# 7.4 - Partial Fractions



→ In the past, we have taken something like

$\frac{2}{x-3} + \frac{-1}{x+2}$ and add it together with a common denominator

to get $\frac{x+7}{(x-3)(x+2)}$. In this section, we learn how to go in the opposite direction.

**EX** Consider $\frac{x-2}{x^2+4x+3} = \frac{x-2}{(x+1)(x+3)}$ distinct linear factors in the denominator.

→ If this came from adding 2 ~~fractions~~ rational expressions, one of the fractions had to have a denominator of $x+1$ and the other had to have a denominator of $x+3$. So really it came from

$\frac{A}{x+1} + \frac{B}{x+3}$ where A and B are the appropriate numbers to get the correct numerator in the result.

→ To find A and B, add the rational expressions

$$\frac{A(x+3)}{(x+1)(x+3)} + \frac{B(x+1)}{(x+1)(x+3)} = \frac{Ax+3A+Bx+B}{(x+1)(x+3)}$$

If this is going to be the same thing we started with, the numerators must be the same.

i.e. we need

$$Ax+3A+Bx+B = x-2 \Rightarrow (A+B)x + (3A+B) = 1x-2$$

That Means

$$A + B = 1$$
$$3A + B = -2$$

subtract 2nd from 1st

$$\begin{array}{r} A+B=1 \\ -(3A+B=-2) \\ \hline -2A=3 \end{array}$$

$$\Rightarrow \boxed{A = -3/2}$$

$$\Rightarrow B - 3/2 = 1$$
$$\Rightarrow B = \tfrac{3}{2} + 1$$
$$\Rightarrow \boxed{B = \tfrac{5}{2}}$$

so $$\frac{x-2}{(x+1)(x+3)} = \frac{-3/2}{x+1} + \frac{5/2}{x+3}$$

→ partial fraction decomposition

<u>EX</u> Find Partial fraction decomposition of $\frac{5}{x^2+x-6}$

$$\frac{5}{x^2+x-6} = \frac{5}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x-2)}{(x+3)(x-2)} + \frac{B(x+3)}{(x+3)(x-2)}$$

Distinct linear factors

$$= \frac{Ax - 2A + Bx + 3B}{(x+3)(x-2)}$$

$$= \frac{(A+B)x + (-2A+3B)}{(x+3)(x-2)}$$

Numerators must match, so $(A+B)x + (-2A+3B) = 0x + 5$

so
$$A + B = 0$$
$$-2A + 3B = 5$$

$$\Rightarrow \begin{array}{r} 2A+2B=0 \\ -2A+3B=5 \\ \hline 5B=5 \end{array}$$

$$\Rightarrow \boxed{B = 1}$$

$$A + 1 = 0$$
$$\Rightarrow \boxed{A = -1}$$

Thus the partial fraction decomposition is

$$\frac{5}{(x+3)(x-2)} = \frac{-1}{x+3} + \frac{1}{x-2}$$

EX $\dfrac{4x^2+2x-1}{x^2(x+1)}$ → repeated linear factors → $x$, $x$, and $x+1$

↳ this could have come from 2 rational expressions, one with denominator $x^2$ and one with denominator $x+1$ or from 3 rational expressions, one with denominator $x$, one with denominator $x^2$ and one with denominator $x+1$.

→ we always assume the greatest number possible. So

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{A(x)(x+1)}{x^2(x+1)} + \frac{B(x+1)}{x^2(x+1)} + \frac{Cx^2}{x^2(x+1)}$$

$$= \frac{Ax^2+Ax+Bx+B+Cx^2}{x^2(x+1)}$$

Numerators must match: $(A+C)x^2 + (A+B)x + B = 4x^2+2x-1$

$\Rightarrow$ $A+C=4$
$A+B=2$
$B=-1$

$\boxed{B=-1} \Rightarrow A-1=2 \Rightarrow \boxed{A=3}$

so $3+C=4 \Rightarrow \boxed{C=1}$

Thus partial fraction decomposition is

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}$$

→ Sometimes the denominator can't be factored into a product of linear factors.

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<u>Ex</u> $\dfrac{x^2-1}{x(x^2+1)}$ → $x^2+1$ can't be factored without using imaginary numbers.

→ proceed like this

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

→ linear instead of constant numerator

$$= \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)}$$

$$= \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

so $(A+B)x^2 + Cx + A = x^2 + 0x - 1$

$\Rightarrow A+B = 1$ $\qquad -1+B=1 \Rightarrow \boxed{B=2}$

$C = 0$ $\qquad$ so

$A = -1$

Decomposition is $\boxed{\dfrac{x^2-1}{x(x^2+1)} = \dfrac{-1}{x} + \dfrac{2x}{x^2+1}}$

Ex $\dfrac{2x^2+x+8}{(x^2+4)^2}$ → repeated quadratic factors. Do analogous thing to repeated linear factors.

$$\frac{2x^2+x+8}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$= \frac{(Ax+B)(x^2+4)}{(x^2+4)(x^2+4)} + \frac{Cx+D}{(x^2+4)(x^2+4)}$$

$$\Rightarrow Ax^3+4Ax+Bx^2+4B+Cx+D = 2x^2+x+8$$

$$\Rightarrow Ax^3+Bx^2+(4A+C)x+(4B+D) = 0x^3+2x^2+x+8$$

$\Rightarrow \boxed{A=0}$

$\boxed{B=2}$

$4A+C=1$ → so $4(0)+C=0 \Rightarrow \boxed{C=1}$

$4B+D=8$ → $4(2)+D=8 \Rightarrow \boxed{D=0}$

partial fraction decomposition is

$$\boxed{\frac{2x^2+x+8}{(x^2+4)^2} = \frac{2}{x^2+4} + \frac{x}{(x^2+4)^2}}$$

→ In all our examples so far, the degree of the denominator has been > degree of numerator

→ If Degree denominator ≤ degree numerator, we start out with long division.

Ex $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}$ → degree 3 / → degree 2 → use long division

$$\begin{array}{r} 2x - 7 \phantom{+x+5} \\ x^2 + 3x + 2 \enclose{longdiv}{2x^3 - x^2 + x + 5} \\ -(2x^3 + 6x^2 + 4x) \phantom{+5} \\ \hline -7x^2 - 3x + 5 \\ -(-7x^2 - 21x - 14) \\ \hline 18x + 19 \end{array}$$

so $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2} = 2x - 7 + \dfrac{18x + 19}{x^2 + 3x + 2}$

$$= 2x - 7 + \frac{18x + 19}{(x+1)(x+2)}$$

use partial fractions on remainder.

$$\frac{18x+19}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+1)(x+2)}$$

$$\Rightarrow Ax + 2A + Bx + B = 18x + 19$$

$$\Rightarrow (A+B)x + (2A+B) = 18x + 19$$

so $A + B = 18$

$2A + B = 19$

1st minus second

$$\begin{array}{r} A+B=18 \\ -(2A+B=19) \\ \hline -A=-1 \end{array}$$

$\Rightarrow \boxed{A=1}$

$1 + B = 18 \Rightarrow \boxed{B=17}$

so $$\frac{18x+19}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{17}{x+2}.$$

Thus

$$\boxed{\frac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2} = 2x - 7 + \frac{1}{x+1} + \frac{17}{x+2}}$$