8.2 - Operations with matrices

→ In this section, we talk about matrices in more general terms.

→ An $m \times n$ matrix is an array of numbers with m rows and n columns. The entry in the ith row and j^{th} column is often denoted a_{ij}. → 1st index gives row, 2nd gives column

we say the order of the matrix is $m \times n$ "m by n".

Ex a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix. entry a_{21} is 3. a_{32} is 6

b) $\begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{bmatrix}$ is a 2×3 matrix. entry a_{13} is 5

c) $\begin{bmatrix} -1 & 3 & 2 \\ 4 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ is a 3×3 matrix

→ we say 2 matrices are equal if their corresponding entries are equal.

Ex $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 0 \end{bmatrix}$ means $a_{11} = 1$, $a_{12} = 4$, $a_{21} = -3$, $a_{22} = 0$

→ we add and subtract matrices by adding and subtracting the corresponding entries

⇒ (you can only add/subtract matrices if their orders are the same)

<u>EX</u> a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

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b) $\underbrace{\begin{bmatrix} 1 & 5 & -2 \\ 3 & 4 & 1 \end{bmatrix}}_{\text{order } 2\times 3} + \underbrace{\begin{bmatrix} 1 & 2 \\ 10 & -3 \end{bmatrix}}_{\text{order } 2\times 2} \rightarrow$ can't be done

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→ there are two different types of multiplication. The first is called scalar multiplication. Here, a number multiplies each term in the matrix.

<u>EX</u> a) $3\begin{bmatrix} 1 & 4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(4) \\ 3(-2) & 3(0) \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ -6 & 0 \end{bmatrix}$

b) $-2\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -2(4) & -2(1) \\ -2(-2) & -2(2) \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ 4 & -4 \end{bmatrix}$

→ The distributive property works with scalar multiplication

<u>EX</u> $4\left(\begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}\right) = 4\begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} + 4\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$

$= \begin{bmatrix} -12 & 4 \\ 8 & 16 \end{bmatrix} + \begin{bmatrix} 8 & -8 \\ 12 & -12 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ 20 & 4 \end{bmatrix}$

<u>EX</u> solve for the matrix X in $3X + A = B$ where

$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$

$B - A = \begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$

$\Rightarrow 3X = B - A$

$\Rightarrow X = \frac{1}{3}(B-A)$

$\Rightarrow X = \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$

$\Rightarrow X = \begin{bmatrix} -4/3 & 2 \\ 2/3 & -2/3 \end{bmatrix}$

→ The second type of multiplication is trickier, and you have to be careful of the order of the matrices you're trying to multiply. It is called Matrix Multiplication.

→ If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where the entry $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$

i.e. The ij entry in the product comes from moving across row i of A and down column j of B, multiplying the entries & then adding.

→ So to multiply two matrices, the number of columns in the 1st matrix must be the same as the number of rows in the second matrix

Ex

$$\begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -1(1) + 3(0) & -1(2) + 3(7) \\ 4(1) - 5(0) & 4(2) - 5(7) \\ 0(1) + 2(0) & 0(2) + 2(7) \end{bmatrix}$$

3×2 $\quad$ 2×2 $= \begin{bmatrix} -1 & 19 \\ 4 & -27 \\ 0 & 14 \end{bmatrix}$

match

$$\begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}$$ → not possible

2×2 $\quad$ 3×2

no match

Ex a)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-2)+0(1)+3(-1) & 1(4)+0(0)+3(1) \\ 2(-2)-1(1)-2(-1) & 2(4)-1(0)-2(1) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}$$

2×3 $\quad$ 3×2 $\quad\quad$ 2×2

match

product

b)

$$\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(1)+4(2) & -2(0)+4(-1) & -2(3)+4(-2) \\ 1(1)+0(2) & 1(0)+0(-1) & 1(3)+0(-2) \\ -1(1)+1(2) & -1(0)+1(-1) & -1(3)+1(-2) \end{bmatrix}$$

3×2 $\quad$ 2×3

match

product

$$= \begin{bmatrix} 6 & -4 & -14 \\ 1 & 0 & 3 \\ 1 & -1 & -5 \end{bmatrix}$$

3×3

Ex

$$\begin{bmatrix} 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(2)+3(-1)+4(0) \end{bmatrix}$$

1×3 $\quad$ 3×1

$$= \begin{bmatrix} -1 \end{bmatrix}$$

1×1

→ There is a special matrix called the identity matrix. It is square, has ones on its main diagonal, and zeros everywhere else.

Ex a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is the 3×3 identity matrix

b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ is the 5×5 identity matrix.

→ We've seen one application of matrices in writing down and solving systems of linear equations in a more concise and organized way.

→ One of the very powerful things about matrices is their ability to store large amounts of information in a way that doesn't take up much space. Here's a baby example of this.

EX Manufacturer makes 3 types of CD players that are shipped to two different warehouses. Number of units of model i shipped to warehouse j is given by entry a_{ij} in the matrix

$$A = \begin{bmatrix} 5{,}000 & 4{,}000 \\ 6{,}000 & 10{,}000 \\ 8{,}000 & 5{,}000 \end{bmatrix}$$

(6,000 of model 2 shipped to warehouse 1. etc.)

Prices per unit represented by

$B = [\$39.50 \quad \$44.50 \quad \$56.50]$ (model 1 costs \$39.50)

→ Compute BA

$$\underset{1\times 3}{[39.5 \quad 44.5 \quad 56.5]} \underset{3\times 2}{\begin{bmatrix} 5000 & 4000 \\ 6000 & 10000 \\ 8000 & 5000 \end{bmatrix}}$$

$$= [39.5(5000) + 44.5(6000) + 56.5(8000) \qquad 39.5(4000) + 44.5(10{,}000) + 56.5(5000)]$$

→ what is BA?

↳ entry 1 is total value of the merchandise in warehouse 1

entry 2 is total value of the merchandise in warehouse 2