8.1 → Matrices and Systems of Equations

→ It's tedious to write out equations over and over to solve systems of equations like we did in 7.3. We can write things a little more ~~[illegible]~~ concisely using something called an augmented matrix. Basically, we keep track of the numbers without writing all the variables.

<u>Ex</u> Solve

$$\begin{aligned} x \quad\quad -3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \begin{array}{l} \\ -3R_1+R_2 \rightarrow \\ -2R_1+R_3 \rightarrow \end{array} \left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right]$$

2 steps of elimination to eliminate the x variables in the 2nd 2 equations

$$\begin{array}{cccc} -3 & 0 & 9 & 6 \\ 3 & 1 & -2 & 5 \\ \hline 0 & 1 & 7 & 11 \end{array} \qquad \begin{array}{cccc} -2 & 0 & 6 & 4 \\ 2 & 2 & 1 & 4 \\ \hline & 2 & 7 & 8 \end{array}$$

$$\begin{array}{cccc} 0 & -2 & \cancel{-14} & -22 \\ 0 & 2 & 7 & 8 \\ \hline 0 & 0 & -7 & -14 \end{array}$$

1 step of elimination

$$\begin{array}{l} \\ \\ -2R_2+R_3 \rightarrow \end{array} \left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$-\tfrac{1}{7}R_3 \rightarrow \left[\begin{array}{ccc:c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

→ the matrix is in row echelon form

→ 1st entry in each row is a 1. Every # below that 1 is a zero

→ The matrix corresponds to the 3 equations

$$x \quad -3z = -2 \rightarrow x - 6 = -2 \rightarrow \boxed{x = -4}$$

$$y \quad = 11 \rightarrow y + 14 = 11 \Rightarrow \boxed{y = 3}$$

$$\boxed{z = 2}$$

so $\boxed{-4, 3, 2}$

→ This process is called Gaussian Elimination with back substitution. The Gaussian Elimination part is putting the matrix in row echelon form. The back substitution part is solving the corresponding equations.

Gaussian Elimination

1) Choose which equation you aren't going to change and put it in the first row. (Usually the one with a coefficient of 1 for the x term)
2) Remove the 1st entry from the 2nd & 3rd rows using the first row.
3) Remove the 2nd entry from the 3rd row using the 2nd row.
4) Solve the system using back substitution.

Ex Solve using Gaussian Elimination with back substitution

$$\begin{aligned} 2x - y - z &= 0 \\ x \quad\; - 2z &= 1 \\ x + y - 5z &= 3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & -5 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1+R_2 \to \\ -2R_1+R_3 \to \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$$\begin{array}{r} \\ -R_2 \\ -\frac{1}{3}R_3 \to \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & +1 & -3 & +2 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ → row echelon form (any all zero rows on bottom)

$$\begin{aligned} x + y - 5z &= 3 \\ y - 3z &= 2 \\ 0z &= 0 \end{aligned}$$ → infinitely many solutions

let $\boxed{z=a}$

$y - 3a = 2 \Rightarrow \boxed{y = 2+3a}$

$x + 2 + 3a - 5a = 3 \Rightarrow x + 2 - 2a = 3$

$\Rightarrow \boxed{x = 1+2a}$ → $\boxed{(1+2a,\ 2+3a,\ a)}$

→ we can go further with the matrix and not do back substitution if we want to

<u>Ex</u> Solve

$$\begin{aligned} 3x - 2y + z &= 15 \\ -x + y + 2z &= -10 \\ x - y - 4z &= 14 \end{aligned} \qquad \left[\begin{array}{ccc|c} 3 & -2 & 1 & 15 \\ -1 & 1 & 2 & -10 \\ 1 & -1 & -4 & 14 \end{array}\right] \qquad R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -1 & -4 & 14 \\ -1 & 1 & 2 & -10 \\ 3 & -2 & 1 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ R_1+R_2 \rightarrow \\ -3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -4 & 14 \\ 0 & 0 & -2 & -4 \\ 0 & 1 & 13 & -27 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -1 & -4 & 14 \\ 0 & 1 & 13 & -27 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

Row echelon form →

$$\begin{array}{r} \\ \\ -\frac{1}{2}R_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -4 & 14 \\ 0 & 1 & 13 & -27 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\begin{array}{r} 4R_3 + R_1 \\ -13R_3 + R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$R_2 + R_1 \rightarrow \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]}_{\text{Reduced Row echelon form}}$$

→ leading entry in each row is 1 and is only nonzero entry in the column.

corresponding equations

$$\boxed{\begin{aligned} 1x &= 5 \\ 1y &= -1 \\ 1z &= -2 \end{aligned}}$$

→ Taking the augmented matrix all the way to reduced row echelon form is called Gauss-Jordan Elimination

Ex Solve

$$-x+y=-22$$
$$3x+4y=4$$
$$4x-8y=32$$

$$\left[\begin{array}{cc|c} -1 & 1 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc} 1 & -1 & 22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1+R_2 \\ -4R_1+R_3 \end{array}\left[\begin{array}{cc|c} 1 & -1 & 22 \\ 0 & 7 & -62 \\ 0 & -4 & -56 \end{array}\right]$$

$$\begin{array}{r} \\ \frac{1}{7}R_2 \\ \\ \end{array}\left[\begin{array}{cc|c} 1 & -1 & 22 \\ 0 & 1 & -62/7 \\ 0 & -4 & -56 \end{array}\right]$$

$$\begin{array}{r} \\ \\ +4R_2+R_3 \end{array}\left[\begin{array}{cc|c} 1 & -1 & 22 \\ 0 & 1 & -62/7 \\ 0 & 0 & \frac{248}{7}-56 \end{array}\right]$$

3rd equation says $0x+0y=\frac{-248}{7}-56$

↳ this can't be true

the $\boxed{\text{system is inconsistent}}$ → there is no solution.

→ Usually if we have more equations than variables, the ~~solution~~ system will be inconsistent.