9.3 - Geometric Sequences and Series

→ In this section we look at another specific type of sequence, called a geometric sequence.

→ A sequence is geometric if the ratio of successive term is always the same.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots = r \rightarrow \text{common ratio}$$

Ex a) $2, 4, 8, 16, \dots$

$\frac{4}{2} = \frac{8}{4} = \dots = 2 \rightarrow r$ → $a_n = 2^n$
$= 2 \cdot 2^{n-1}$

b) $-\frac{1}{4}, +\frac{1}{16}, -\frac{1}{64}, +\frac{1}{256}, \dots$

$\frac{\frac{1}{16}}{-\frac{1}{4}} = -\frac{1}{4}, \quad \frac{-\frac{1}{64}}{\frac{1}{16}} = \frac{-16}{64} = -\frac{1}{4}$, etc $\quad r = -\frac{1}{4}$ → $a_n = \left(-\frac{1}{4}\right)^n$
$= -\frac{1}{4}\left(-\frac{1}{4}\right)^{n-1}$

c) $12, 36, 108, 324, \dots$

$\frac{36}{12} = \frac{108}{36} = \frac{324}{108} = 3 \rightarrow r$ → $a_n = 12(3^{n-1})$

General Form

The general form for a geometric sequence is $a_n = a_1 r^{n-1}$

→ Terms are $a_1 = a_1, \ a_2 = a_1 r, \ a_3 = a_1 r^2, \ a_4 = a_1 r^3, \dots$

Ex A geometric sequence with 1st term 2 and common ratio -3 is

$$a_n = 2(-3)^{n-1}$$

$$= 2, -6, 18, -54, \ldots$$

Find 10th term: $a_{10} = 2(-3)^{10-1} = 2(-3)^9 = 2(-19683) = -39366$

→ We can do problems like the ones we did in the previous section.

Ex A geometric sequence has 4th term 125 and 10th term $\frac{125}{64}$. What is the 14th term.

$a_4 = 125$, $a_{10} = \frac{125}{64}$ → a_4 gets multiplied by r 6 times to get to a_{10}.

$$a_{10} = a_4 r^6 \Rightarrow \frac{125}{64} = 125 \cdot r^6 \Rightarrow \frac{1}{64} = r^6 \Rightarrow \frac{1}{2} = r$$

Thus $a_n = a_1 \left(\frac{1}{2}\right)^{n-1}$ → to get a_1, use a_4

$$a_4 = a_1 \left(\frac{1}{2}\right)^{4-1} \Rightarrow 125 = a_1 \left(\frac{1}{2}\right)^3 \Rightarrow 125 = \frac{a_1}{8} \Rightarrow a_1 = 1000.$$

Then $a_n = 1000 \left(\frac{1}{2}\right)^{n-1}$

So $a_{14} = 1000 \left(\frac{1}{2}\right)^{14-1} \Rightarrow a_{14} = 1000 \cdot \frac{1}{8192} \Rightarrow \boxed{a_{14} = \frac{125}{1024}}$

→ We may want to find the sum of ~~the~~ terms in a geometric sequence. There is a nice formula for this.

→ Consider the product $\left(\overbrace{a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{n-1}}^{\sum_{i=1}^{n} a_1 r^{i-1}}\right)(1-r)$

$$= (a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{n-1}) - (a_1 r + a_1 r^2 + a_1 r^3 + \dots + a_1 r^n)$$

$$= a_1 - a_1 r^n$$

$$= a_1 (1 - r^n)$$

That means that

$$a_1 + a_1 r + a_1 r^2 + \dots + a_1 r^{n-1} = \sum_{i=1}^{n} a_1 r^{i-1} = a_1 \left(\frac{1-r^n}{1-r} \right)$$

Ex Find the twelfth partial sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$

$$\sum_{i=1}^{12} 4(0.3)^{i-1} = 4\left(\frac{1-0.3^{12}}{1-0.3} \right) \approx 5.714$$

→ Be sure your sum has the right form to apply the formula.

want $i-1$, not i

Ex $\sum_{i=1}^{12} 4(0.3)^{i}$

using rules of exponents

$$= \sum_{i=1}^{12} \underbrace{4(0.3)^1}_{a_1} \underbrace{(0.3)^{i-1}}_{r^{i-1}} = 4(0.3)\left(\frac{1-0.3^{12}}{1-0.3} \right) \approx 1.714$$

→ if $|r|<1$, than as $n\to\infty$, $r^n \to 0$ so for $|r|<1$, $a_1\left(\frac{1-r^n}{1-r}\right) \to a_1\left(\frac{1-0}{1-r}\right)$ as $n\to\infty$.

→ Thus for $|r|<1$, the sum of the infinite geometric series is

$$\underbrace{\sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}}$$

$$\hookrightarrow = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$$

→ we can write this as $\sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1-r}$

→ These are 2 equivalent ways to express an infinite geometric series.

Ex Evaluate

a) $$\sum_{i=1}^{8} 5\left(-\frac{3}{2}\right)^{i-1} = 5\left(\frac{1-\left(-\frac{3}{2}\right)^8}{1-\left(-\frac{3}{2}\right)}\right) = 5\left(\frac{1-\frac{6561}{256}}{\frac{5}{2}}\right) = 5\left(\frac{\frac{-6305}{256}}{\frac{5}{2}}\right)$$

$$= 5\cdot\frac{-6305}{256}\cdot\frac{2}{5} = \frac{-6305}{112}$$

b) $$\sum_{i=1}^{8} 5\left(-\frac{3}{2}\right)^{i} = \sum_{i=1}^{8} 5\left(-\frac{3}{2}\right)^1\left(-\frac{3}{2}\right)^{i-1} = 5\left(-\frac{3}{2}\right)\left(\frac{1-(-3/2)^8}{1-\left(-\frac{3}{2}\right)}\right) = -\frac{3}{2}\left(\frac{-6305}{112}\right) = \frac{18915}{256}$$

c) $$\sum_{n=0}^{40} 5\left(\frac{3}{5}\right)^n = 5 + 5\left(\frac{3}{5}\right) + 5\left(\frac{3}{5}\right)^2 + \cdots + 5\left(\frac{3}{5}\right)^{40}$$

need 1 not 0

$$= \sum_{n=1}^{41} 5\left(\frac{3}{5}\right)^{n-1} = 5\left(\frac{1-\left(\frac{3}{5}\right)^{40}}{1-3/5}\right) \approx 12.5$$

d) $$\sum_{n=0}^{\infty} 5\left(\frac{3}{5}\right)^n = \frac{5}{1-3/5} = \frac{5}{2/5} = 5\cdot\frac{5}{2} = \frac{25}{2} = 12.5$$

e) $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = \frac{2}{1-2/3} = \frac{2}{1/3} = 6$

Ex A deposit of \$100 is made at the beginning of each month on an account with an annual interest rate of 6% compounded monthly. What is the balance after 5 years?

→ If you deposit \$100 at the beginning of the month, at the end of the month, you have

$$\underbrace{100}_{\text{deposit}} + \underbrace{100 \cdot \left(\frac{.06}{12}\right)}_{\text{one month interest}} = 100\left(1 + \frac{0.06}{12}\right)^1$$

at the end of the second month you have (from the 1st deposit)

$$\underbrace{100\left(1+\frac{0.06}{12}\right)^1}_{\text{start with this}} + \underbrace{100\left(1+\frac{0.06}{12}\right)\left(\frac{0.06}{12}\right)}_{\text{interest}} = 100\left(1+\frac{0.06}{12}\right)\left(1+\frac{0.06}{12}\right)$$

$$= 100\left(1+\frac{0.06}{12}\right)^2$$

→ The deposit you make at the beginning of the 5 years is compounded $12 \cdot 5 = 60$ times after 5 years and has grown to

$$100\left(1+\frac{0.06}{12}\right)^{60}$$

→ The deposit of 100 you make at the beginning of the second month of the first year is compounded

59 times and grows to $100\left(1+\frac{0.06}{12}\right)^{57}$

Deposit at start of 3rd month grows to $100\left(1+\frac{0.06}{12}\right)^{58}$

→ This continues each month until the deposit of 100 you make at the start of the 59^{th} month is compounded once and grows to $100\left(1+\frac{0.06}{12}\right)$

So the total amount you have is

$$A = 100\left(1+\frac{0.06}{12}\right) + 100\left(1+\frac{0.06}{12}\right)^2 + 100\left(1+\frac{0.06}{12}\right)^3 + \dots + 100\left(1+\frac{0.06}{12}\right)^{60}$$

$$= \sum_{i=1}^{60} 100\left(1+\frac{0.06}{12}\right)^i = \sum_{i=1}^{60} \underbrace{100\left(1+\frac{0.06}{12}\right)}_{a}\left(1+\frac{0.06}{12}\right)^{i-1}$$

$$= 100\left(1+\frac{0.06}{12}\right)\left(\frac{1-\left(1+\frac{0.06}{12}\right)^{60}}{1-\left(1+\frac{0.06}{12}\right)}\right)$$

$\approx$ \$7011.89 — (you deposited \$6000)