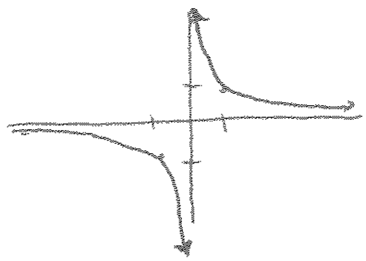
2.6 - Rational Functions

→ Recall the graph of the function $f(x) = \frac{1}{x}$



as $x \to 0$ from the right, $f(x) \to \infty$.
as $x \to 0$ from the left, $f(x) \to -\infty$

→ we call this a vertical asymptote at $x=0$
→ vertical asymptotes come at places we would divide by zero in the simplified rational function.

as $x \to \infty$ and $x \to -\infty$, $f(x) \to 0$.
→ we call this a horizontal asymptote at $y=0$

Asymptotes

write $f(x) = \frac{N(x)}{D(x)}$ where N & D have no common factors

1) f has vertical asymptotes at zeros of $D(x)$

2) if degree $N <$ degree D, there is a horizontal asymptote at $y=0$
2) if degree $N =$ degree D, there is a horizontal asymptote at $y = a_n / b_n$ where a_n is leading coeff. of N and b_n is leading coeff of D.

3) if degree $N >$ degree D, there is a slant asymptote (no horizontal asymptote)

Ex a) $f(x) = \frac{2x^2}{x^2-1} = \frac{2x^2}{(x-1)(x+1)}$

vertical asymptotes at $x=1, -1$

horizontal asymptote at $y=2$

b) $f(x) = \frac{3x^2}{2x^2+1}$

→ no vertical asymptotes

→ horizontal asymptote $y = \frac{3}{2}$

c) $f(x) = \frac{x-1}{x^2-5x+6} = \frac{x-1}{(x-2)(x-3)}$

→ vertical asymptotes $x = 2, 3$

→ horizontal asymptote $y=0$

Plotting Rational Functions : $f(x) = \frac{N(x)}{D(x)}$

1) simplify
2) find vertical asymptotes where $D(x)=0$
3) find horizontal/slant asymptotes
4) find zeros of function where $N(x)=0$
5) Plot a additional points between the zeros & vertical asymptotes

Ex Plot $f(x) = \frac{3}{x-2}$

→ vertical asymptote : $x=2$

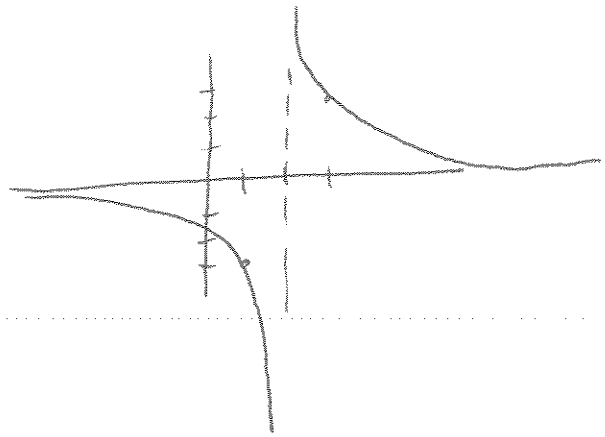
→ horizontal asymptote : $y=0$

→ no zeros

→ additional points

$x=1: \frac{3}{1-2} = -3$

$x=3: \frac{3}{3-2} = 3$



Ex plot $f(x) = \frac{x^2}{x^2-9} = \frac{x^2}{(x-3)(x+3)}$

vertical asymptotes: $x=3,-3$

horizontal: $y=1$

zeros: $x=0$

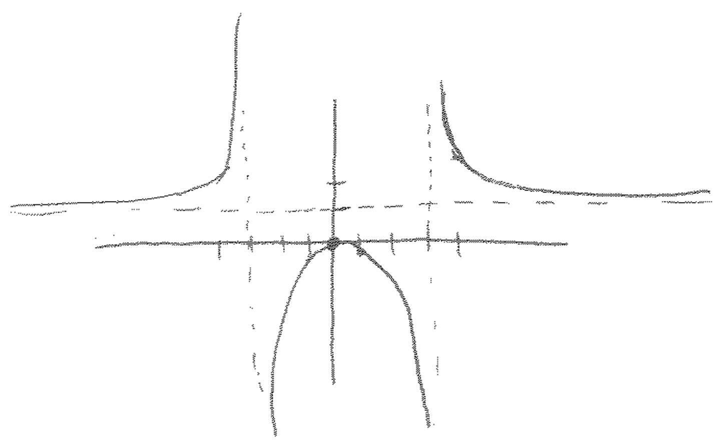
additional points:

$x=-4: \frac{(-4)^2}{(-4-3)(-4+3)} = \frac{16}{(-7)(-1)} = \frac{16}{7}$

$x=-1: \frac{(-1)^2}{(-1)^2-9} = \frac{1}{1-9} = -\frac{1}{8}$

$x=1: \frac{1^2}{1^2-9} = \frac{1}{1-9} = -\frac{1}{8}$

$x=4: \frac{4^2}{4^2-9} = \frac{16}{16-9} = \frac{16}{7}$



Slant Asymptotes $\quad f = \dfrac{N(x)}{D(x)}$

when the degree $N(x) >$ degree $D(x)$, $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to \infty$ and $x \to -\infty$. It follows a slant asymptote as it does so.

→ To find a slant asymptote, use long division

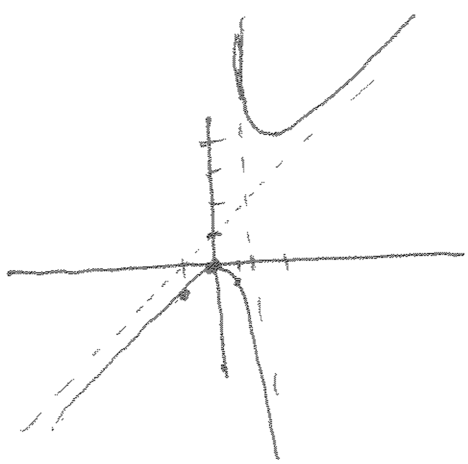
Ex plot $h(x) = \dfrac{x^2}{x-1}$

asymptote at $x = 1$

- no horizontal asymptote
- zero: $x = 1$
- slant asymptote: $\quad$ do $x^2 \div x-1$

$$\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ & & 1 & 1 \\ \hline & 1 & 1 & 1 \end{array} \qquad \text{so } \frac{x^2}{x-1} = x+1+\frac{1}{x-1}$$

→ as $x \to \infty$ or $x \to -\infty$, $\dfrac{1}{x-1} \to 0$, so $f(x)$ looks like $y = x+1$



Test points: $x = -1$: $\dfrac{(-1)^2}{-1-1} = \dfrac{1}{-2}$

$x = 1/2$: $\dfrac{(1/2)^2}{1/2-1} = \dfrac{\frac{1}{4}}{\left(-\frac{1}{2}\right)} = -\dfrac{1}{8}$

$x = 2$: $\dfrac{(2)^2}{2-1} = 4$

Note: you can cross horizontal and slant asymptotes because they only describe the behavior as $x \to \infty$ and $x \to -\infty$.

You cannot cross a vertical asymptote because that would mean dividing by zero.

Ex plot $g(x) = \dfrac{x^3}{2x^2-8} = \dfrac{x^3}{2(x-2)(x+2)}$

V.A. at $x=2$, $x=-2$

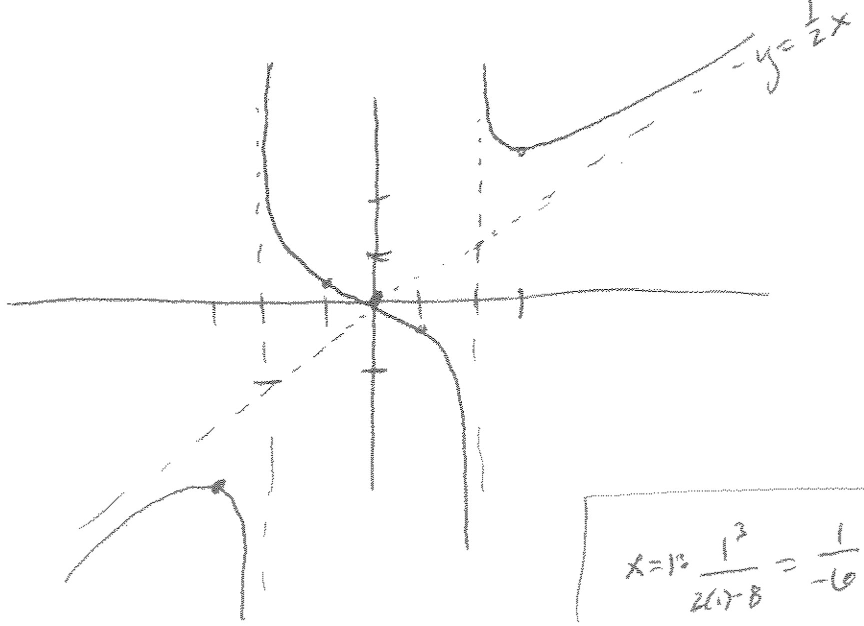
no horiz. asymp.

Zeros: $x=0$

slant asymptote

$$\begin{array}{r} \tfrac{1}{2}x \\ 2x^2+0x-8 \overline{\smash{)}\, x^3+0x^2+0x+0} \\ -(x^3+0x^2-4x) \\ \hline 4x \end{array}$$

so $\dfrac{x^3}{2x^2-8} = \underbrace{\tfrac{1}{2}x}_{\text{slant asymptote}} + \dfrac{4x}{2x^2-8}$ → goes to zero as $x \to \infty$ and $x \to -\infty$



Test pts:

$x=-3$: $\dfrac{(-3)^3}{2(-3)^2-8} = \dfrac{-27}{18-8} = \dfrac{-27}{10}$

$x=-1$: $\dfrac{(-1)^3}{2(-1)^2-8} = \dfrac{-1}{2-8} = \dfrac{1}{6}$

$x=1$: $\dfrac{1^3}{2(1)-8} = \dfrac{1}{-6}$

$x=3$: $\dfrac{3^3}{2(3)^2-8} = \dfrac{27}{10}$