8.4 → Determinant of a Square Matrix

→ Each square matrix can be associated with a real number called its determinant. We'll see a few applications of determinants in the next section.

→ The determinant of a 2×2 matrix is given by the following.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The determinant of A written $\det(A)$ or $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is

$$|A| = ad - bc$$

Ex $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$. $\det A = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2(3) = -7$

$B = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$. $|B| = \begin{vmatrix} 2 & 4 \\ -1 & 6 \end{vmatrix} = 2(6) - (-1)(4) = 16$

→ To take the determinant of a larger square matrix, we break it down into the determinants of a bunch of 2×2 matrices using a process called cofactor expansion.

→ we need to define 2 things called minors and cofactors.

→ If A is square, the minor M_{ij} of entry a_{ij} is the determinant of the matrix obtained from deleting the i^{th} row and j^{th} column of A. The cofactor C_{ij} of entry a_{ij} is the minor with an associated sign.

$$C_{ij} = (-1)^{i+j} M_{ij}$$

→ Sign pattern for cofactors:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}, \text{etc.}$$

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ -2 & 1 & -1 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} = 0(-1) - 1(4) = -4$, $C_{11} = (-1)^{1+1} M_{11} = -4$

$M_{31} = \begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix} = 2(4) - (0)(-1) = 8$, $C_{31} = (-1)^{3+1} M_{31} = 8$

$M_{23} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1(1) - (-2)(2) = 5$, $C_{23} = (-1)^{2+3} M_{23} = -5$

→ The determinant of a square matrix is the sum of the entries in any row or column multiplied by their respective cofactors

Ex $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & -2 \end{bmatrix}$

$\det(A) = 1 \cdot C_{11} + 0 \cdot C_{21} + 2C_{31}$ → cofactor expansion down 1st column

$= 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 4 \\ 0 & -2 \end{vmatrix} + 0 \cdot (-1)^{2+1} \begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + 2 \cdot (-1)^{3+1} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$

$= (1)(1)(-2 - 0(4)) + (0)(-1) \cdot (3(-2) - 0(2)) + (2)(1)(3 \cdot 4 - 1 \cdot 2)$

$= -2 + 0 + 20 = 18$

$\det(A) = 0 \cdot C_{21} + 1 \cdot C_{22} + 4 \cdot C_{23}$ → cofactor expansion across 2nd row

$= (0)(-1)^{2+1} \begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + (1)(-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + 4(-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$

$= 0 + 1(-2) - 2(2) + (4)(-1)(1(0) - 2(3))$

$= 0 - 6 + 24 = 18$

→ you get the same answer regardless of what column or row you use, so do the easiest one!

Ex $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & 2 \end{bmatrix}$ → use 3rd column to take determinant

$$\det(A) = 3M_{13} - 0M_{23} + 0M_{33} - 0M_{43}$$

$$= 3\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$ → now do cofactor expansion on this

$$= 3\left(-0\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right)$$

$$= 3\left(2(-2-6) - 3(-4-3)\right) = 3\left(2(-8) - 3(-7)\right)$$

$$= 3(-16+21) = 3(5) = 15$$

Ex $$\begin{vmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{vmatrix} = -2(-1)\begin{vmatrix} 6 & -5 & 4 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{vmatrix} + 0 - 6\begin{vmatrix} 3 & 6 & 4 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \end{vmatrix} + 0$$

$$= 2\left(6\begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} - (-5)\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}\right) - 6\left(3\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 6 & 4 \\ 3 & -1 \end{vmatrix} + 0\right)$$

$$= 2\left(6(-2-(-2)) + 5(-1-6) + 4(-1-6)\right) - 6\left(3(-1-6) - 1(-6-12)\right)$$

$$= 2\left(6(0) + 5(-7) + 4(-7)\right) - 6\left(3(-7) - 1(-18)\right)$$

$= 2\left(-35-28\right)-6\left(-21+18\right)$ 5

$= 2(-63) - 6(-3) = \boxed{-108}$ whew!