3.1. → Exponential Functions and Their Graphs

→ up until this point, we have had variables raised to numbers i.e. the exponent was a number, not a variable. Now we start considering functions, called exponential functions, that have a variable in the exponent.

→ The exponential function with base a is

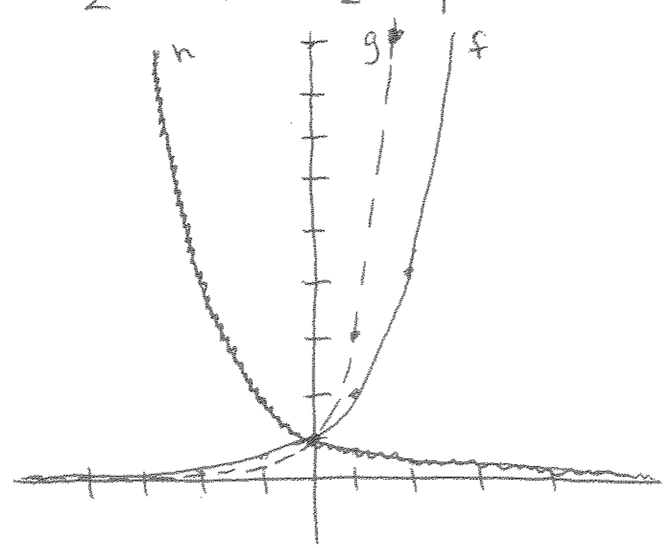
$$f(x) = a^x$$

where $a > 0$, $a \neq 1$ and x is a variable

→ we can plot an exponential function in the same way we plot any function, plot points and connect the dots.

Ex $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = \left(\frac{1}{2}\right)^x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$h(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



→ when $a > 1$, all exponential fuctions look like f and g (steepness depends on value of a)

→ when $0 < a < 1$, all exponential functions look like h

Observations:

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Increasing on $(-\infty, \infty)$ if $a > 1$

Decreasing on $(-\infty, \infty)$ if $a < 1$

All exponential functions cross the y-axis at $y = 1$

All are one-to-one (pass horizontal line test) so they have inverses

↳ The inverses are called logarithms → next section

▷ Because exponential functions are 1-to-1, if we know the output, we should be able to calculate the input. This allows us to solve simple equations.

Ex Solve $16 = 4^{x-1}$

$\Rightarrow 4^2 = 4^{x-1}$ so $2 = x - 1 \Rightarrow \boxed{3 = x}$

Ex $\left(\frac{1}{3}\right)^x = 27$

$\Rightarrow \frac{1}{3^x} = 27$

$\Rightarrow 3^{-x} = 27$

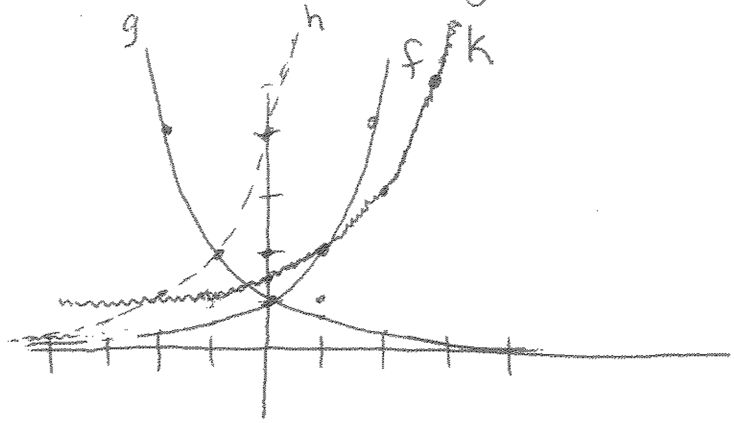
$\Rightarrow 3^{-x} = 3^3$

$\Rightarrow -x = 3$

$\Rightarrow \boxed{x = -3}$

→ The same rules for shifts/flips/etc. apply to exponential functions as other functions.

Ex Plot $f(x) = 2^x$, $g(x) = 2^{-x}$, $h(x) = 2^{x+2}$, $k(x) = 2^{x-1} + 1$



→ The most commonly used exponential function has a specific base, the number

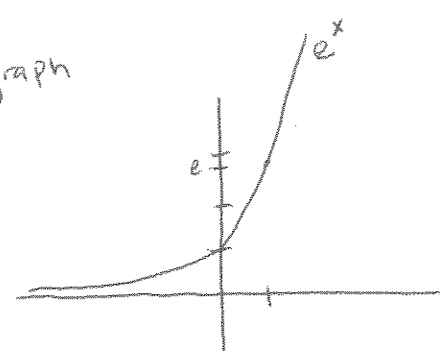
$$e \approx 2.718281828\ldots$$

→ This is an irrational number like π that just seems to show up a lot.

→ The function $f(x) = e^x$ is called the natural exponential.

→ this function is really important and widely used in calculus.

graph



→ looks like all the other ones where $a > 1$.

→ Exponential functions have a lot of practical applications. Two of them are exponential growth and exponential decay.

Compound Interest

Suppose you invest P_0 dollars at an annual interest rate of r percent compounded annually.

year	Balance
0	P_0
1	$P_1 = P_0 + rP_0 = P_0(1+r)$
2	$P_2 = P_1 + rP_1 = P_1(1+r) = P_0(1+r)(1+r) = P_0(1+r)^2$
3	$P_3 = P_2 + rP_2 = P_2(1+r) = P_0(1+r)^2(1+r) = P_0(1+r)^3$
⋮	
t	$P_t = P_0(1+r)^t$

→ Suppose you compound n times per year instead of once. Each time you multiply by $\frac{r}{n}$ instead of r and it happens nt times in t years. So the amount you have is

$$A(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$$

↳ compound n times per year at annual interest rate of r.

<u>Ex</u> You invest \$1000 at an annual interest rate of 5%. How much do you have after 5 years if your investment is compounded

a) quarterly: $A(5) = 1000\left(1 + \frac{.05}{4}\right)^{4(5)} = 1000\left(\frac{4.05}{4}\right)^{4(5)} \approx \$1282.04$

b) monthly: $A(5) = 1000\left(1 + \frac{.05}{12}\right)^{12(5)} = 1000\left(\frac{12.05}{12}\right)^{12(5)} \approx \$1283.36$

c) daily: $A(5) = 1000\left(1 + \frac{.05}{365}\right)^{365(5)} = 1000\left(\frac{365.05}{365}\right)^{365(5)} \approx \$1284.00$

→ As the number of times we compound increases, i.e. as $n \to \infty$, the equation approaches

$$A(t) = P_0 e^{rt}$$

↳ compounded continuously at annual interest rate of r.

d) compounded continuously. $A(5) = 1000\, e^{(.05)(5)} \approx \$1284.03$

→ Another example of exponential functions is exponential decay.

<u>EX</u> Carbon 14 is radioactive and has a half-life of 5715 years. The amount in grams $Q(t)$ of a sample remaining after t years is given by

$$Q(t) = 10\left(\tfrac{1}{2}\right)^{t/5715}$$

what was the original amount of the sample?

$$Q(0) = 10\left(\tfrac{1}{2}\right)^{0/5715} = 10\left(\tfrac{1}{2}\right)^{0} = 10(1) = 10 \text{ grams}$$

How much is left after 2000 years?

$$Q(2000) = 10\left(\tfrac{1}{2}\right)^{2000/5715} \approx 10\left(\tfrac{1}{2}\right)^{0.34996} \approx 7.846 \text{ grams}$$