2$, ${}_7C_3$, etc.

so 4th term has $\quad {}_7C_3\, x^{7-3}(-10z)^3$

(1st term has x^7, 2nd $x^{7-1}(-10z)^1$, etc.)

$${}_7C_3 = \frac{7!}{(7-3)!\,3!} = \frac{7!}{4!\,3!} = \frac{7\cdot6\cdot5\cdot4!}{4!\,3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$$

$\Rightarrow$ 4th term is $\quad 35x^4(-10z)^3 = -35{,}000\,x^4z^3$

Ex In expansion of $(2x-3y)^8$, find the coefficient a of the term ax^6y^2

$\to$ 3rd term in expansion

$\quad\hookrightarrow$ looks like $\quad {}_8C_2(2x)^{8-2}(-3y)^2$

$${}_8C_2 = \frac{8!}{(8-2)! \cdot 2!} = \frac{8!}{6!2!} = \frac{8 \cdot 7 \cdot 6!}{6!\ 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

→ So the term is $28 \cdot (2x)^6 (-3y)^2$

$$= 28 \cdot 64 \cdot 9 \cdot x^6 y^2$$

$$= 16{,}128\, x^6 y^2$$

so $\boxed{a = 16{,}128}$

<u>Ex</u> Evaluate $(1.1)^5$

→ use the binomial theorem!

$$(1.1)^5 = (1 + .1)^5 = 1 + 5(1)^4(.1) + 10(1)^3(.1)^2 + 10(1)^2(.1)^3 + 5(1)(.1)^4 + (.1)^5$$

$$= 1 + 5(.1) + 10(.01) + 10(.001) + 5(.0001) + .00001$$

$$= 1 + .5 + .1 + .01 + .0005 + .00001$$

$$= 1.61051$$