8.3 - The Inverse of a Square Matrix

→ In this section, ~~we talk about finding an inverse of a matrix.~~

↳ In order to have an inverse, a matrix must be square.

↳ Some, but not all matrices have inverses.

↳ multiplying by an inverse is analogous to dividing

→ Let A be an $n \times n$ matrix and I_n be the $n \times n$ identity matrix. If there exists another matrix A^{-1} where

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

then A^{-1} is the inverse of A.

Ex let $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} -1(1) + 2(1) & -1(-2) + 2(-1) \\ -1(1) + 1(1) & -1(-2) + 1(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-1) - 2(-1) & 1(2) - 2(1) \\ 1(-1) - 1(-1) & 1(2) - 1(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So A and B are inverses.

→ why would we want to use an inverse? we can solve matrix equations.

→ Suppose A, B, and X are matrices ~~there~~ where X is an unknown matrix and A is square

- If A has an inverse, then

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B$$
→ Do the same thing to both sides. Multiply on the left by A^{-1}

$$\Rightarrow IX = A^{-1}B$$
→ multiplying by the identity matrix is like multiplying by 1.

$$\Rightarrow X = A^{-1}B$$

Ex find the inverse of $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

→ find $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ such that $AX = I$

$$\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_{11} + 2x_{21} & -x_{12} + 2x_{22} \\ -x_{11} + x_{21} & -x_{12} + x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -x_{11} + 2x_{21} &= 1 \\ -x_{11} + x_{21} &= 0 \end{aligned} \xrightarrow{\text{augmented matrix}} \left[\begin{array}{cc:c} -1 & 2 & 1 \\ -1 & 1 & 0 \end{array}\right]$$

→ If we put this in reduced row echelon form then the # in the top right position will be x_{11} and in the bottom right position will be x_{21}

and

$$\begin{aligned} -x_{12} + 2x_{22} &= 0 \\ -x_{12} + x_{22} &= 1 \end{aligned} \longrightarrow \left[\begin{array}{cc:c} -1 & 2 & 0 \\ -1 & 1 & 1 \end{array}\right]$$

→ similarly here, but we'll get x_{12} and x_{22}

→ Since the numbers on the left are the same, we could combine and only have to do the row operations once.

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

→ if we put the left side in reduced row echelon form, the right side will give us X.

$$\xrightarrow{-R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

So the inverse is $A^{-1} = X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

→ we already verified that this is the inverse.

→ We can do this process to try to find the inverse of a general square matrix A.

1) Write $[A \,\vdots\, I]$
2) If possible, do operations to change A to I.
3) The result is $[I \,\vdots\, A^{-1}]$

↳ Some times A can be changed to I because we get a row of zeros. This means A is not invertible.

Ex Solve

$x + 2y + 2z = 1$
$3x + 7y + 9z = 1$ using an inverse Matrix
$-x - 4y - 7z = 1$

In matrix form, we have

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B}$$

$$AX = B$$

so $A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$

Find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -3R_1+R_2 \\ R_1+R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_2+R_1 \\ \\ 2R_2+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 7 & -2 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

$$\begin{array}{l} 4R_3+R_1 \\ -3R_3+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -13(1)+6(1)+4(1) \\ 12(1)-5(1)-3(1) \\ -5(1)+2(1)+1(1) \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$$

$\boxed{\text{Solution is } (-3, 4, -2)}$

→ This may seem harder than just using an augmented matrix, but imagine if you were solving $AX = B$ a bunch of times with the same A but a different B each time. Instead of going through the row operations for each new B, you could just do $A^{-1}B$ to solve.

→ Finding an inverse is a long and tedious process, but luckily there's a shortcut for 2x2 matrices.

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

↳ obviously A^{-1} only exists if $ad-bc = 0$

EX a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ then $A^{-1} = \dfrac{1}{(1)(7)-(2)(3)} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

$$= \frac{1}{7-6} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$$

↳ check to see that it's true!

b) $B = \begin{bmatrix} 1 & 3 \\ -3 & -9 \end{bmatrix}$ $B^{-1} = \dfrac{1}{(1)(-9)-(3)(-3)} \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

↳ divide by zero!

B has no inverse! If we reduced $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -3 & -9 & 0 & 1 \end{array}\right]$ we would get a row of all zeros to the left of the dotted line.